Nonexistence of Negative Weight Derivations on Graded Artin Algebras: A Conjecture of Halperin

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1. INTRODUCTION

A classical result of A. Borel [Bo] states that the Serre spectral sequence for rational cohomology of the universal bundle $G/H \to B_H \to B_G$ collapses if $G/H$ is a homogeneous space of equal rank pairs $(G, H)$ of compact Lie groups. In 1976 S. Halperin made the following conjecture [H, Me].

Halperin Conjecture. Suppose that $F \to E \to B$ is a fibration with simply connected base $B$ and that the (rational) cohomology algebra of the fibre is an Artin algebra of the form

$$H^*(F, Q) = Q[x_0, x_1, \ldots, x_n]/(f_0, f_1, \ldots, f_n).$$

Then the Serre spectral sequence for this fibration collapses.

Here we note that $x_0, x_1, \ldots, x_n$ are cohomology classes and we can take their dimensions to be their weights. In this weight type, $f_0, f_1, \ldots, f_n$ are relations of these cohomology classes in some cohomology groups. Therefore they are weighted homogeneous polynomials. Actually the above conjecture is equivalent to the following conjecture about the nonexistence of negative weight derivations [H, Me].

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Halperin Conjecture (Equivalent Form). Let $x_0, x_1, \ldots, x_n$ be weighted variables and $f_0, f_1, \ldots, f_n$ be weighted homogeneous polynomials in $P = C[x_0, x_1, \ldots, x_n]$. Suppose that $R$ is an Artin algebra of the form $C[x_0, x_1, \ldots, x_n]/(f_0, f_1, \ldots, f_n)$. Then there is no non-zero negative weight derivation on $R$.

Here we note that $R$ and $\text{Der}(R, R)$, the $R$-module of derivations on $R$, are naturally graded by the (weighted) degrees (see Definition 2.1) on $C[x_0, x_1, \ldots, x_n]$.

The (non)existence of negative weight derivations appears in many other contexts which may also serve as a motivation for the present work. For example, their nonexistence has also been conjectured by J. Wahl for positive dimensional quasi-homogeneous isolated singularities [Wa1] and by S. Yau for moduli algebras of isolated quasi-homogeneous hypersurface singularities [Ya1, Ch-Xu-Ya]. For a study of the (non)existence of negative weight derivations from the point of view of singularity theory, we refer the reader to [Ka1, Ka2, Wa, Wa2, M-S, Ch-Xu-Ya, Ch1].

The Halperin conjecture is true when the fibre $F$ is homogeneous [Bo, Me, H]. Also this conjecture has been proved to be true for the two-variable case [Th1, Th2]. The purpose of this note is to give a proof for the Halperin conjecture for the three-variable case. This is a generalization of the work of [Ch-Xu-Ya]. On the other hand we should mention that the Halperin conjecture has been proved in the case of “large” graded Artin algebras (i.e., when the degrees of $f_0, f_1, \ldots, f_n$ are sufficiently large) in our other paper [Ch2].

2. NEW WEIGHT TYPES

Definition 2.1. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be positive integers. A monomial $x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n}$ in the polynomial ring $P = C[x_0, x_1, \ldots, x_n]$ is said to be of (weighted) degree $i_0 \alpha_0 + i_1 \alpha_1 + \cdots + i_n \alpha_n$. A polynomial in $P$ is said to be weighted homogeneous of degree $d$ if it is a $C$-linear combination of monomials of degree $d$.

It is clear that $P$ is graded. If $I \subset P$ is a homogeneous ideal, i.e., an ideal generated by weighted homogeneous polynomials, we know that $R = P/I = \bigoplus R_k$ is also graded. Let $\text{Der}(R, R)$ be the $R$-module of derivations on $R$. There is a natural grading on $\text{Der}(R, R)$ defined by $\text{Der}(R, R)_k = \{D \in \text{Der}(R, R): D(R_i) \subset R_{i+k} \text{ for any possible } i\}$. Derivations in $\text{Der}(R, R)_k$ for $k < 0$ are called negative weight derivations. The main problem considered in this paper is whether $\text{Der}(R, R)_k$ for $k < 0$ is zero for some $k$ and some $R$. We denote the weight of a derivation $D$ by $\text{wt}(D)$. 

It is well known that derivations on $R$ can be thought of as derivations on $P$ preserving the ideal $I$. On the other hand, any derivation on $P$ is of the form

$$D = p_0 \frac{\partial}{\partial x_0} + p_1 \frac{\partial}{\partial x_1} + \cdots + p_n \frac{\partial}{\partial x_n},$$  \hspace{1cm} (2.1)$$

where $p_0, p_1, \ldots, p_n$ are polynomials in $P$. Thus we only consider the negative weight derivations on $P$ which preserve the ideal $I$ in the following part of this note.

From now on we always assume $\alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_n$, which we may do without loss of generality. If a derivation $D$ as in (2.1) on $P$ is of negative weight, it is clear that each $p_i$ is a weighted homogeneous polynomial of degree less than $\alpha_i$. Thus we know that $p_i$ is a polynomial of variables $x_{i+1}, x_{i+2}, \ldots, x_n$ for $i = 0, 1, \ldots, n$, i.e., the negative weight derivation $D$ on $P$ has to be the following form for some $0 \leq k \leq n$,

$$D = p_0(x_1, \ldots, x_n) \frac{\partial}{\partial x_0} + p_1(x_2, \ldots, x_n) \frac{\partial}{\partial x_1} + \cdots + p_k(x_{k+1}, \ldots, x_n) \frac{\partial}{\partial x_k},$$  \hspace{1cm} (2.2)$$

where $p_k$ is a non-zero polynomial.

The purpose of this section is to introduce some new weight types for any given negative weight derivation $D$ on the weighted polynomial ring $P$. These new weight types are the natural generalization of the original weight type $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ and will play a crucial role in this paper and our other paper (Ch2).

**Definition 2.2.** Let $D$ be a negative weight derivation on the weighted polynomial ring $P$ as in (2.2). The following weight types $(l_0, l_1, \ldots, l_n)$ controlled by parameters $(e_0, e_1, \ldots, e_k, \mu_{k+1}, \ldots, \mu_n)$ are called new weight types associated with $D$,

$$l_n = \alpha_n \mu_n$$

$$l_{n-1} = \alpha_{n-1} \mu_{n-1}$$

$$\cdots$$

$$l_{k+1} = \alpha_{k+1} \mu_{k+1},$$  \hspace{1cm} (2.3)$$

where $\mu_{k+1}, \ldots, \mu_n$ are real parameters. If $l_{q+1}, l_{q+2}, \ldots, l_n$ are defined, $l_q$ is defined as follows

(i) If the coefficient $p_q$ of $\frac{\partial}{\partial x_q}$ is zero

$$l_q = \alpha_q \mu_q,$$
where \( \mu \) is a real parameter;

(ii) If the coefficient \( p_q \) of \( \partial / \partial x_q \) is a non-zero polynomial

\[
l_q = \epsilon_q + \min\{Q = \text{degrees of monomials } x_{q+1}^i \cdots x_n^i \text{ appearing in the expansion of } p_q(x_{q+1}, \ldots, x_n)\}, \quad (2.4)
\]

where \( \epsilon_q \) is a real parameter and the \( Q \)-degrees are defined with respect to the weight type \((l_{q+1}, \ldots, l_n)\).

As in the original weight type \((\alpha_0, \alpha_1, \ldots, \alpha_n)\) we can define \( Q \)-degrees of monomials with respect to this new weight type \((l_0, l_1, \ldots, l_n)\) and \( Q \)-weighted homogeneous polynomials.

**Example 2.1.** For any negative weight derivation \( D \) on the weighted polynomial ring \( P \), the new weight type associated with \( D \) is the original weight type \((\alpha_0, \alpha_1, \ldots, \alpha_n)\) if we take the \( \mu \)'s parameters to be 1 and the \( \epsilon \)'s parameters to be \(-\text{wt}(D)\).

**Example 2.2.** Let \( P = C[x_0, x_1, x_2] \) be the weighted polynomial ring of three variables with the weight type \((\alpha_0, \alpha_1, \alpha_2)\) and \( D \) be a negative weight derivation on \( P \) of the form

\[
D = p_0(x_1, x_2) \partial / \partial x_0 + cx_1^m \partial / \partial x_1, \quad (2.5)
\]

where \( p_0 \) is a weighted homogeneous polynomial of degree \( \alpha_0 + \text{wt } D \), \( c \) is a non-zero constant and \( m \) is the positive integer \( \alpha_1 + \text{wt } D \). We consider the new weight type associated with \( D \) controlled by the parameters

\[
\begin{align*}
\epsilon_0 &= -m \alpha_2 \\
\epsilon_1 &= -m \alpha_2 \\
\mu_2 &= 1.
\end{align*} \quad (2.6)
\]

Then we have

\[
\begin{align*}
l_2 &= \alpha_2 \\
l_1 &= 0 \\
l_0 &= -m \alpha_2 + \min\{i_2 \alpha_2 : x_1^i x_2^{i_2} \text{'s appearing in the expansion of } p_0(x_1, x_2)\}. \quad (2.7)
\end{align*}
\]

This new weight type will be used in our proof for the main result, Theorem 3.1. In our proof for the main result, we can introduce a coordinate change; by this coordinate change we only need to prove the
We can assume \( u < m \) without loss of generality (see Section 3). Hence we have

\[
\begin{align*}
l_0 &= -m + u \\
l_1 &= 0 \\
l_2 &= 1.
\end{align*}
\]

We note \( l_0 < 0 \).

Let \( A_k \) be the set of weighted homogeneous polynomials of degrees \( k \) in \( P \), \( A_k^l \) be the set of polynomials in \( A_k \) with \( Q \)-degrees not less than \( l \), and \( C_k^l \) be the set of weighted homogeneous polynomials with respect to both weight types \((\alpha_0, \alpha_1, 1)\) and \((l_0, l_1, l_2)\) of degrees \( k \) and \( Q \)-degrees \( l \).

**Proposition 2.1.** (i) Suppose that \( l \geq \alpha_1 + \text{wt}(D) \). Then we have \( A_k^l \subset D(A_{k - \text{wt}(D)}) \);

(ii) Suppose that \( l < \alpha_1 + \text{wt}(D) \). Then we can take elements of the form \( x_0^u x_1^v x_2^z \), with \( u \alpha_0 + v \alpha_1 + w = k, u l_0 + w = l \), and \( w < m \), as the base of \( C_k^l / C_k^l \cap (A_k^l + D(A_{k - \text{wt}(D)})) \). Moreover \( x_0^{i_0 + i_1 x_1^{i_2}} \) (for \( i_2 - l_0 \geq m \)) = \( (i_0/i_1 + 1)x_0^{i_0 - 1}x_1^{x_1^{i_2}} + x_0^{x_0^{i_2}} \) in \( C_k^l / C_k^l \cap (A_k^l + D(A_{k - \text{wt}(D)})) \), where \( c \) is a non-zero constant only depending on \( p_0(x_1, x_2) \) and \( c \).

**Proof.** (i) Let \( x_0^i x_1^j x_2^k \in A_k^l \) with \( l \geq \alpha_1 + \text{wt}(D) \). Note \( k = l \geq \alpha_1 + \text{wt}(D) = m \), we have \( x_0^{i_0} \in D(e^i x_0 x_1 x_2) \) for a non-zero constant \( e \). Hence \( x_0^{i_0} \in D(A_{k - \text{wt}(D)}) \). For any \( x_1^{j_1} x_2^{k_2} \in A_k^l \) with \( l \geq \alpha_1 + \text{wt}(D) = m \), we have \( l = j + m \) and \( x_0^{j_1} x_1^{j_2} \in A_k^{l - \text{wt}(D)} \). Thus \( x_0^{i_0} x_1^{j_1} x_2^{k_2} = D(e^i x_0 x_1^{j_1} x_2^{k_2}) \) for a non-zero constant \( e \). For any \( x_0^{i_0} x_1^{j_1} x_2^{k_2} \) in the expansion of \( p_0(x_1, x_2) \), we have \( v - m + i_2 = v - (m - i_2) \geq v + l_0 \) (by the definition of \( l_0 \)). Hence \( e^{x_0 x_1 x_2} = D(x_0^{i_0} x_1^{j_1} x_2^{k_2}) \) and terms \( x_0^{i_0} x_1^{j_1} x_2^{k_2} \) with \( i_2 \geq m \). Therefore we have \( x_0^{i_0} x_1^{j_1} x_2^{k_2} \in D(A_{k - \text{wt}(D)}) \). Assume we have proved that \( x_0^{i_0} x_1^{j_1} x_2^{k_2} \)'s, with arbitrary \( u, v, \) fixed \( h \in A_k^{l - \text{wt}(D)} \), and \( l \geq \alpha_1 + \text{wt}(D) = m \), are in \( D(A_{k - \text{wt}(D)}) \).

Now we want to prove that \( x_0^{j_1} x_1^{j_2} \) (in \( A_k^{l - \text{wt}(D)} \)) with \( l \geq 1 + \text{wt}(D) \) is in \( D(A_{k - \text{wt}(D)}) \).

We have \( D(x_0^{j_1} x_1^{j_2}) = (h + 1)p_0(x_1, x_2)x_0^{j_1} x_1^{j_2} + e^m x_0^{x_0^{j_1} x_1^{j_2}} \), for a non-zero constant \( e^m \). As explained above we find that every monomial in the first term is in \( D(A_{k - \text{wt}(D)}) \). Hence the second term is in \( D(A_{k - \text{wt}(D)}) \). We get the conclusion.
(ii) For any monomials \(x_0^{h+1}x_1^{l_0}x_2^{l_1}\) with \(v' \geq m\), we note that \(x_0^{h+1}x_1^{l_0}x_2^{l_1} = \text{linear combination of terms } x_0^{h_{\text{wt}}}x_1^{l_{\text{wt}}} + x_2^{l_{\text{wt}}} \mod D(A_{k-\text{wt}D})\) by the above argument. Moreover we have that monomials of the form \(x_0^{h}x_1^{l}x_2^{m}\) with \(v \geq m\) are in \(D(A_{k-\text{wt}D})\). Therefore the only possible monomials in the base of \(C_k^l/C_k^l \cap (A_{k-\text{wt}D}^l + D(A_{k-\text{wt}D}))\) are of the form indicated in (ii). The other conclusion of (ii) follows directly from the above argument. Q.E.D.

The following proposition gives a clear description about the set \(C_k^l\).

**Proposition 2.2.** (i) Suppose \(C_k^l \neq 0\). We have \(k - l\) can be divided by \(a_1\).

(ii) Suppose that \(C_k^l \neq 0\). We have that

(a) \((l_0/a_0)k \leq l \leq k\), if \(k\) can be divided by \(a_0\);

(b) \(p^m + pl_0 \leq l \leq k\), if \(k\) cannot be divided by \(a_0\). Here \(k = pa_0 + p^m\), where \(p = [k/a_0]\) is the integral part of \(k/a_0\) and \(0 < p' < a_0\), \(p' = p^m\alpha_1 + p^m\) where \(p^m = \{p'/\alpha_1\}\) and \(0 \leq p^m < \alpha_3\);

(iii) For any pair of positive integers \(k\) and \(l\) such that \(k - l\) can be divided by \(a_1\), we have that

(a) if \(k\) can be divided by \(a_0\) and \((l_0/a_0)k + (h + 1)\alpha_1 \leq l \leq 0\) or \(0 < l \leq k - h(t + 1)\alpha_1\), then \(\dim C_k^l \geq h + 1\);

(b) if \(k\) cannot be divided by \(a_0\) and \(p^m + pl_0 + (h + 1)\alpha_1 \leq l \leq 0\) or \(0 < l \leq k - h(t + 1)\alpha_1\), then \(\dim C_k^l \geq h + 1\).

Proof. (i) We note \(a_0 - l_0 = t\alpha_1 + u - \text{wt} D + m - u = t\alpha_1 + u - \text{wt} D + \alpha_1 + \text{wt} D - u = (t + 1)\alpha_1\). The conclusion follows directly.

(ii) For (a), we note \(l_0 < 0\), \(l_1 = 0\), and \(l_2 = 1\). Thus \(Q - \deg x_0^{(l_0/a_0)} = \min(Q\text{-degrees of monomials in } C_k^l)\). For (b) the conclusion follows similarly.

(iii) For fixed positive integer \(k\) divided by \(a_0\), i.e., \(k = pa_0\) and \(l = pl_0 + ra_1\) (note the condition in (iii)), we have \(1 + h \leq r \leq p\) from the condition \(pa_0 + (h + 1)\alpha_1 \leq l \leq 0\). Then \(x_0^{p-r+i}x_1^{r-h(t+1)}x_2^{r-w}D^{w} \in C_k^l\) for \(i = 0, 1, \ldots, h\). We have \(\dim C_k^l \geq h + 1\). In the case that \(0 < l \leq k - h(t + 1)\alpha_1\), let \(k - l = h(t + 1)\alpha_1 + wa_1\) with a positive integer \(w\) (note the condition in (iii)). We have \(x_0^{h}x_1^{h}x_2^{h} \in C_k^l\) for \(i = 0, 1, \ldots, h\). Therefore \(\dim C_k^l \geq h + 1\). The conclusion of (b) can be proved similarly. Q.E.D.

### 3. The Halperin Conjecture for the Three-Variable Case

In this section we give the proof for the main result—the Halperin conjecture in the three-variable case. We first prove a weak result in
Proposition 3.1. From this result we can see that it is natural to introduce the new weight types.

**Proposition 3.1.** Let \( f_0, f_1, \ldots, f_n \in \mathbb{C}[x_0, x_1, \ldots, x_n] \) be weighted homogeneous polynomials. Suppose that \( R = \mathbb{C}[x_0, x_1, \ldots, x_n] / (f_0, f_1, \ldots, f_n) \) is an Artin algebra and let \( D = p_0(x_{i+1}, \ldots, x_n) \partial / \partial x_i \) be a negative weight derivation on \( R \). Then \( D \) is a zero-derivation on \( R \).

**Proof.** By local duality [A-G-V, Theorem 2 in 5.11] we know \( \dim R_M = 1 \) and \( R_{>M} = 0 \). Moreover \( R_i \times R_{M-i} \rightarrow R_M = C \) is a non-degenerate bilinear form. We note that for every weighted homogeneous polynomial \( g \) of degree \( M = \deg f_0 + \deg f_1 + \cdots + \deg f_n - \alpha_0 - \alpha_1 - \cdots - \alpha_n \), there exists a weighted homogeneous polynomial \( h \) of degree \( M - \deg(D) \) such that \( \partial h / \partial x_i = g \). Thus \( h \in (f_0, f_1, \ldots, f_n) \) from the fact \( R_{>M} = 0 \) and \( p_i g = Dh \in (f_0, f_1, \ldots, f_n) \) since \( D \) is a derivation on \( R \). Therefore \( p_i = 0 \) in \( R_{a_i + \deg(D)} \) since \( g \) is arbitrary in \( R_{M - \alpha_i - \deg(D)} \). The conclusion is proved. Q.E.D.

**Corollary 3.1 [Th1, Th2].** The Halperin conjecture is true for the two-variable case.

**Proof.** The statement follows immediately from the analysis (2.2) of negative weight derivations and the above Proposition 3.1.

**Theorem 3.1.** The Halperin conjecture is true for the three-variable case.

**Proof.** Let

\[
D = p_0(x_1, x_2) \partial / \partial x_0 + p_1(x_2) \partial / \partial x_1 + p_2 \partial / \partial x_2 \quad (3.1)
\]

be a negative weight derivation on the weighted polynomial ring \( P \) which preserves the ideal \((f_0, f_1, f_2)\). We want to prove that it is a zero-derivation on \( R = P / (f_0, f_1, f_2) \). There are three possibilities here

1. \( p_2 \neq 0 \);
2. \( p_2 = 0 \), \( p_0 \) and \( p_1 \) are non-zero polynomials;
3. \( p_2 = 0 \), one of \( p_0 \) and \( p_1 \) is the zero-polynomial.

In case (1), we can find a coordinate change which preserves the original weight type as in [Ch1]. In this new coordinate system \( D \) is of the form \( c \partial / \partial x_2 \) (where \( c \) is a non-zero constant). Hence we get the conclusion in cases (1) and (3) from Proposition 3.1. The only remaining case is (2). In case (2), \( D \) is a negative weight derivation as in Example 2.2, (2.5). If \( p_0(x_1, x_2) \) can be divided by \( x_2^m \), i.e., \( u \geq m \), \( D = x_2^m(p_0 \partial / \partial x_0 + c \partial / \partial x_1) \) (where \( c \) is a non-zero constant). Hence we can find a coordinate change as in [Ch1] to put \( D \) into the form \( x_2^m \partial / \partial x_2 \). The conclusion follows from
Proposition 3.1 immediately. If the weight \( \alpha_2 > 1 \) in the case (2), we can introduce a new variable \( x'_2 \) with weight 1 as

\[
D' = p_0(x_1, (x'_2)^{\alpha_2}) \frac{\partial}{\partial x_0} + c(x'_2)^{m_{\alpha_2}} \frac{\partial}{\partial x_1}
\]

\[
f'_0(x_0, x_1, x'_2) = f_0(x_0, x_1, (x'_2)^{\alpha_2})
\]

\[
f'_1(x_0, x_1, x'_2) = f_1(x_0, x_1, (x'_2)^{\alpha_2})
\]

\[
f'_2(x_0, x_1, x'_2) = f_2(x_0, x_1, (x'_2)^{\alpha_2}).
\]

Then \( D' \) is a negative weight derivation on \( R = C(x_0, x_1, x'_2)/(f'_0, f'_1, f'_2) \).

We note that \( R \) is an Artin algebra from Proposition 2 in p. 198 of [A-G-V]. Hence we can assume \( \alpha_2 = 1 \) without loss of generality. From now on we only consider the situation in Example 2.2 continued.

Let \( \deg f_i = d_i \) and \( M = d_0 + d_1 + d_2 - \alpha_0 - \alpha_1 - 1 \). We have the following conclusions from local duality (see [A-G-V])

(i) \( R = \bigoplus_{k \geq 0} R_k \) and \( R_k = 0 \) for \( k > M \);

(ii) \( \dim R_M = 1 \) and \( R_M \) is generated over \( C \) by \( \det(\partial f_i/\partial x_0)_{0 \leq i, j \leq 3} \);

(iii) \( R_i \times R_{M-i} \rightarrow R_M \), defined by the multiplication in the Artin algebra \( R \), is a non-degenerate bilinear form.

In the following part of the proof we want to prove \( A_M \subset D(A_{M - \wtd}) + (f_0, f_1, f_2) \), i.e., \( R_M = 0 \) (note \( R_{M - \wtd} = 0 \) for \( \wtd < 0 \)), which is a contradiction.

Let \( x_1^i x_2^j \) be the monomial in the expansion of \( p_0(x_1, x_2) \) with the smallest possible exponent \( u \) of \( x_2 \) as in Example 2.2 (continued). It is clear that \( A_u \) is generated over \( C \) by \( x_2^u \) since \( u < m < \alpha_1 \leq \alpha_0 \). Therefore we can reduce to the case that \( u = 0 \) by shifting \( -u \) in the exponents of the variable \( x_2 \) and local duality (iii) (note \( x_2^0 \neq 0 \) in \( R_0 \), otherwise, \( D = p_0(x_1, x_2) \frac{\partial}{\partial x_0} \) in \( R \) and the conclusion follows directly from Proposition 3.1). We assume \( u = 0 \) in the following part of the proof. In this case, if \( x_2^0 x_1^l x_2^j \) and \( x_2^0 x_1^l x_2^j \) are two elements in \( C_k \) we have \( (i_0 - i_0)_0 + (i_2 - i_2) = l - l = 0 \) and \( i_2 - i_2 \) can be divided by \( m \) by \( i_0 = -m \). Thus we have \( \dim(C_k \cap (A_k^0 + D(A_{k - \wtd}))) = 1 \) from Proposition 2.1(ii).

Let \( h = \max(l; A_M^0/A_M \cap (D(A_{M - \wtd}) + (f_0, f_1, f_2)) \neq 0) \). By Proposition 2.1(i) we know \( h < \alpha_1 + \wtd \). Here we want to prove \( C_M^h \subset C_M^h \cap ((A_M^0 + D(A_{M - \wtd})) + (f_0, f_1, f_2)) \), which is a contradiction and thus \( A_M^0 \subset D(A_{M - \wtd}) + (f_0, f_1, f_2) \).

We note that there is a term of the form \( x_2^0 \) in the expansion of one of \( f_0, f_0, f_2 \), otherwise, we have \( (f_0, f_1, f_2) \subset (x_1, x_2) \), which is a contradiction.
to the fact that $P/(f_0, f_1, f_2)$ is an Artin algebra. Therefore we can assume without loss of generality that $f_0$ contains a term $x_0^j$ in its expansion.

Case (i). $h \leq (l_0/\alpha_0)d_0 + M - d_0$.

Let $M = p\alpha_0 + p'$ and $p' = p''\alpha_1 + p'''$ as in Proposition 2.2. We have $p_0 + p'' \leq h$ by Proposition 2.2. Note that $d_0 = v\alpha_0$. We have $M - d_0 = (p - v)\alpha_0 + p'$ with $p' < \alpha_0$ and $p' = p''\alpha_1 + p'''$ with $p''' < \alpha_1$. Therefore $(p - v)l_0 + p''' \leq h - vl_0 \leq M - d_0$ from the condition of case (i). Whenever $h - vl_0$ is positive or negative we have $\dim C_{M-d_0}^{h-vl_0} \geq 1$ from Proposition 2.2. Thus we can find a monomial $x_0^{l_0}x_1^{l_1}x_2^{l_2}$ in $C_{M-d_0}^{h-vl_0}$.

Let $f_0(x_0, x_1, x_2) = x_0^{l_0} + f_0^1 + f_0^2 + \cdots$ be the expansion of $f_0$ according to $A_0 = \oplus_{i \geq v} C_i^{l_0}$. It is clear that $\dim A_{d_0}^{l_0} = 1$ and $A_{d_0}^{l_0}$ is spanned by the monomial $x_0^{l_0}$. Multiplying the $x_0^{l_0}x_1^{l_1}x_2^{l_2}$ in $C_{M-d_0}^{h-vl_0}$ as above on both sides we get

$$x_0^{l_0}x_1^{l_1}x_2^{l_2}f_0 = x_0^{l_0} + x_0^{l_0}x_1^{l_1}x_2^{l_2}f_0^1 + \cdots. \quad (3.3)$$

Note that the terms after the first term in (3.3) are in $A_{d_0}^{l_0}$. On the other hand $j_i - vl_0 \geq \alpha_i$ from Proposition 2.2(i). Thus $j_i + h - vl_0 \geq h + \alpha_i$. We have all those terms are in $D(A_M - wt.D) + (f_0, f_1, f_2)$ by the definition of $h$. Therefore we have

$$x_0^{l_0}x_1^{l_1}x_2^{l_2}f_0 = x_0^{l_0}x_1^{l_1}x_2^{l_2}$$

in $C_M^{h} / (C_M^{h} \cap \left[ A_M^{h} + D(A_M - wt.D) + (f_0, f_1, f_2) \right])$. \quad (3.4)

On the other hand $x_0^{l_0}x_1^{l_1}x_2^{l_2} = ex_0^{l_0}x_1^{l_1}x_2^{l_2}$ for a non-zero constant $e$ and the base monomial $x_0^{l_0}x_1^{l_1}x_2^{l_2}$ of $C_M^{h} / (C_M^{h} \cap \left( A_M^{h} + D(A_M - wt.D) \right))$ as in Proposition 2.1. Here we should note that the dimension of the $C_M^{h} / (C_M^{h} \cap \left( A_M^{h} + D(A_M - wt.D) \right))$ is 1 as remarked above. Thus by (3.4) we have $x_0^{l_0}x_1^{l_1}x_2^{l_2} \in (f_0, f_1, f_2)$ in $C_M^{h} / (C_M^{h} \cap \left( A_M^{h} + D(A_M - wt.D) \right))$. We get the conclusion.

Case (ii). $h > (l_0/\alpha_0)d_0 + M - d_0$.

Here we first assume that $d_0 \geq 3\alpha_0$ and $d_1 \geq (\alpha_0 + \alpha_1 + \alpha_2) + (\alpha_1 - 1) = \alpha_0 + 2\alpha_1$. The low-degree case will be treated later. In case (ii) we cannot use $f_0$ to get the conclusion as in case (i). Let us assume $d_2 = \deg f_2 = \min(d_0, d_1, d_2)$ without loss of generality. Consider the expansion of $f_2$ according to $A_{d_2} = \oplus_{i \geq (l_0/\alpha_0)d_2} C_i^{d_2}$.

$$f_2(x_0, x_1, x_2) = f_2^0 + f_2^1 + \cdots, \quad (3.5)$$

where the terms after $f_2^0$ have $Q$-degrees $\geq Q - \deg f_2^0 + \alpha_1$ (see Proposition 2.2(i)). This $f_2^0$ will be used to lead to a contradiction.
Suppose that there are $w + 1 (w \geq 0)$ monomials in the expansion in the expansion of $f^p_0$. By Proposition 2.2 and its proof we know that all monomials $x_0^i x_1^r x_2^s$ in $C^h_{M-d_1}$ have their $i_0$’s distinct. Let $g = \max(i_0; x_0^i x_1^r x_2^s$ in the expansion of $f^p_0$). Hence we have $g \geq w$ and for the monomial $x_0^i x_1^r x_2^s$ in the expansion of $f^p_0$, we have $i_2 \geq -wl_0$. Thus $d_2 \geq g \alpha_0 - wl_0$ and $d_2 - w(t + 1) \alpha_1 \geq p \geq g \alpha_0 - w l_0$.

By Proposition 2.1(i) and the definition of $h$ we have $h - p < \alpha_1 + wt D - gl_0 + wl_0 \leq \alpha_1 + wt D + (g - w) \alpha_0 \leq d_2 + \alpha_1 + wt D + w(l_0 - \alpha_0) = d_2 + \alpha_1 + wt D - w(t + 1) \alpha_1$. Since $d_0 \geq 3 \alpha_0$ is assumed we have $M - d_2 \geq d_2 + \alpha_1 + wt D$ and $h - p < M - d_2 - w(t + 1) \alpha_1$.

On the other hand $p \leq d_2 - w(t + 1) \alpha_1$ as argued above. We have $h - p \geq vl_0 + M - d_0 - d_2 + w(t + 1) \alpha_1$ from the assumption of $h$ in case (ii). Therefore $h - p \geq vl_0 + d_1 - (\alpha_0 + \alpha_1 + \alpha_2) + w(t + 1) \alpha_1 \geq vl_0 + \alpha_1 - 1 + w(t + 1) \alpha_1$ by $d_i \geq \alpha_0 + 2 \alpha_1$ is assumed. Let $M - d_2 = [(M - d_2)/\alpha_0] \alpha_0 + p'$ with $p' < \alpha_0$ and $p' = p'' \alpha_1 + p'''$ with $p''' < \alpha_1$, as in Proposition 2.2. We have $[(M - d_2)/\alpha_0] \geq v$ since $M - d_2 \geq d_0$ from the condition $d_1 \geq \alpha_0 + 2 \alpha_1$. Therefore $h - p \geq [(M - d_2)/\alpha_0] \alpha_0 + \alpha_1 - 1 + w(t + 1) \alpha_1$.

From the above argument whenever $h - p$ is positive or negative, it satisfies the condition in Proposition 2.2(iii). Hence we have dim $C^h_{M-d_1} \geq w + 1$. A similar argument as in case (i) gives us that $x_0^i x_1^r x_2^s f_0, \ldots, x_0^i x_1^r x_2^s f_2 \in C^h_M \cap (A^h_M + D(A_{M-wtD} + (f_0, f_1, f_2)))$, where $x_0^i x_1^r x_2^s, \ldots, x_0^i x_1^r x_2^s$ are $w + 1$ distinct monomials in $C^h_{M-d_2}$ whose existences have been proved as above. As in Proposition 2.1 and the remark before case (i), let $x_0^i x_1^r x_2^s$ be the base monomial of the space $C^h_M/C^h_M \cap (A^h_M + D(A_{M-wtD}))$.

We have that
\[
\begin{align*}
    x_0^i x_1^r x_2^s f_0^0 &= e_0 x_0^0 x_1^r x_2^s \\
    \ldots \\
    x_0^i x_1^r x_2^s f_2^0 &= e_w x_0^i x_1^r x_2^s
\end{align*}
\]
where $e_0, \ldots, e_w$ are constants. By Proposition 2.1(ii) we have
\[
\begin{pmatrix}
    e_0 \\
    e_1 \\
    \ldots \\
    e_w
\end{pmatrix} = E
\begin{pmatrix}
    a_0 \\
    a_1 \\
    \ldots \\
    a_w
\end{pmatrix},
\]
where $E$ is a non-singular matrix and $(a_0, a_1, \ldots, a_w)^t$ is a non-zero vector only depending on $f_2$ and the derivation $D$. Finally we get the conclusion $x_0^i x_1^r x_2^s \in C^h_M \cap (A^h_M + D(A_{M-wtD} + (f_0, f_1, f_2)))$. This is what we want to prove.
In the following part we treat the low-degree case \( d_0 < 3\alpha_0 \) or \( d_1 < \alpha_0 + 2\alpha_1 \). The main point here is that we can analyze the \( Q \)-degrees of monomials in the expansion of \( f_0 \) and \( f_1 \) if their ordinary degrees are small. In the case \( d_1 < \alpha_0 + 2\alpha_1 \), let us assume that the smallest \( Q \)-degree component in \( f_1 \) is of the form \( x_0 x_1 x_2^q \) with an integer \( 0 \leq q \leq \alpha_1 \).

As in the previous arguments what we want to prove here is that \( C_{M - p, d_1} \neq 0 \) where \( p = Q - \deg(x_0 x_1 x_2^q) = q + l_0 \). Note that \( Df_1 \) cannot be zero here (otherwise \( (c_0 x_1^q \partial / \partial x_0 + c_1 x_1^q \partial / \partial x_1)(x_0 x_1 x_2^q) = x_2^q (c_0 x_1^q + c_1 x_0 x_1^{-1}) \) (since \( Df_1 = g f_2 \) from the condition \( D(f_0, f_1, f_2) \subset (f_0, f_1, f_2) \) and the assumption \( d_1 \geq d_1 \geq d_2 \)).

Let us suppose that the smallest \( Q \)-degree component of \( f_2 \) is \( c_0 x_1^{q+1} + c_1 x_0 x_2^{l_0} \). Hence \( d_2 = \alpha_0 - l_0 \alpha_2 = \alpha_0 - l_0 \). Therefore \( M - d_1 = d_0 + d_2 - (\alpha_0 + \alpha_2 + 1) = (v - 1)\alpha_0 + d_2 - \alpha_1 - 1 \) (here \( v = d_0 / \alpha_0 \) by the fact \( x_0^q \) is in the expansion of \( f_0 \)). What we want to prove here is that \( x_0^q x_1^{q-1} \) (note that it has degree \( M - d_1 \)) has its \( Q \)-degree \( \leq h - p \). We have \( q - 1 + v l_0 \leq v l_0 + M - d_0 - q - l_0 \leq h - p \) by the assumption of \( h \) in case (ii). Thus we have \( h - p = Q - \deg(x_0^q x_1^{q-1}) + r \alpha_1 \) with \( r \leq v \) and \( x_0^q x_1^{q-1} x_2^{1-r} \neq 0 \). We get the conclusion. The other possibilities can be proved similarly. Q.E.D.

In the arbitrary \( n + 1 \) variable case we cannot have a clear description of \( C_k \) (i.e., Propositions 2.1 and 2.2). Thus only a partial result which gives a positive answer to the Halperin conjecture for “large” graded Artin algebras can be obtained [Ch2].

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**REFERENCES**


