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Expansions, Free Inverse Semigroups, and Schützenberger Product

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In this paper we shall present a new construction of the free inverse monoid on a set X. Contrary to the previous constructions of [9, 11], our construction is symmetric and originates from classical ideas of language theory. The ingredients of this construction are the free group on X and the relation that associates to a word w of the free monoid on X, the set of all pairs (u, v) such that $uv = w$ It follows at once from our construction that the free inverse monoid on X can be naturally embedded into the Schützenberger product of two free groups of basis X. We shall also give some connections with the theory of expansions as developed by Rhodes and Birget [2, 3].

1. THE SCHÜTZENBERGER PRODUCT

The Schützenberger product of two monoids was introduced by Schützenberger in view of applications to language theory. Given two monoids M_1 and M_2, the Schützenberger product $\diamond(M_1, M_2)$ is defined as follows. Let $M = M_1 \times M_2$. Then $\mathscr{P}(M)$, the set of finite subsets of M, is a semiring with union as addition and the usual multiplication of subsets as multiplication. Then $\diamond(M_1, M_2)$ is the set of all 2×2 matrices P over this semiring satisfying the following conditions:

(1) $P_{21} = \varnothing$, that is, P is upper triangular.

(2) $P_{11} = \{(m_1, 1)\}$ for some $m_1 \in M_1$.

(3) $P_{22} = \{(1, m_2)\}$ for some $m_2 \in M_2$.

The multiplication is the usual multiplication of matrices over a semiring.

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Conditions (2) and (3) above show that an element of $\Diamond(M_1, M_2)$ can be represented by matrices of the form

$$\begin{pmatrix} m_1 & Q \\ \varnothing & m_2 \end{pmatrix},$$

where $m_1 \in M_1$, $m_2 \in M_2$, and $Q \subset M_1 \times M_2$. If we denote union by $+$ (and the empty set by 0) the multiplication of two such matrices is given by

$$\begin{pmatrix} m_1 & Q \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} n_1 & R \\ 0 & n_2 \end{pmatrix} = \begin{pmatrix} m_1 n_1 & m_1 R + Q n_2 \\ 0 & m_2 n_2 \end{pmatrix},$$

where $m_1 R = \{(m_1 x_1, x_2) | (x_1, x_2) \in R\}$ and $Q m_2 = \{(x_1, x_2 m_2) | (x_1, x_2) \in Q\}$.

Here is a first connection between Schützenberger products and inverse monoids.

PROPOSITION 1.1. *Let G and H be groups. Then $\Diamond(G, H)$ is an inverse monoid.*

Proof. Let $P = \left(\begin{smallmatrix} g & X \\ 0 & h \end{smallmatrix}\right)$ and $\bar{P} = \left(\begin{smallmatrix} \bar{g} & \bar{X} \\ 0 & \bar{h} \end{smallmatrix}\right)$. Then $P\bar{P}P = P$ and $\bar{P}P\bar{P} = \bar{P}$ if and only if $\bar{g} = g^{-1}$, $\bar{h} = h^{-1}$, $X + g\bar{X}h = X$, and $\bar{X} + \bar{g}X\bar{h} = \bar{X}$. The last two equations are equivalent to $\bar{X} = g^{-1}Xh^{-1}$ since $g\bar{X}h \subset X$ and $g^{-1}Xh^{-1} \subset \bar{X}$ imply $X = g(g^{-1}Xh^{-1})\,h \subset g\bar{X}h \subset X$. It follows that

$$\bar{P} = \begin{pmatrix} g^{-1} & g^{-1}Xh^{-1} \\ 0 & h^{-1} \end{pmatrix}$$

is the unique inverse of P. Thus $\Diamond(G, H)$ is an inverse monoid. ∎

This last result is also a consequence of the following proposition since it is well known that a semidirect product of a group by a semilattice is inverse.

PROPOSITION 1.2. *Let G and H be groups. Then $\Diamond(G, H)$ is isomorphic to a semidirect product $S * (G \times H)$, where S is the semilattice of subsets of $G \times H$ under union.*

Proof. If $X \in S$ and $(g, h) \in G \times H$, set

$$(g, h)\,X = gXh^{-1} = \{(gx, yh^{-1}) | (x, y) \in X\}.$$

This defines a left action of $G \times H$ on S and we can form the semidirect product $S * (G \times H)$. Define a function

$$\varphi\colon S * (G \times H) \to \Diamond(G, H)$$

by

$$(X, (g, h))\,\varphi = \begin{pmatrix} g & Xh \\ 0 & h \end{pmatrix}.$$

It is bijective since

$$(Xh^{-1}, (g, h))\,\varphi = \begin{pmatrix} g & X \\ 0 & h \end{pmatrix}.$$

Now a simple verification shows that φ is a morphism. Thus $\Diamond(G, H)$ is isomorphic to $S * (G \times H)$. ∎

COROLLARY 1.3. *If G and H are groups, $\Diamond(G, H)$ is isomorphic to a reverse semidirect product $(G \times H) *_r S$.*

Proof. Indeed, if S is a semigroup and G is a group, every semidirect product $S * G$ is isomorphic to a reverse semidirect product $G *_r S$ and vice versa. See [8], for instance. ∎

2. THE FREE INVERSE MONOID

In this paper, we view an inverse monoid M as a monoid with an involution, denoted by "$^-$", satisfying the axioms

(1) $x\bar{x}x = x$,

(2) $(\overline{xy}) = \bar{y}\bar{x}$,

(3) $\bar{\bar{x}} = x$,

(4) $x\bar{x}y\bar{y} = y\bar{y}x\bar{x}$.

Axioms (1), (2), and (3) say that $\bar{x}$ is an inverse of x and (4) says that idempotents commute. If S is a subset of M, the inverse submonoid of M generated by S is the smallest submonoid of M containing $S \cup \bar{S}$, where $\bar{S} = \{\bar{s} \mid s \in S\}$. This is also the smallest inverse monoid containing S.

Let X be a set. We denote by $F(X)$ the free group on X and by $FI(X)$ the free inverse semigroup on X. $F(X)$ can be constructed as follows. Let $\bar{X}$ be a disjoint copy of X and let $A = X \cup \bar{X}$. Then $F(X)$ is the quotient of the free monoid A^* by the relations $x\bar{x} = \bar{x}x = 1$, for all $x \in X$. In the sequel we shall denote by $\varphi\colon A^* \to F(X)$ the canonical morphism. A word of A^* is reduced if it does not contain a factor of the form $x\bar{x}$ or $\bar{x}x$ with $x \in X$. Notice that a factor of a reduced word is also reduced. Since the restriction of φ to the set of all reduced words is one-to-one, we shall identify $F(X)$ with the set of all reduced words of A^*.

For the free inverse monoid, a number of constructions are known [9–11]. Let us briefly review the construction of [10, 11]. A subset S of $F(X)$ is called a Schreier system [4, 5] if for all words u and v of A^*, $uv \in S$ implies $u \in S$. (In the terminology of [10] this is a closed subset of $F(X)$.) The next proposition is a slightly modified version of Theorem 4.2 of [9].

PROPOSITION 2.1. *If S and T are two Schreier systems and if g is an element of S, then $S \cup gT$ is a Schreier system.*

It follows that the set

$$M = \{(S, g) | S \text{ is a finite Schreier system and } g \in S\}$$

is a monoid under the multiplication

$$(S, g)(T, h) = (S \cup gT, gh).$$

In fact M is the free inverse monoid on X [9]. The inverse of (S, g) is $(g^{-1}S, g^{-1})$. Furthermore, M is generated (as an inverse monoid) by the set

$$\{(\{1, x\}, x) | x \in X\}.$$

Let A be an alphabet and let $\psi: A^* \to M$ be a surjective morphism onto a monoid M. We now define the 2-expansion of M relative to ψ. It was first introduced by K. Henckell (private communication) and considered in more detail by Birget [2].

Let $\tau: A^* \to \mathcal{P}(M \times M)$ be the function defined by

$$w\tau = \{(u\psi, v\psi) | uv = w\}.$$

In the terminology of language theory [1], it is called a transduction from A^* to $M \times M$. Notice that $u\tau = v\tau$ implies $u\psi = v\psi$. Moreover

LEMMA 2.2. *The equivalence $\sim_\tau$ defined on A^* by $u \sim_\tau v$ iff $u\tau = v\tau$ is a congruence on A^*.*

Proof. By duality it is sufficient to prove that $\sim_\tau$ is a right congruence. Let $u, v \in A^*$ be such that $u\tau = v\tau$ and let $w \in A^*$. Let $(m_1, m_2) \in (uw)\tau$. Then there exists a factorization $uw = x_1 x_2$ such that $x_1\psi = m_1$ and $x_2\psi = m_2$. Now one of the following cases hold.

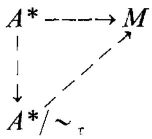
(1) There exists a word $u' \in A^*$ such that $u = x_1 u'$ and $u'w = x_2$.

(2) There exists a word $w' \in A^*$ such that $x_1 = uw'$ and $w = w'x_2$.

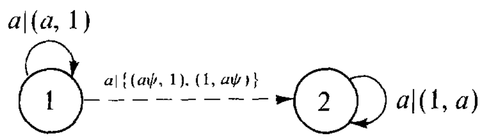
In the first case, $(x_1\psi, u'\psi) \in u\tau = v\tau$ and hence there exist $v_1, v_2 \in A^*$ such that $v_1 v_2 = v$ and $v_1\psi = x_1\psi = m_1$ and $u'\psi = v_2\psi$. It follows that

$(v_1\psi, (v_2 w)\psi) \in (vw)\tau$. But $v_1\psi = m_1$ and $(v_2 w)\psi = (v_2\psi)(w\psi) = (u'\psi)(w\psi) = (u'w)\psi = x_2\psi = m_2$ and hence $(m_1, m_2) \in (vw)\tau$. In the second case, $((vw')\psi, x_2\psi) \in (vw)\tau$. But $(vw')\psi = (v\psi)(w'\psi) = (u\psi)(w'\psi) = x_1\psi = m_1$ and $x_2\psi = m_2$. Therefore $(m_1, m_2) \in (vw)\tau$. Thus $(uw)\tau \subset (vw)\tau$ and similarly $(vw)\tau = (uw)\tau$. It follows that $\sim_\tau$ is a right congruence. ∎

By definition, $A^*/\sim_\tau$ is the 2-expansion of M relative to ψ, denoted by $M(2, \psi)$. Since $u \sim_\tau v$ implies $u\psi = v\psi$, there is a surjective morphism $A^*/\sim_\tau \to M$ and we have the commutative diagram



In fact there is a much better description of $M(2, \psi)$ using the matrix representation of τ. (The reader is referred to [1, Chap. 3] for the basic definitions on transductions.) A transducer realizing τ is represented by



for all $a \in A$.

The matrix representation of this transducer is the monoid morphism

$$\mu: A^* \to \mathscr{P}(M \times M)^{2\times 2}$$

defined by

$$w\mu = \begin{pmatrix} \{(w\psi, 1)\} & w\tau \\ 0 & \{(1, w\psi)\} \end{pmatrix}.$$

We claim that $M(2, \psi) = A^*\mu$. Indeed if $u\mu = v\mu$ then $(u\mu)_{12} = (v\mu)_{12}$, that is, $u\tau = v\tau$. Conversely if $u\tau = v\tau$, $u\psi = v\psi$ and thus $u\tau = v\tau$.

It follows that one can identify $M(2, \psi)$ with a submonoid of $\diamond(M, M)$. Thus elements of $M(2, \psi)$ have the form

$$\begin{pmatrix} m & X \\ 0 & m \end{pmatrix},$$

where X is a finite subset of $M \times M$. However, we do not claim that all matrices of this form are in $M(2, \psi)$. Since ψ is onto, there is a natural surjective morphism $\pi\colon M(2, \psi) \to M$ defined by

$$\begin{pmatrix} m & X \\ 0 & m \end{pmatrix} \pi = m.$$

Clearly the following diagram is commutative:

$$\begin{array}{ccc} A^* & \xrightarrow{\ \mu\ } & M(2, \psi) \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

We come back to the case where $A = X \cup \bar{X}$ and $\psi = \varphi\colon A^* \to F(X)$ is the natural morphism onto the free group on X. Then we have

PROPOSITION 2.3. *The 2-expansion of $F(X)$ relative to φ is an inverse monoid.*

Proof. Let N be the 2-expansion of $F(X)$ relative to φ. Then N is a submonoid of $\diamond(F(X), F(X))$, which is inverse by Proposition 1.1. We need only show that every generator of N has an inverse in N. N is generated as a monoid by $\{a\mu \mid a \in A\}$. Set

$$\bar{a} = \begin{cases} \bar{x} & \text{if } \ a = x \in X \\ x & \text{if } \ a = \bar{x} \in \bar{X}. \end{cases}$$

Then we have

$$\begin{aligned} (a\tau)(\bar{a}\tau)(a\tau) &= (a\bar{a}a)\,\tau \\ &= \{(1\varphi, (a\bar{a}a)\,\varphi), (a\varphi, (\bar{a}a)\,\varphi), ((a\bar{a})\,\varphi, a\varphi), ((a\bar{a}a)\,\varphi, 1\varphi)\} \\ &= \{(1, a\varphi), (a\varphi, 1)\} = a\tau \end{aligned}$$

and similarly $(\bar{a}a\bar{a})\,\tau = \bar{a}\tau$. It follows that $\bar{a}\tau$ is an inverse of $a\tau$. ∎

We are now ready to state our main result.

THEOREM 2.4. *$FI(X)$ is isomorphic to the 2-expansion of $F(X)$ relative to φ.*

Proof. Since N is an inverse monoid generated—as an inverse monoid—by $\{x\tau \mid x \in X\}$, it suffices to prove that there is a surjective morphism $N \to FI(X)$. We use the representation of $FI(X)$ given above.

First define a function $\gamma: A^* \to \mathscr{P}(F(X))$ by

$$u\gamma = \{v\varphi \mid v \in A^* \text{ and there exists } w \in A^* \text{ such that } vw = u\}.$$

Thus $u\gamma$ is the set of left factors of the reduced word $v\varphi$, where v is a left factor of u. It follows that $u\gamma$ is a finite Schreier system such that $u\varphi \in u\gamma$.

Let $\theta: A^* \to FI(X)$ be the function defined by

$$u\theta = (u\gamma, u\varphi).$$

Then θ is a morphism since for all $u \in A^*$ and for all $a \in A$

$$\begin{aligned}(u\theta)(a\theta) &= (u\gamma, u\varphi)(a\gamma, a\varphi) = (u\gamma \cup (u\varphi)(a\gamma), (u\varphi)(a\varphi)) \\ &= (u\gamma \cup (u\varphi)\{1, a\varphi\}, (ua)\,\varphi) \\ &= (u\gamma \cup \{(ua)\,\varphi\}, (ua)\,\varphi) \qquad (\text{since } u\varphi \in u\gamma) \\ &= ((ua)\,\gamma, (ua)\,\varphi) \\ &= (ua)\,\theta.\end{aligned}$$

Moreover θ is surjective since $FI(X)$ is generated (as a monoid) by the set $\{a\theta \mid a \in A\}$. ▌

COROLLARY 5.3. *$FI(X)$ is a submonoid of $\diamond(F(X), F(X))$.*

Proof. Follows immediately from Theorem 5.2. ▌

Corollary 5.3 gives a two-sided construction of $FI(X)$. In fact Theorem 5.2 is just a proof that this two-sided construction is equivalent to an oriented construction based on left factors of words in A^*. Just to cover all political bases, we mention that a construction based on right factors is also possible. These correspond to embeddings of $FI(X)$ into semidirect and reverse semidirect products of $F(X)$ with an appropriate semilattice. We also remark that the notion of a left (right) factor expansion of a morphism can be refined to produce the Rhodes expansion, which plays a prominent role in finite semigroup theory and has recently been generalized to all semigroups. Finally, let us mention the existence of an n-expansion of a morphism $\varphi: A^* \to M$. It can be embedded into the n-fold Schützenberger product $\diamond(M,..., M)$ as defined in [12]. See [2, 3] for further details.

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