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Flow of viscous fluid along a nonlinearly stretching curved surface

K.M. Sanni [a,b], S. Asghar [a,c], M. Jalil [a,*], N.F. Okechi [a,b]

[a] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Chak Shahzad, 44000 Islamabad, Pakistan
[b] National Mathematical Centre, Abuja, Nigeria
[c] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

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ABSTRACT

This paper focuses on the flow of viscous fluid over a curved surface stretching with nonlinear power-law velocity. The boundary layer equations are transformed into ordinary differential equations using suitable non-dimensional transformations. These equations are solved numerically using shooting and Runge-Kutta (RK) methods. The impact of non-dimensional radius of curvature and power-law indices on the velocity field, the pressure and the skin friction coefficient are investigated. The results deduced for linear stretching are compared with the published work to validate the numerical procedure. The important findings are: (a) Slight variation of the curvature of the stretching sheet increases the velocity and the skin friction coefficient significantly. (b) The nonlinearity of the stretching velocity increases the skin friction. (c) The results for linear stretching and the flat surface are the special cases of this problem.

Introduction

Stretching of surface is widely discussed in literature due to its importance and wide range of applications in engineering and industries. The stretching of sheets has definite impact on the quality of finished products in manufacturing processes. Therefore, several real processes take place under different stretching velocities; especially in a flow generated in hot rolling, rubber sheet, glass blowing, fiber spinning, continuous casting and drawing of annealing wires, paper product, glass fiber and polymer sheet extrusion from dye and so on. This has necessitated the consideration of various stretching velocities such as linear, non-linear and exponential stretching velocities.

Crane [1] produced an exact solution for the flow generated by linear stretching of the sheet in the earliest work which is rarely seen in the solution of Navier Stokes equations of fluid dynamics. His work has been extended in many ways along with assumed physical features including heat and mass transfer along flat plate, effect of suction and injection in vertical direction and many more. Gupta and Gupta [2] stressed that the linear stretching of sheet or surface may not necessary be realistic which led to the genesis of non-linear stretching which as of today have made series of contributions in the literature. Bank [3] obtained numerical solution for viscous fluid flow over power-law stretching. Magyari and Keller [4] investigated the flow behavior and heat transfer due to exponentially stretching of surface. Ahmad and Asghar [5] found the analytical and numerical solution for the flow and heat transfer over hyperbolic stretching surface. In addition to Newtonian fluid several researchers investigated the flow of non-Newtonian fluid over a stretching surface. Rajagopal and Gupta [6] presented an excellent exact solution for a boundary layer flow of non-Newtonian fluid flow past an infinite plate. Anderson and Kumaran [7] obtained analytical and numerical solution for non-Newtonian power-law fluid over power-law stretching sheet. Analytical solutions for the flow of power-law fluid over a power law stretching of flat surface was given by Jalil et al. [8]. Jalil and Asghar [9] also presented analytical and numerical solution for flow of power-law fluid over exponentially stretching surface. Hayat et al. [10] obtained the analytical solution for the boundary layer flow of Walters' B fluid using homotopic approach. Ali et al. [11] discussed the flow of Jeffrey fluid over an oscillatory stretching surface.

All the preceding papers address Newtonian and non-Newtonian fluid over linear and nonlinear stretching of a flat surface, plate or sheet as the case may be. However, the flow of viscous fluid past curved surface has been scarcely attended. Sajid et al. [12] presented linear stretching on a curved surface and showed that the boundary layer thickness is greater for a curved surface as compared to flat surface. They indicated the reduction of drag force in moving fluid on a curved surface as compared to flat surface. Likewise, they stressed the importance of pressure variation and of course the application which may be found useful in curving jaw in production of machines.

* Corresponding author.
E-mail address: mudassarjalil@yahoo.com (M. Jalil).

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It has been observed that no study has taken place which blends together curved surface and the nonlinear stretching of the surface. These considerations have great advantage from mathematical physics and applied point of views. The objective of this work is to study the flow of viscous fluid due to nonlinear stretching of the curved surface. The Navier-Stokes (NS) equations are formulated for which the viscous term is modified that takes into account the curvature effects. Mathematically the nonlinearity in NS equations appears due to the curvilinear nature of the curved boundary besides the convective part of the NS equations. Analytic solution for these non-linear equations is highly improbable and hence the numerical solution is presented. An appropriate dimensionless transformation is defined (the first time for such boundary value problem) reducing the partial differential equation into ordinary differential equation. The numerical solution of the resulting non-linear equations is obtained and presented graphically. The following clear objectives are achieved: (a) Reduction of governing partial differential equations to ordinary differential equation. (b) Numerical solution of the velocity profile and skin friction coefficients are calculated numerically and presented graphically. The generalized results for non-linear stretching velocity and curvature are compared with the existing literature. The particular case of $m = 1$ [12] can be recovered as a special case of the generalized stretching considered here.

Statement of the problem

We consider the flow of an incompressible viscous fluid passing over a stretching curved surface. The surface is stretched with nonlinear velocity ($u = as^m$) along the s-direction with the fluid forming a boundary layer in the r-direction. The distance of surface from the origin R determines the shape of the curved surface, i.e., the surface tends to flat for large value of R. The geometry of the problem and the coordinate axes are shown in the Fig. 1. The governing boundary layer equations of the problem satisfying the equations of continuity and momentum are expressed as Sajid et al. [12]:

$$\frac{\partial}{\partial r}[(r+R)v] + R\frac{\partial u}{\partial s} = 0, \tag{1}$$

$$\frac{u^2}{r+R} = \frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2}$$

$$u\frac{\partial v}{\partial r} + \frac{R}{r+R}u\frac{\partial u}{\partial s} + \frac{uv}{r+R} = -\frac{1}{\rho}\frac{R}{r+R}\frac{\partial p}{\partial s} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r+R}\frac{\partial u}{\partial r} - \frac{u}{(r+R)^2}\right]. \tag{3}$$

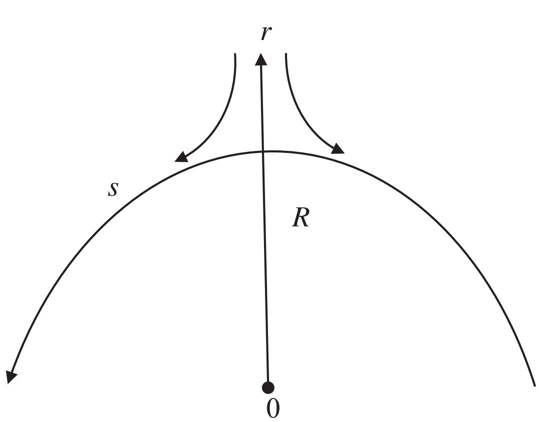

Fig. 1. Flow geometry for a curved stretching surface.

The appropriate boundary conditions corresponding to nonlinear stretching are:

$$u = as^m, \quad v = 0 \quad \text{at} \quad r = 0$$
$$u \to 0, \quad \frac{\partial u}{\partial r} \to 0 \quad \text{as} \quad r \to \infty, \tag{4}$$

where u and v are the components of velocity in r and s directions, respectively, p is the pressure, ν is the kinematic viscosity of the fluid and ρ is the fluid density.

We observe that the governing equations and the boundary conditions are both non-linear in nature. In addition, these are partial differential equations. To find the solution of these equations is quite difficult job; however, using the following non-dimensional transformations we can find solutions of the equations. The non-dimensional variables are defined as:

$$\eta = \sqrt{\frac{as^{m-1}}{\nu}}r, \; v = \frac{-R}{r+R}\sqrt{a\nu s^{m-1}}\left\{\frac{(m+1)}{2}f(\eta) + \frac{(m-1)}{2}\eta f'(\eta)\right\},$$
$$u = as^m f'(\eta), \; p = \rho a^2 s^{2m} P(\eta), \; K = \sqrt{\frac{as^{m-1}}{\nu}}R. \tag{5}$$

With the help of above transformations Eqs. (1)–(4) are transformed into the following ordinary differential equations and the boundary conditions:

$$\frac{\partial P}{\partial \eta} = \frac{f'^2}{\eta + K} \tag{6}$$

$$\frac{(m-1)\eta K}{2(\eta+K)}\frac{\partial P}{\partial \eta} + \frac{2mK}{\eta+K}P = f''' + \frac{f''}{\eta+K} - \frac{f'}{(\eta+K)^2} - \frac{(1+m)\eta + 2mK}{2(\eta+K)^2}Kf'^2 + \frac{(m+1)K}{2(\eta+K)}ff'' + \frac{(m+1)K}{2(\eta+K)^2}ff' \tag{7}$$

$$f'(0) = 1, f(0) = 0, f'(\infty) = 0, f''(\infty) = 0. \tag{8}$$

It is worth mentioning that the transformation introduced by Eq. (5) is presented in the literature for the first time and the similarity transformation used in[12]can be deduced by taking $m = 1$. Eqs. (8) and (6) together gives an additional boundary equation $P'(0) = 1/\xi$.

Eliminating $P(\eta)$ from the Eqs. (6) and (7) yields the following equation:

$$f^{iv} + \frac{2f'''}{\eta+K} - \frac{f''}{(\eta+K)^2} + \frac{f'}{(\eta+K)^3} + \frac{(m+1)K}{2(\eta+K)}ff''' + \frac{(m+1)K}{2(\eta+K)^2}ff'' - \frac{(m+1)K}{2(\eta+K)^3}ff'$$
$$- \frac{(3m-1)}{2(\eta+K)^2}Kf'^2 - \frac{(3m-1)}{2(\eta+K)}Kf'f'' = 0, \tag{9}$$

with the boundary conditions given by Eq. (8).

Numerical results and discussion

In this section, numerical results for the field quantities are presented by implementing the shooting method using Runge–Kutta (RK) algorithm in MATLAB. The effects of radius of curvature and the power law index on the velocity and pressure profiles are shown graphically and the skin friction coefficients are presented in the tabular form. The numerical procedure is validated by making a comparison with the published work[12] and [13] for $m = 1$ and different values of dimensionless radius of curvature K. The comparison shows a very good match (Table 1).

We recall that the effects of the dimensionless radius of curvature K and the stretching power law index m are the prime objectives of this study. The velocity and pressure profiles for different K

Table 1
Comparison of present results of the skin friction coefficient $-Re_s^{1/2}C_f$ with the published results for $m = 1$.

K	Sajid et al. [12]	Zaheer et al. [13]	Present
5	0.7576	1.1576	1.1576
10	0.8735	1.0735	1.0734
20	0.9356	1.0356	1.0355
30	0.9569	1.0235	1.0235
40	0.9676	1.0176	1.0176
50	0.9741	1.0141	1.0140
100	0.9870	1.0070	1.0070
200	0.9936	1.0036	1.0036
1000	0.9988	1.0008	1.0008
∞	1	1	1

and fixed $m = 3$ are plotted in Fig. 2. This reveals that the velocity and pressure profiles and the momentum boundary layer increase with the decrease of the radius of curvature- the sheet becomes more curved (curvature increases). It can be explained on the basis the curvature of the sheet gives rise to a secondary flow due to the curvilinear nature of the fluid flow under the action of centrifugal force as the fluid particles traverse the curved path along the surface of the sheet. The secondary flow is thus superimposed on the primary flow to enhance the velocity field. The impact of the curvature is small along the plate (x-direction) and is significant in the r-direction due to the centrifugal force directed towards the origin.

The behavior of pressure for increasing radius of curvature is also seen from Fig. 2. Increasing curvature makes the curved surface flat and the pressure approach zero at $K = 1000$.

The influence of nonlinearity in the stretching velocity for different values of m and fixed radius of curvature is shown in Fig. 3. We observe that both the components of velocity f and f decrease with m, which is as expected physically. However, the pressure has a different behavior in that it shows decreasing trend near the surface and increasing trend away from the surface as m increases. This cut off is noticed at $\eta = 0.5$ for the particular choice of K and m. This is an interesting and somewhat intriguing observation that must be explained by some physical reasoning. We notice that increasing the stretching index (increasing stretching velocity) increases the radius of curvature that in turn increases the flatness of the curved surface. As argued earlier the pressure will approach zero but slowly for increasing m. Thus for higher values of m the pressure goes to zero earlier (closer to the plate) and thus crosses over to the pressures for smaller values of m. The crossover of pressures for $m = 1$ and $m = 3$ is at approximately $\eta = 0.5$ which varies slightly for other choice of m.

The effects of dimensionless radius of curvature K and stretching index m on the skin friction coefficient $-Re_s^{1/2}C_f$ are shown in Table 2. It is observed that the skin friction coefficient decreases with increase in K. This is because an increase in K tends to make the surface more flat which results in the decrease of skin friction coefficient. It is also observed that increases in nonlinearity of the stretching curved surface (m) increases the skin friction coefficient which is physically obvious.

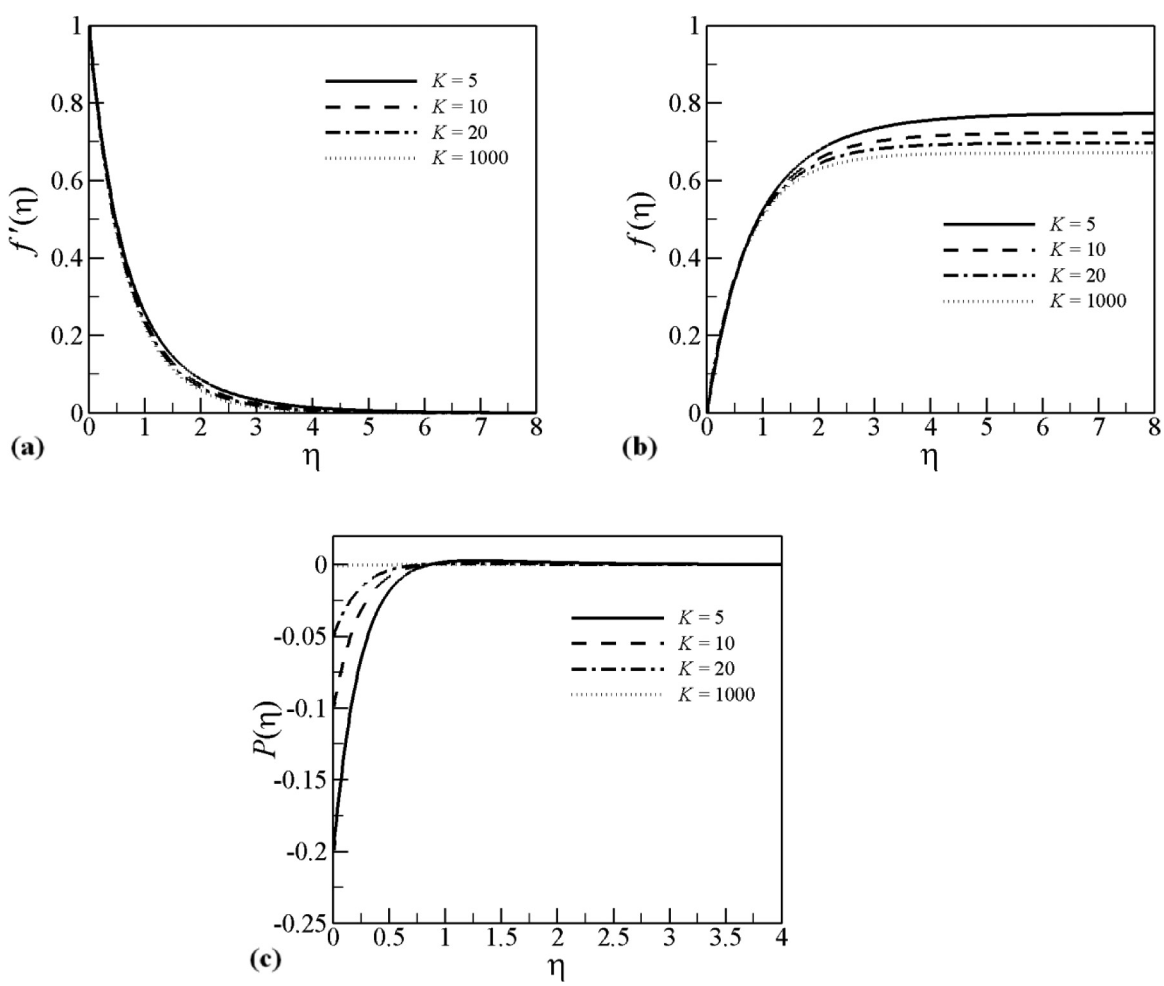

Fig. 2. Effects of dimensionless radius of curvature K on horizontal and vertical components of velocity profiles (a) $f'(\eta)$ (b) $f(\eta)$ and (c) dimensionless pressure $P(\eta)$.

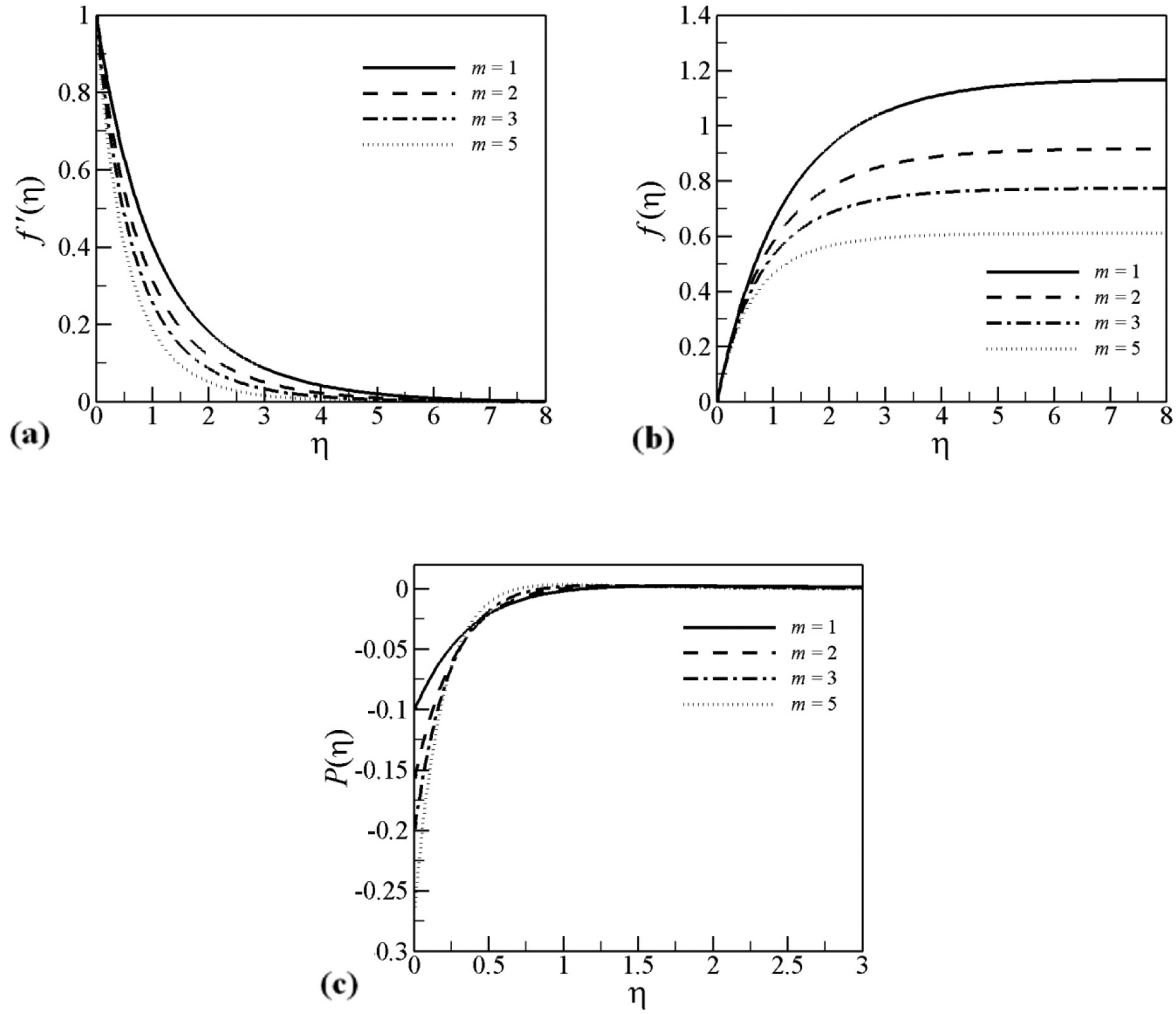

Fig. 3. Effects of power-law stretching index m on horizontal and vertical components of velocity profiles (a) $f'(\eta)$ (b) $f(\eta)$ and (c) dimensionless pressure $P(\eta)$.

Table 2
Variation of the skin friction coefficient $-Re_s^{1/2}C_f$ for various values of dimensionless radius of curvature K and power-law stretching index m.

K	$m = 2$	$m = 3$	$m = 4$	$m = 5$
5	1.4913	1.7613	1.9938	2.2009
10	1.4166	1.6904	1.925	2.1333
20	1.3818	1.6569	1.892	2.1007
50	1.3616	1.6372	1.8727	2.0815
100	1.355	1.6308	1.8663	2.0752
200	1.3517	1.6276	1.8631	2.072
1000	1.3491	1.625	1.8605	2.0695

Conclusion

In this work we investigate the flow of a Newtonian fluid over a curved surface due to nonlinear stretching for quadratic and polynomial power-law stretching indices. New dimensionless transformations are introduced that transforms the boundary layer equations into ordinary differential equation leading to numerical results. We observe that stretching a curved surface leads to positive effect on velocity of flow due to changes in pressure; and the nonlinear stretching reduces the flow velocity as the pressure drops. In other words, nonlinear stretching of curved surface has been found useful to correct the deficiencies experienced in medical devices for propelling a fluid or other flowing dispensable product out of a container; for example, in a fluid dispensing apparatus with prestressed bladder. This justifies our interest in considering a more generalized and realistic situations of curved surface and non-linear stretching in the wake of useful applications.

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