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On the Jacobi Last Multiplier, integrating factors and the Lagrangian formulation of differential equations of the Painlevé–Gambier classification

A. Ghose Choudhury [a], Partha Guha [b,c,*], Barun Khanra [d]

[a] *Department of Physics, Surendranath College, 24/2 Mahatma Gandhi Road, Calcutta-700009, India*
[b] *Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany*
[c] *S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India*
[d] *Sailendra Sircar Vidyalaya, 62A Shyampukur Street, Calcutta-700004, India*

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ABSTRACT

We use a formula derived almost seventy years ago by Madhav Rao connecting the Jacobi Last Multiplier of a second-order ordinary differential equation and its Lagrangian and determine the Lagrangians of the Painlevé equations. Indeed this method yields the Lagrangians of many of the equations of the Painlevé–Gambier classification. Using the standard Legendre transformation we deduce the corresponding Hamiltonian functions. While such Hamiltonians are generally of non-standard form, they are found to be constants of motion. On the other hand for second-order equations of the Liénard class we employ a novel transformation to deduce their corresponding Lagrangians. We illustrate some particular cases and determine the conserved quantity (first integral) resulting from the associated Noetherian symmetry. Finally we consider a few systems of second-order ordinary differential equations and deduce their Lagrangians by exploiting again the relation between the Jacobi Last Multiplier and the Lagrangian.

1. Introduction

The study of nonlinear ordinary differential equations (ODEs) has been an ongoing endeavor for well over two centuries now with significant contributions by many of the greatest mathematicians of all times, such as Euler, Lie, Painlevé and Poincaré to mention just a few. Their contributions have ranged from finding explicit solutions of ODEs, to developing general methods of classifications, to the qualitative analysis of their solutions etc. These in turn have often led to the opening of entirely new branches of study in algebra, topology, geometry and have shed new light on several physical phenomena.

Over the years many techniques have been developed to obtain closed-form solutions of various kinds of ODEs. However, there does not exist any single common method for obtaining their solutions. Nevertheless the apparently different techniques share one common feature: they somehow tend to exploit the symmetries of ODEs. Consequently symmetry analysis of ODEs has become one of the most powerful tools for analyzing them. The foundations of this method are contained in the works of Sophus Lie [1,2].

* Corresponding author at: S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India.
E-mail addresses: a_ghosechoudhury@rediffmail.com (A. Ghose Choudhury), partha.guha@mis.mpg.de (P. Guha), barunkhanra@rediffmail.com (B. Khanra).

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It is also well known that the existence of a sufficient number of first integrals greatly simplifies the process of solving any ODE. Having said so, it is not always quite obvious what these first integrals are. Indeed their determination is, in general, a nontrivial task. In the case of conservative mechanical systems, one often has just a single first integral – the energy. In this context the semialgorithmic procedure developed by Prelle and Singer deserves mention [6]. In its original version it applied to first-order ODEs involving rational functions with coefficients belonging to the field of complex numbers $\mathbb{C}$. Subsequently their method, which involved the use of Darboux polynomials, was extended by Duarte et al. [7,8] and Chandrasekhar et al. in a series of papers [3–5].

As is often the case in a field which has been so thoroughly investigated over the years, some aspects often tend to fade out only to resurface after many years when new results point to a mysterious link with those of the past. One such result, which has appeared in the recent literature on differential equations, concerns the Jacobi Last Multiplier. The credit for resurrecting this has to go to Nucci and Leach, who have shown how it may be used to determine the first integrals and also Lagrangians of a wide variety of nonlinear differential equations. It appears that the connection of the Jacobi Last Multiplier to the existence of Lagrangian functions was the subject of investigation by a few authors in the early 1900s. However, the precise nature of their interrelation was brought out by Rao in [9] in the 1940s. Thereafter this did not attract much attention among researchers in the field of differential equations.

Recently there has been a renewal of interest in this area and it appears that Jacobi's Last Multiplier can be incorporated in the formalism initiated by Lie for the study of differential equations.

1.1. Motivation and plan

It is clear that Rao's formula can be used to deduce the Lagrangian of a second-order ODE or even a system of such ODEs once the last multiplier is known. Unlike the Hamiltonian structure of the six Painlevé equations, which have received much attention [17], the Lagrangian formulation has not been sufficiently nurtured. In a recent paper Wolf and Brand [18] proposed a Lagrangian for Painlevé VI. In this paper we investigate the Lagrangians for the majority of the Painlevé equations, using Rao's formula and also indicate its applicability to other equations of the Painlevé–Gambier classification.

The organization of the paper is as follows. In Section 2 we introduce the basic ideas underlying the Jacobi Last Multiplier and state its defining equation for an nth-order ODE or an equivalent system of first-order ODEs. The connection between the last multiplier and symmetries is also mentioned. It also contains a discussion of certain geometrical aspects underlying Jacobi's Last Multiplier. Section 3 constitutes the main body of this paper and explains the relationship between the Jacobi Last Multiplier and the Lagrangian description of second-order ODEs. It includes a deduction of the Lagrangians for four of the six Painlevé equations and also their corresponding Hamiltonians. It also briefly outlines the procedure for other equations of the Painlevé–Gambier classification. In Section 4 we analyze in this context second-order equations of the Liénard type. It contains a specific example of a generic equation of nonlinear oscillator type. Finally in Section 5 we apply the technique to a coupled system of second-order ODEs, which has not been very extensively studied, and also summarize the results for a couple of other more well-known systems.

2. The Jacobi Last Multiplier

Consider the nth-order ODE in the normal form

$$y^{(n)} = w\big(x, y, y', \ldots, y^{(n-1)}\big). \tag{2.1}$$

Corresponding to this ODE there exists an equivalent first-order partial differential equation (PDE) in $(n+1)$ variables,

$$\widetilde{D} f = (\partial_x + y'\partial_y + y''\partial_{y'} + \cdots + w\partial_{y^{(n-1)}}) f = 0, \tag{2.2}$$

in which the quantities $y', y'' \ldots$ are treated as independent variables at par with x and y.

Their equivalence is provided by the first integrals of (2.1). By definition a first integral is a global function, $I = I(x, y, y', \ldots, y^{(n-1)})$, that is constant along the solutions of (2.1), i.e.,

$$\frac{dI}{dx} = \widetilde{D} I = I_x + y' I_y + y'' I_{y'} + \cdots + w I_{y^{(n-1)}} = 0. \tag{2.3}$$

Having determined a first integral, say $I = I(x, y, y', \ldots, y^{(n-1)}) = I_0$, one can invert it to obtain

$$y^{(n-1)} = w_1\big(x, y, y', \ldots, y^{(n-2)}; I_0\big)$$

provided $I_{y^{(n-1)}} \neq 0$. This shows that the existence of a first integral allows for the reduction in the order of the differential equation by one. Furthermore it is evident that every first integral is a solution of the linear PDE (2.2) and conversely.

Assume that ϕ^α, $\alpha = 1, \ldots, n$ denote a set of n functionally independent solutions of (2.2). As each ϕ^α is a first integral, one has

$$\phi^\alpha\big(x, y, y', \ldots, y^{(n-1)}\big) = I_0^\alpha, \quad \alpha = 1, 2, \ldots, n. \tag{2.4}$$

Consequently by eliminating all derivatives from (2.4) one arrives at the general solution of (2.1) in the form

$$y = y\big(x; I_0^1, \dots, I_0^n\big),$$

the I_0^α's being essentially constants of integration.

As we mentioned above, the determination of even a single first integral is in most cases a nontrivial task. Hence, while in principle the above procedure is fine, its practical application is often a daunting task, to say the least.

Nevertheless, assuming we have at our disposal $(n-1)$ solutions ϕ^α of the linear PDE $\widetilde{D} f = 0$, by means of the Jacobi Last Multiplier the nth solution can be obtained by a quadrature. The formal definition of the Jacobi Last Multiplier is as follows.

Definition 2.1. Given an nth-order ODE or its equivalent linear PDE in $(n+1)$ variables

$$\widetilde{D} f = (\partial_x + y'\partial_y + y''\partial_{y'} + \dots + w\partial_{y^{(n-1)}}) f = 0,$$

the Jacobi Last Multiplier M is defined by

$$M\widetilde{D} f := \frac{\partial(f, \phi^1, \phi^2 \dots \phi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})} = \det\begin{pmatrix} f_x & f_y & \dots & f_{y^{(n-1)}} \\ \phi^1_x & \phi^1_y & \dots & \phi^1_{y^{(n-1)}} \\ \vdots & \vdots & \dots & \vdots \\ \phi^{(n-1)}_x & \phi^{(n-1)}_y & \dots & \phi^{(n-1)}_{y^{(n-1)}} \end{pmatrix} = 0. \tag{2.5}$$

From the above definition it follows that the Jacobi Last Multiplier (JLM) can be varied by selecting a different set of $(n-1)$ independent solutions $\psi^1, \psi^2, \dots, \psi^{n-1}$ of (2.2). If the corresponding JLM be $\widetilde{M}$, then

$$\widetilde{M}\widetilde{D} f = \frac{\partial(f, \psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})} = \frac{\partial(f, \phi^1, \phi^2, \dots, \phi^{n-1})}{\partial(x, y, y', \dots, y^{(n-1)})}\frac{\partial(\psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(\phi^1, \phi^2, \dots, \phi^{n-1})} = M\frac{\partial(\psi^1, \psi^2, \dots, \psi^{n-1})}{\partial(\phi^1, \phi^2, \dots, \phi^{n-1})}.$$

Indeed each JLM, as defined above, turns out to be a solution of the following linear PDE

$$\frac{\partial M}{\partial x} + \sum_{k=1}^{n} \frac{\partial\big(y^{(k)} M\big)}{\partial y^{(k-1)}} = 0 \quad \text{on } y^{(n)} = w\big(x, y, y', \dots, y^{(n-1)}\big) \tag{2.6}$$

or, if the ODE be expressed as a system of first-order ODEs of the form

$$\dot{x}_k = W_k(t, x_1, \dots, x_n), \quad k = 1, 2, \dots, n, \tag{2.7}$$

as a solution of the equation

$$\frac{d}{dt}\log M + \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{2.8}$$

It is evident that the classical definition of the JLM is overly restrictive, requiring as it does almost complete knowledge of the system. However, being dependent on first integrals, it is natural to expect that it should bear some connection to the symmetries of the equation under investigation. This connection was unravelled by Lie and its formulation in terms of the generators of the Lie symmetry algebra is outlined below.

For the ODE in Eq. (2.1) or its equivalent PDE given by (2.2) let $X_i = \xi_i\partial_x + \eta_i\partial_y$ denote n Lie point symmetry generators of the equation. The vector field associated with $\widetilde{D} f = 0$ has $(n+1)$ components $(1, y', \dots, w)$ on $y^{(n)} = w(x, y, \dots, y^{(n-1)})$. Using standard methods for constructing the prolongations of these generators X_i up to the $(n-1)$th-order, *viz.*

$$X_i^{(n-1)} = \xi_i\partial_x + \eta_i\partial_y + \eta_i^{(1)}\partial_{y'} + \dots + \eta_i^{(n-1)}\partial_{y^{(n-1)}}, \quad i = 1, 2, \dots, n,$$

consider the determinant

$$\Delta = \det\begin{pmatrix} 1 & y' & y'' & \dots & f_{y^{(n-1)}} & w \\ \xi_1 & \eta_1 & \eta_1^{(1)} & \dots & \eta_1^{(n-2)} & \eta_1^{(n-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \dots & \eta_n^{(n-2)} & \eta_n^{(n-1)} \end{pmatrix}. \tag{2.9}$$

If $\Delta \neq 0$, then the JLM is given by $M = \Delta^{-1}$.

Similarly for a system of n first-order ODEs given by (2.7) the associated vector field has components $(1, W_1, \ldots, W_n)$. If we can find n symmetry generators of the form $X_i = \sum_{j=0}^{n} a_{ij} \frac{\partial}{\partial x_j}$, $i = 1, \ldots, n$, with $x_0 = t$, then the last multiplier is given by

$$M^{-1} = \Delta = \det \begin{pmatrix} 1 & W_1 & \cdots & W_n \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n0} & a_{n1} & \ldots & a_{nn} \end{pmatrix}. \tag{2.10}$$

Since M satisfies (2.8), it follows that the ratio of two last multipliers is a first integral, i.e.,

$$\frac{d}{dt}\left(\frac{M_1}{M_2}\right) = 0.$$

In other words the ratios of the Δ_i's ($i \geqslant 2$) provide us first integrals for the system of equations (2.7).

2.1. Geometric description of Jacobi's Last Multiplier

Let $M = M(\mathbf{x})$ be a non-negative C^1 function non-identically vanishing on some open subset of $\mathbb{R}^n$. Then Jacobi's Last Multiplier is a solution of the linear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial(MW_i)}{\partial x_i} = 0, \tag{2.11}$$

where $\mathbf{W} = \sum_{i=1}^{n} W_i \partial_{x_i}$ is the vector field of the system of first-order ODEs. Essentially, if a Jacobi multiplier is known together with $(n-2)$ first integrals, then we can reduce locally to a $2D$ vector field on the intersection of the $(n-2)$ level sets formed by first integrals.

Let $\Omega = dx_1 \wedge \cdots \wedge dx_n$ be a volume form on $\mathbb{R}^n$. Define an inner product, $\langle \cdot, \cdot \rangle$, between 1-forms and $(n-1)$-forms on $\mathbb{R}^n$ as

$$\omega_1 \wedge \omega_2 = \langle \omega_1, \omega_2 \rangle \Omega.$$

So both the space of vectors and the space of $(n-1)$-forms are dual to the space of 1-forms. Hence there is a natural isomorphism between the space of vectors and the space of $(n-1)$-forms. Let $\mathbf{W}$ be a vector field. Then $\mathbf{W}$ corresponds to $(n-1)$ form $\omega_{\mathbf{W}}$ under isomorphism

$$i_{\mathbf{W}}\Omega = \omega_{\mathbf{W}} = \sum_{i=1}^{n} (-1)^{(i-1)} W_i \, dx^1 \wedge \cdots \wedge \hat{d}x_i \wedge \cdots \wedge dx_n. \tag{2.12}$$

Thus the condition for Jacobi's Last Multiplier can be manifested as [11,12]

$$0 = d(M\omega_{\mathbf{W}}) = \left(\sum_{i=1}^{n} \frac{\partial(MW_i)}{\partial x_i} \right) \Omega.$$

Therefore an element, M, is called a Jacobi's Last Multiplier for an ODE if

$$d(M\omega_{\mathbf{W}}) = dM \wedge \omega_{\mathbf{W}} + M d\omega_{\mathbf{W}} = 0. \tag{2.13}$$

Using

$$L_{\mathbf{W}}\omega_{\mathbf{W}} = (\operatorname{div}_\Omega \mathbf{W})\omega_{\mathbf{W}} \tag{2.14}$$

and $(dg) \wedge \omega_{\mathbf{W}} = (\mathbf{W}g)\Omega$ $(\forall g \in C^\infty(\mathbb{R}^n))$ we can show that the Jacobi Last Multiplier M satisfies

$$\mathbf{W}M + M \operatorname{div}_\Omega \mathbf{W} = 0. \tag{2.15}$$

This equation reveals that M is a last multiplier for the divergence-free vector field $\mathbf{W}$ if and only if M is a first integral of $\mathbf{W}$. In general the vector field $\mathbf{W}$ is not divergence-free and in this situation the theory of multipliers, namely, the ratio of two multipliers is a first integral etc., holds good. In fact the set of last multipliers measures how far away $\mathbf{W}$ is from the divergence-free condition.

The theory of the Jacobi Last Multiplier is also connected to another important method namely adjoint symmetry equation in determining explicit integrating factors and first integrals of nonlinear ODEs. One can define the action of the adjoint vector field $\mathbf{W}^*$ corresponding to $\mathbf{W}$ on functions [13] as

$$\mathbf{W}^*(M) = -\mathbf{W}(M) - M \operatorname{div}_\Omega \mathbf{W} = 0. \tag{2.16}$$

Thus solving the adjoint equation one can obtain Jacobi's Last Multiplier. This is the essential feature of adjoint method.

We state another geometrical idea related to the last multiplier. Characterization of Jacobi's Last Multiplier can be obtained in terms of the Marsden differential [14]. Let $m \in C^{\infty}(M)$. Then the Marsden differential,

$$d^m : \Lambda^*(M) \to \Lambda^{*+1}(M)$$

is given by

$$d^m(\eta) = \frac{1}{m} d(m\eta). \tag{2.17}$$

Thus M is a Jacobi Last Multiplier if and only if $\omega_{\mathbf{W}}$ is d^M-closed.

A vector field, $\mathbf{S}$, is called a symmetry of an ODE given by a vector field, $\mathbf{W}$, if

$$L_{\mathbf{S}}\mathbf{W} = [\mathbf{W}, \mathbf{S}] = \lambda \mathbf{W}, \quad \lambda \in C^{\infty}. \tag{2.18}$$

Let $\mathbf{S}_1, \ldots, \mathbf{S}_{n-1}$ be $(n-1)$ symmetries. Define

$$h = i_{S_{n-1}} \cdots i_2 i_1 \omega_{\mathbf{W}}. \tag{2.19}$$

Then $M = h^{-1}$ is a last multiplier for $\omega_{\mathbf{W}}$, i.e., $d(M\omega_{\mathbf{W}}) = 0$. This can be proved using the symmetry condition.

$$L_{\mathbf{W}} h = L_{\mathbf{W}} i_{S_{n-1}} \cdots i_2 i_1 \omega_{\mathbf{W}} = (i_{[\mathbf{W}, \mathbf{S_{n-1}}]} + i_{S_{n-1}} L_{\mathbf{W}}) i_{S_{n-2}} \cdots i_{S_2} i_{S_1} \mathbf{W}.$$

The first term in the expression above vanishes. Thus recursively one can prove that

$$L_{\mathbf{W}} h = h \, \mathrm{div}_{\Omega} \, \mathbf{W}, \tag{2.20}$$

where the function $M = h^{-1}$ is called an inverse multiplier.

At last we wish to outline a connection between last multiplier and Nambu mechanics. Consider a special case of (2.14), a divergence-free condition

$$\mathrm{div}_{\Omega} \, \mathbf{W} = \sum_{i=1}^{n} \frac{\partial W_i}{\partial x_i} = 0. \tag{2.21}$$

In this situation Eq. (2.7) can be mapped to Nambu dynamical systems, i.e. systems of time-autonomous ODEs of the form [15], with a special value of W_i

$$\dot{x}^i = W_i(x) = \epsilon_{j_1, \ldots, j_n} \delta^i_{j_1} \frac{\partial H_2}{\partial x^{j_2}} \cdots \frac{\partial H_n}{\partial x^{j_n}}. \tag{2.22}$$

In other words a system obeying Nambu mechanics automatically satisfies the Liouville condition. In fact by duality the vector field $\mathbf{W} = W_i(x)\partial/\partial x_i$ maps to an $(n-1)$-differential form given by

$$\begin{aligned}
\omega_{\mathbf{W}} &= \frac{1}{(n-1)!} \epsilon_{j_1, \ldots, j_n} W_{j_1} \, dx^{j_2} \wedge \cdots \wedge dx^{j_n} \\
&= \frac{1}{(n-1)!} \epsilon_{k_1, \ldots, k_n} \epsilon_{j_1, \ldots, j_n} \delta_{k_1 j_1} \left[\frac{\partial H_2}{\partial x_{j_2}} dx^{j_2} \right] \wedge \cdots \wedge \left[\frac{\partial H_n}{\partial x_{j_n}} dx^{j_n} \right] \\
&= dH_2 \wedge \cdots \wedge dH_n.
\end{aligned}$$

Thus $\omega_{\mathbf{W}}$ is a decomposable and closed $(n-1)$-form and the set of $(n-1)$ independent functions, $H_2, \ldots, H_n$, are such that every integral curve is given by an equation of the form $H_2(\mathbf{x}) = C_2, \ldots, H_n(\mathbf{x}) = C_n$.

3. Lagrangians and the last multiplier

In a series of recent papers Leach, Nucci and Tamizhmani (for example, [16,19–21] and references therein) have investigated the relation between integrating factors and the Hessian. It appears that this connection has a long history, which can be traced to Jacobi's attempts to obtain the last multiplier [23,24]. In 1874 Lie [1,2] showed that point symmetries could be used to determine Jacobi's Last Multiplier (JLM). The explicit nature of the relation between the JLM and Hessian was clarified by Rao in a article [9] and is also mentioned in Whittaker's book on analytical dynamics [10].

3.1. Second order equations

For a second-order ODE $y'' = w(x, y, y')$ which admits a Lagrangian function $L(x, y, y')$ the Jacobi Last Multiplier, M, is given by

$$M = \frac{\partial^2 L}{\partial y'^2}. \tag{3.1}$$

On the other hand, given a system of first order equations

$$y'_k = f_k(x, y), \qquad y = (y_1, y_2, \ldots, y_n),$$

the JLM is a solution of the equation

$$\frac{d\log M}{dx} + \sum_{k=1}^{n} \frac{\partial f_k}{\partial y_k} = 0.$$

It follows that, if a solution of this equation is obtained, then from a knowledge of the JLM one can construct the Lagrangian function as

$$L(x, y, y') = \int\left(\int M\,dy'\right) + f_1(x, y)y' + f_2(x, y). \tag{3.2}$$

3.2. Lagrangians for the Painlevé equations

A large number of second-order ODEs in the Painlevé–Gambier classification system belong to the following class of equations, namely

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + \phi_t\dot{x} + B(t, x) = 0. \tag{3.3}$$

Writing this equation in the form

$$\ddot{x} = \mathcal{F}(t, x, \dot{x}) = -\left[\frac{1}{2}\phi_x\dot{x}^2 + \phi_t\dot{x} + B(t, x)\right],$$

the Jacobi Last Multiplier M for (3.3) is given by the solution of

$$\frac{d}{dt}\log M = -\frac{\partial\mathcal{F}}{\partial\dot{x}}. \tag{3.4}$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial\dot{x}^2} = \exp\bigl[\phi(t, x)\bigr]. \tag{3.5}$$

By (3.2) we then obtain the Lagrangian as

$$L(t, x, \dot{x}) = \frac{1}{2}e^{\phi(t,x)}\dot{x}^2 + f_1(t, x)\dot{x} + f_2(t, x). \tag{3.6}$$

To determine the unknown functions, f_1 and f_2, we substitute this Lagrangian into the Euler–Lagrange equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) = \frac{\partial L}{\partial x} \tag{3.7}$$

and use (3.3) to get

$$f_{1t} - f_{2x} = e^{\phi}B(t, x).$$

The making of a gauge transformation $f_1 = G_x$ and $f_2 = G_t + f_3(t, x)$ allows us to satisfy the last equation when

$$f_3(t, x) = -\int e^{\phi}B(t, x)\,dx. \tag{3.8}$$

Consequently the final Lagrangian for (3.3) becomes

$$L(t, x, \dot{x}) = e^{\phi(t,x)}\frac{\dot{x}^2}{2} - \int e^{\phi}B(t, x)\,dx + \frac{dG}{dt}. \tag{3.9}$$

The total derivative term obviously is of little consequence. Hence we may safely discard it.

The conjugate momentum may be defined by

$$p = \frac{\partial L}{\partial\dot{x}} = e^{\phi}\dot{x} \quad \text{which implies} \quad \dot{x} = e^{-\phi}p$$

and leads to the Hamiltonian

$$H = e^{-\phi}\frac{p^2}{2} + \int e^{\phi} B(t,x)\,dx$$

by the usual Legendre transformation. It is clear that the Lagrangian obtained in the above manner is a non-standard one.

One can attempt to bring it closer to the standard form by means of the transformation

$$\dot{y} = e^{\phi/2}\dot{x} \text{ or } y(t,x) = \int e^{\phi(t,x)/2}\,dx. \tag{3.10}$$

We illustrate this by a specific example in the sequel.

3.2.1. The Painlevé III equation

The P_{III} equation may be written as

$$\ddot{x} - \frac{1}{x}\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{3.11}$$

where $B(t,x) = -[\frac{1}{t}(\alpha x^2 + \beta) + \gamma x^3 + \frac{\delta}{x}]$. Comparison with (3.3) shows that $\phi_x = -2/x$ and $\phi_t = 1/t$ which yields for the last multiplier $M = \exp\phi = t/x^2$. Then from (3.9) we obtain

$$L_{III} = \frac{t}{x^2}\frac{\dot{x}^2}{2} + \alpha x - \frac{\beta}{x} + t\left(\frac{\gamma x^2}{2} - \frac{\delta}{2x^2}\right) \tag{3.12}$$

and the Hamiltonian as

$$H_{III} = \frac{x^2}{t}\frac{p^2}{2} + \left(\frac{\beta}{x^2} - \alpha x\right) + \frac{t}{2}\left(\frac{\delta}{x^2} - \gamma x^2\right). \tag{3.13}$$

3.2.2. The Painlevé V equation

The P_V equation may be written as

$$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 + \frac{1}{t}\dot{x} + B(t,x) = 0, \tag{3.14}$$

where

$$B(t,x) = -\left[\frac{(x-1)^2}{t^2}(\alpha x + \frac{\beta}{x}) + \frac{\gamma x}{t} + \frac{\delta x(x+1)}{x-1}\right].$$

Following the same procedure as before we obtain for the Jacobi Last Multiplier

$$M = \frac{t}{x(x-1)^2}$$

and the Lagrangian

$$L_V = \frac{t}{x(x-1)^2}\frac{\dot{x}^2}{2} + \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) - \frac{\gamma}{x-1} - \delta\frac{tx}{(x-1)^2}. \tag{3.15}$$

The corresponding Hamiltonian is

$$H_V = \frac{x(x-1)^2}{t}\frac{p^2}{2} - \frac{1}{t}\left(\alpha x - \frac{\beta}{x}\right) + \frac{\gamma}{x-1} + \delta\frac{tx}{(x-1)^2}. \tag{3.16}$$

3.2.3. The Painlevé IV equation

The P_{IV} equation may be written as

$$\ddot{x} - \frac{1}{2x}\dot{x}^2 + B(t,x) = 0, \tag{3.17}$$

where

$$B(t,x) = -\left[\frac{3}{2}x^3 + 4tx^2 + 2(t^2-\alpha)x + \frac{\beta}{x}\right].$$

Unlike the previous two Painlevé equations, here we have $\phi_t = 0$ so that the last multiplier is now time independent. Indeed for the P_{IV} equation we have $M = 1/x$ while the corresponding Lagrangian is

$$L_{IV} = \frac{1}{x}\frac{\dot{x}^2}{2} + \left[\beta\ln|x| + (t^2-\alpha)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{3.18}$$

The associated Hamiltonian is

$$H_{IV} = \frac{xp^2}{2} - \left[\beta \ln|x| + \left(t^2 - \alpha\right)x^2 + \frac{4}{3}tx^3 + \frac{3}{8}x^4\right]. \tag{3.19}$$

3.2.4. The Painlevé VI equation

The P_{VI} equation is perhaps one of the most well-studied equations of the Painlevé class. It may be written as

$$\ddot{x} - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\dot{x}^2 + \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\dot{x} + B(t,x) = 0, \tag{3.20}$$

where

$$-B(t,x) = \frac{(x-1)(x-1)(x-t)}{t^2(t-1)^2}\left[\alpha + \frac{\beta t}{x^2} + \frac{\gamma(t-1)}{(x-1)^2} + \frac{\delta t(t-1)}{(x-t)^2}\right].$$

In this case we have

$$\phi_x = -\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right) \quad \text{and} \quad \phi_t = \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)$$

so that the last multiplier is given by

$$M = e^{\phi} = \frac{t(t-1)}{x(x-1)(x-t)}. \tag{3.21}$$

The Lagrangian for the P_{VI} equation is found to be

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \int \frac{t(t-1)}{x(x-1)(x-t)}\left(-B(t,x)\right)dx + \frac{dG}{dt},$$

$$L_{VI}(t,x,\dot{x}) = \frac{t(t-1)}{x(x-1)(x-t)}\frac{\dot{x}^2}{2} + \frac{\alpha x}{t(t-1)} - \frac{\beta}{x(t-1)} - \frac{\gamma}{t(x-1)} - \frac{\delta}{x-t} + \frac{dG}{dt}. \tag{3.22}$$

Let p be the conjugate momentum. With

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{t(t-1)}{x(x-1)(x-t)}\dot{x}$$

the corresponding Hamiltonian is

$$H_{VI} = \frac{t(t-1)}{x(x-1)(x-t)}\frac{p^2}{2} - \frac{\alpha x}{t(t-1)} + \frac{\beta}{x(t-1)} + \frac{\gamma}{t(x-1)} + \frac{\delta}{x-t}. \tag{3.23}$$

Besides the Painlevé equations many other equations of the Painlevé–Gambier classification may also be treated in a similar manner. We illustrate this below.

3.2.5. The Painlevé–Gambier equations XXI

This equation is of the form

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{3.24}$$

The Jacobi Last Multiplier is given by $M = x^{-3/2}$ and the corresponding Lagrangian is

$$L_{21} = x^{-3/2}\frac{\dot{x}^2}{2} + 2x^{3/2}. \tag{3.25}$$

The associated Hamiltonian H_{21} provides a first integral (i.e., $\frac{dH_{21}}{dt} = 0$), namely

$$H_{21} = x^{-3/2}\frac{\dot{x}^2}{2} - 2x^{3/2}. \tag{3.26}$$

It is interesting to note that L_{21} and H_{21} both have a 'wrong relative sign'. Consider the transformation

$$x \mapsto y = 4x^{1/4} \quad \text{so that} \quad \dot{y} = x^{-3/4}\dot{x}. \tag{3.27}$$

Under this transformation the Lagrangian L_{21} assumes the more familiar form

$$L_{21}(t,y,\dot{y}) = \left[\frac{1}{2}\dot{y}^2 + \left(2(y/4)^6\right)\right].$$

4. Equations of the Liénard type

In a series of interesting papers Chandrasekhar et al. have made a thorough study of many nonlinear equations of the oscillator type, using an extension of the Prelle–Singer method [3–5]. We investigate below one such generic equation of the Liénard type,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{4.1}$$

from the perspective of the Jacobi Last Multiplier.

4.1. Lagrangian for second-order Liénard type of equations

From (3.4) the last multiplier for Eq. (4.1) is given by $M = \exp(\int f(x)\,dt)$. Following [20] we introduce a new variable v by setting

$$\int f(x)\,dt = \log\left(v^{-\alpha^{-1}}\right) \tag{4.2}$$

which implies

$$\dot{v} + \alpha f(x) v = 0, \tag{4.3}$$

with α being a nonzero scalar to be determined. As a result we have

$$M = v^{-\alpha^{-1}}. \tag{4.4}$$

Indeed, if we can map the original equation, (4.1), to the first-order Eq. (4.3) in terms of the variable v, then a suitable Lagrangian can be easily deduced. It is obvious that v must be linear in $\dot{x}$. In fact it is shown in [20] that such a map exists and is given by

$$v = \dot{x} + \frac{g}{\alpha f} \tag{4.5}$$

provided f and g satisfy the condition

$$\frac{d}{dx}\left(\frac{g}{f}\right) = \alpha(1-\alpha) f. \tag{4.6}$$

From (4.4), since $M = \partial^2 L/\partial \dot{x}^2$, we find that

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})} v^{2-\alpha^{-1}} + f_1 v + f_2. \tag{4.7}$$

Now we substitute this into the Euler–Lagrange equation leads to the condition

$$f_{1t} - f_{2x} = \frac{d}{dx}\left(f_1 \frac{g}{\alpha f}\right),$$

which may be satisfied by setting $f_1 = G_x$ and $f_2 = G_t + f_3$ yielding

$$f_{3x} = -\frac{d}{dx}\left(G_x \frac{g}{\alpha f}\right) \quad \Rightarrow \quad f_3 = -G_x \frac{g}{\alpha f}.$$

The simple choice $G_x = 0$, i.e., $f_1 = 0$ gives, $f_3 = 0$ and $f_2 = dG/dt$. Thus

$$L = \frac{1}{(2-\alpha^{-1})(1-\alpha^{-1})}\left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha} + \frac{dG}{dt}, \quad \alpha \neq 0, \frac{1}{2}, 1. \tag{4.8}$$

We can rescale the Lagrangian to get rid of the inconsequential scalar factors and also drop the total time derivative to get it into the neater form

$$L = \left(\dot{x} + \frac{g}{\alpha f}\right)^{2-1/\alpha}. \tag{4.9}$$

This Lagrangian, being invariant under time translation, admits a Noether symmetry with corresponding conserved quantity or first integral (disregarding overall scalar factors)

$$I = \left(\dot{x} + \frac{g}{\alpha f}\right)^{1-1/\alpha} \frac{(\alpha-1) f\dot{x} - g}{f}. \tag{4.10}$$

4.2. Example: A generic equation of nonlinear oscillator type

Consider the following equation

$$\ddot{x} + \left(k_1 x^q + k_2\right)\dot{x} + \left(k_3 x^{2q+1} + k_4 x^{q+1} + k_5 x\right) = 0. \tag{4.11}$$

This is a generic equation of nonlinear oscillator type, which includes many subcases depending upon the choice of the k_i, which are parameters. The case $q = 0$ corresponds to a damped harmonic oscillator, while $q = 1$ corresponds to the force-free Helmholtz oscillator. Substituting f and g from (4.11) into the condition (4.6), we obtain the following equations from the different coefficients of x.

$$\alpha(1-\alpha) = (q+1)\frac{k_3}{k_1^2}, \tag{4.12}$$

$$\alpha(1-\alpha) = \frac{k_5}{k_2^2}, \tag{4.13}$$

$$k_1 k_4 + k_2 k_3(2q+1) = \alpha(1-\alpha)k_1^2 k_2, \tag{4.14}$$

$$k_1 k_5(1-q) + k_2 k_4(1+q) = 3\alpha(1-\alpha)k_1 k_2^2. \tag{4.15}$$

Equating (4.12) and (4.13) we find that

$$q + 1 = \frac{k_1^2 k_5}{k_2^2 k_3}. \tag{4.16}$$

Using this value of q in the remaining Eqs. (4.14) and (4.15) while eliminating α by means of (4.13), we get

$$k_5 = \frac{k_2}{k_1^2}(k_1 k_4 - k_2 k_3). \tag{4.17}$$

The constant α is determined from the quadratic equation (4.13) and is

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_5}{k_2^2}}\right), \tag{4.18}$$

where k_5 is given by (4.17). Given q there exists another relation between the k_i $(i = 1, \ldots, 5)$ derivable from (4.16) and (4.17), *viz.*

$$\frac{k_1 k_4}{k_2 k_3} = q + 2. \tag{4.19}$$

Thus of the five parameters k_i $(i = 1, \ldots, 5)$ only three are independent and to summarize we have the following relations:

$$k_4 = \frac{k_2 k_3}{k_1}(q+2),$$

$$k_5 = \frac{k_2^2 k_3}{k_1}(q+1),$$

$$\alpha = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}(q+1)}\right).$$

4.2.1. Special cases

When $q = 0$, we have $k_1 k_4 = 2k_2 k_3$ and $k_5 = k_2^2 k_3 / k_1^2$. Consequently $\alpha = \frac{1}{2}(1 \pm \sqrt{1 - \frac{4k_3}{k_1^2}})$ and the equation $\ddot{x} + (k_1 + k_2)\dot{x} + (k_3 + k_4 + k_5)x = 0$, which is simply the damped harmonic oscillator, has Lagrangian

$$L = \left(\dot{x} + \frac{(k_3 + k_4 + k_5)x}{\alpha(k_1 + k_2)}\right)^{2-1/\alpha}.$$

When $q = 1$, we have $k_1 k_4 = 3k_2 k_3$ and $k_5 = 2k_2^2 k_3 / k_1^2$ while $\alpha = \frac{1}{2}(1 \pm \sqrt{1 - 8k_3/k_1^2})$. The Lagrangian for the equation, $\ddot{x} + (k_1 x + k_2)\dot{x} + k_3(x^3 + 3k_2/k_1 x^2 + 2k_2^2/k_1^2 x) = 0$ is

$$L = \left\{\dot{x} + \frac{k_3}{\alpha k_1}\left(x^2 + 2k_2/k_1 x\right)\right\}^{2-1/\alpha}.$$

From this Lagrangian one can easily compute the conjugate momentum to obtain the corresponding Hamiltonian.

5. A system of second-order coupled equations

The extension of the above technique to a system of second-order ODEs is also possible under certain conditions. We describe below the formulation as presented in [22]. In the case of a system of n degrees of freedom the Lagrangian $L = L(t, q, \dot{q})$, where $q = \{q_1, \dots, q_n\}$ and $\dot{q} = \{\dot{q}_1, \dots, \dot{q}_n\}$ define the generalized coordinates and corresponding velocities, we may define the ijth Jacobi Last Multiplier by

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \quad i, j = 1, \dots, n. \tag{5.1}$$

It is assumed that the equations of motion:

$$\ddot{q}_k = w_k(t, q, \dot{q}), \quad k = 1, \dots, n, \tag{5.2}$$

are derivable from the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \dots, n. \tag{5.3}$$

It is evident that the conjugate momenta are

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = p_j(t, q, \dot{q}), \quad j = 1, \dots, n,$$

which implies

$$\frac{dp_j}{dt} = \frac{\partial p_j}{\partial t} + \sum_{k=1}^{n}\left(\dot{q}_k \frac{\partial p_j}{\partial q_k} + w_k \frac{\partial p_j}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_j}.$$

This means

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{k=1}^{n}\left(\dot{q}_k \frac{\partial^2 L}{\partial q_k \partial \dot{q}_j} + w_k \frac{\partial p_j}{\partial \dot{q}_k \partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}, \quad j = 1, \dots, n. \tag{5.4}$$

Differentiating (5.4) with respect to $\dot{q}_i$ and using the definition of the last multiplier given in (5.1) we find

$$\frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ij}) + \frac{\partial}{\partial \dot{q}_k}(w_k M_{ij})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i} M_{kj} - \frac{\partial w_k}{\partial \dot{q}_k} M_{ij}\right) + \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} = 0. \tag{5.5}$$

Interchanging i and j in (5.5) we get

$$\frac{\partial M_{ji}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ji}) + \frac{\partial}{\partial \dot{q}_k}(w_k M_{ji})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_j} M_{ki} - \frac{\partial w_k}{\partial \dot{q}_k} M_{ji}\right) + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i} - \frac{\partial^2 L}{\partial \dot{q}_j \partial q_i} = 0. \tag{5.6}$$

Adding (5.5) and (5.6) and making use of the fact that $M_{ij} = M_{ji}$ we have

$$\frac{\partial M_{ij}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ij}) + \frac{\partial}{\partial \dot{q}_k}(w_k M_{ij})\right) + \sum_{k=1}^{n}\left(\frac{1}{2}\left(\frac{\partial w_k}{\partial \dot{q}_i} M_{kj} + \frac{\partial w_k}{\partial \dot{q}_j} M_{ki}\right) - \frac{\partial w_k}{\partial \dot{q}_k} M_{ij}\right) = 0. \tag{5.7}$$

It is evident that M_{ij} satisfies the defining relation (2.6) for the JLM whenever

$$\sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i} M_{kj} + \frac{\partial w_k}{\partial \dot{q}_j} M_{ki}\right) = 2\sum_{k=1}^{n} \frac{\partial w_k}{\partial \dot{q}_k} M_{ij} \quad \text{for each } k = 1, \dots, n. \tag{5.8}$$

A trivial way to ensure this condition is satisfied is to assume the w_k's to be velocity independent,

$$\frac{\partial w_k}{\partial \dot{q}_l} = 0 \quad \text{for all } k, l = 1, \dots, n.$$

On the other hand, when $i = j$, the last two terms in (5.5) cancel leaving

$$\frac{\partial M_{ii}}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial}{\partial q_k}(\dot{q}_k M_{ii}) + \frac{\partial}{\partial \dot{q}_k}(w_k M_{ii})\right) + \sum_{k=1}^{n}\left(\frac{\partial w_k}{\partial \dot{q}_i} M_{ki} - \frac{\partial w_k}{\partial \dot{q}_k} M_{ii}\right) = 0. \tag{5.9}$$

Here also M_{ii} satisfies (2.6) when the last sum of (5.9) vanishes, which may be ensured by choosing the w_k's to be velocity independent. Under these circumstances all the M_{ij}'s satisfy the equation

$$\frac{\partial M_{ii}}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial}{\partial q_k}(\dot{q_k} M_{ii}) + \frac{\partial}{\partial \dot{q_k}}(w_k M_{ii}) \right) = 0, \tag{5.10}$$

as they should, provided $\partial w_k / \partial \dot{q_j} = 0$ for all $k, j = 1, \dots, n$. With this assumption equations (5.7) and 5.10) always admit the solution $M_{ij} =$ constant. The following examples illustrate how simple choices of M_{ij} can be made to obtain the Lagrangians of second-order ODEs satisfying the above velocity-independent criterion.

Example 1. Consider the system

$$\ddot{x} + \frac{\alpha}{x^2} g(y/x) - \frac{\lambda}{x^3} = 0,$$
$$\ddot{y} + \frac{\beta}{x^2} f(y/x) - \frac{\mu}{y^3} = 0.$$

Here $w_1(x, y) = -\alpha g(y/x)/x^2 + \lambda/x^3$ and $w_2(x, y) = -\beta f(y/x)/x^2 + \mu/y^3$ respectively. On the other hand α, β, λ and μ are arbitrary parameters while g and f are functions with argument $u = y/x$. Notice that w_1 and w_2 are independent of the velocities. The Jacobi Last Multiplier for this system is therefore a solution of the equation,

$$\frac{\partial M}{\partial t} + \frac{\partial (M\dot{x})}{\partial x} + \frac{\partial (M\dot{y})}{\partial y} + \frac{\partial (M w_1)}{\partial \dot{x}} + \frac{\partial (M w_2)}{\partial \dot{y}} = 0,$$

and admits constant solutions. We choose them as follows:

$$M_{xy} = M_{yx} = 0 \quad \text{and} \quad M_{xx} = M_{yy} = 1.$$

These yield the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + h_1(t, x, y)\dot{x} + h_2(t, x, y)\dot{y} + h_3(t, x, y).$$

Substitution of this into the Euler–Lagrange equations for x and y gives, upon using the original equations of motion,

$$h_{1t} - h_{3x} + w_1 + (h_{1y} - h_{2x})\dot{y} = 0, \tag{5.11}$$
$$h_{2t} - h_{3y} + w_2 + (h_{2x} - h_{1y})\dot{x} = 0. \tag{5.12}$$

Equating the coefficients of $\dot{x}$ and $\dot{y}$ respectively we get the following set of equations:

$$h_{1y} - h_{2x} = 0 \quad \text{which implies} \quad h_1 = G_x, \quad h_2 = G_y \quad \text{and} \tag{5.13}$$
$$h_{1t} - h_{3x} + w_1 = 0, \tag{5.14}$$
$$h_{2t} - h_{3y} + w_2 = 0. \tag{5.15}$$

These in turn give

$$h_{3x} = G_{xt} + w_1 \text{ or } h_3 = G_t + \int w_1\, dx + r(y), \tag{5.16}$$
$$h_{3y} = G_{yt} + w_2 \text{ or } h_3 = G_t + \int w_2\, dy + s(x). \tag{5.17}$$

Consistency for h_3 requires that

$$h_{3xy} = h_{3yx}$$

and translates into the requirement that $w_{1y} = w_{2x}$. This imposes the following condition on the functions f and g which define the second-order system:

$$\frac{\alpha}{\beta} g'(u) + u f'(u) + 2f(u) = 0, \quad \text{where } u = \frac{y}{x}.$$

One can rewrite this as

$$\frac{\alpha}{\beta} u g'(u) + \frac{d}{du}\left(u^2 f(u)\right) = 0. \tag{5.18}$$

When we use the explicit forms of w_1 and w_2 and make use of the last condition, the form of the functions $r(y)$ and $s(x)$ occurring in (5.16) and (5.17) may be fixed and the functional form of h_3 is found to be

$$h_3(t,x,y) = G_t - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta\frac{y}{x}f(y/x)\right)\right].$$

Therefore the Lagrangian is given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left[\frac{\alpha}{2x^2} + \frac{\mu}{2y^2} - \frac{1}{x}\left(\alpha g(y/x) + \beta\frac{y}{x}f(y/x)\right)\right] + \frac{dG}{dt}. \tag{5.19}$$

Again the total derivative term, being of little physical significance in the classical case, may be safely discarded. It is interesting to note that the above second-order system, though similar in some respects to equations of the Ermakov system, is not merely a mathematical artifact. It is similar in structure to the system studied in [25] in the context of the dynamics of stellar systems.

A similar exercise may be carried out for the following:

Example: Generalized Van der Waals Potential

$$\ddot{x} = -\left(2\gamma x + \frac{x}{r^3}\right) = w_1(x,y),$$
$$\ddot{y} = -\left(2\gamma\beta^2 y + \frac{y}{r^3}\right) = w_2(x,y) \quad \text{where } r = \sqrt{x^2+y^2},$$

and γ, β are parameters. In this case the Lagrangian is given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left[\gamma\left(x^2 + \beta^2 y^2\right) - \frac{1}{r}\right] + \frac{dG}{dt}. \tag{5.20}$$

Similarly for the

Example: Henon–Heiles system,

$$\ddot{x} = -(Ax + 2\alpha xy),$$
$$\ddot{y} = -\left(By + \alpha x^2 - \beta y^2\right), \tag{5.21}$$

the Lagrangian is given by

$$L(t,x,\dot{x}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \left(A\frac{x^2}{2} + B\frac{y^2}{2} + \alpha x^2 y - \beta\frac{y^3}{3}\right) + \frac{dG}{dt}. \tag{5.22}$$

6. Outlook

In this paper we have discussed applications of the Jacobi Last Multiplier for the deduction of Lagrangian functions for the second-order ODEs of the Painlevé–Gambier classification. We have specifically deduced the Lagrangians for the majority of the six Painlevé equations as also other prototype equations of the Painlevé–Gambier classification. We have also dwelt on the geometrical background involving the last multiplier. This is an on-going endeavour and we propose to perform further investigations in our future works. In addition we have used the above technique to analyse a particular class of coupled second-order equations. Besides the well-known Henon–Heiles system we have obtained the Lagrangian for a relatively less studied systems occurring in the context of stellar dynamics. The Lagrangians discussed here are found to admit a Noetherian symmetry, with an associated first integral, which are the Hamiltonians of the equations concerned.

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