An adaptive logic for relevant classical deduction

Hans Lycke

Centre for Logic and Philosophy of Science, Universiteit Gent, Belgium

Available online 4 May 2006

Abstract

In this paper, I will show that it is possible to delete Ex Falso Quodlibet from Classical Logic, without depriving it of any of its deductive powers. This is done by means of the ambiguity-adaptive logic AALns, which is equivalent to dCR, the deductive version of Neil Tennant’s CR.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Ex falso quodlibet; Paraconsistency; Adaptive logics

1. Introduction

When applied to an inconsistent premise set, Classical Logic (CL) leads to the trivial consequence set. In other words, Ex Falso Quodlibet (EFQ) is valid in CL:

$$\forall A : \Gamma \vdash_{\text{CL}} A.$$  

However, as there are a lot of inconsistent, but non-trivial (scientific) theories, CL cannot be the logic underlying human reasoning in inconsistent contexts (see [8]). As a consequence, a lot of paraconsistent logics have been proposed in order to replace CL in those contexts. But, although they all avoid EFQ, they all weaken the deductive strength of CL in one way or another. This means that they do not succeed in isolating EFQ from CL, and as such they are unable to derive some genuine classical consequences of the premise set, consequences which do not follow from their premises only because of the joint inconsistency of those premises.¹

For example, consider the premise set $\Gamma = \{ p, \sim p, p \lor \sim p \lor q \}$. It is obvious that we do not need to “abuse” the inconsistency of the premises (which is how EFQ is established) in order to derive $q$ from $\Gamma$. As $q$ is in a sense part of the premises, we should be allowed to use Disjunctive Syllogism (DS) twice in order to derive it, even though $\Gamma$ is inconsistent. But, there is no paraconsistent logic which allows for this derivation, because (1) it is based on an inconsistent premise set (Rescher and Manor [9]), (2) DS is invalidated in the logic (Priest [7], da Costa [6], . . .), or (3) there has been made use of unreliable (read: inconsistent) information (Batens [1]). However, this implies that the
existential paraconsistent logics not only avoid the “abuse” we can make of an inconsistent premise set by means of the classical derivation rules (EFQ-derivation), but that they also limit the “relevant use” we can make of those derivation rules (relevant CL-derivation).

Obviously, the question which follows from this, is whether we can delete Ex Falso Quodlibet from Classical Logic without depriving it of any of its deductive powers. In other words, is it possible to decide for any $\Delta \vdash_{CL} A$ whether $A$ ‘follows’ from $\Delta$ by dint of $\Delta$’s inconsistency, rather than by dint of any genuine deductive connection between $\Delta$ and $A$.$^2$

In [10] and [11], Neil Tennant proved it to be possible to split up CL-proofs into explosive classical proofs and relevant classical proofs.$^3$ The consequences of explosive classical proofs solely depend on the inconsistency of that premise set in order to be derivable, while the consequences of relevant classical proofs follow from the premises by “relevant use” of the classical derivation rules. In other words, relevant classical proofs give us all and only those classical consequences of a premise set that are somehow “in” the premises, as f.e. $q$ is in the premise set $\{p, \neg p, p \lor \neg p \lor q\}$.\footnote{Remark that this notion of relevance does not apply to the implication (as it does for relevant logics), but to the deduction process. This is why Tennant also likes to call it “relevance at the turnstile”.}

In Section 2 below, I will present Tennant’s theory of relevant (classical) proofs, called Compassionate Relevantism (CR) in [12], and I will show how it can be used to capture relevant classical deduction. I will call this theory of relevant classical deduction $dCR$, the deductive version of CR.

In Section 3, I will present the ambiguity-adaptive logic $\mathbb{AAL}_{ns}$, and in Section 4, I will show that $\mathbb{AAL}_{ns}$ also captures relevant classical deduction by proving the equivalence with $dCR$.\footnote{For reasons of simplicity, I will restrict myself to the propositional version of both CR, $dCR$ and $\mathbb{AAL}_{ns}$, although I will show in Section 3.3.4 how propositional $\mathbb{AAL}_{ns}$ can be upgraded to the predicative level.}

## 2. Relevant classical proofs: CR

In this section, I will describe the semantics of Neil Tennant’s CR,\footnote{To be clear, this is my terminology.} and I will show that it can be used to capture relevant classical deduction, because of the fact that CR isolates EFQ from CL (see [10] and [11]).

First of all, I have to mention that Tennant uses set sequents to characterize CR. These are “formulas” of the form $\Delta : \Theta$ with $\Delta$ and $\Theta$ being sets of well-formed formulas of classical logic (CL) in which the order and the repetition of elements are irrelevant. In the following, I will restrict the succedent set $\Theta$ to the singleton $A$. This will not lead to a change in the logic.

Tennant states that $\Delta \vdash_{CR} A$ expresses that there is a relevant classical proof of $A$ from $\Delta$, whenever $\Delta : A$ is an entailment. Whether or not a sequent is an entailment, depends on the following definitions:

**Definition 1.** A valid sequent $\Delta : A$ is a sequent, of which there exists no CL-model which makes all elements of $\Delta$ true and $A$ false.

**Definition 2.** A perfectly valid sequent $\Delta : A$ is a sequent which is valid and which has no valid proper subsequents.

**Definition 3.** A proper subsequent of a sequent $\Delta : A$ is

(a) a sequent $\Delta' : A$ with $\Delta' \subset \Delta$ (meaning that not all elements of $\Delta$ are needed in order to derive $A$), or
(b) the sequent $\Delta : \emptyset$ (meaning that $\Delta$ is inconsistent).

**Definition 4.** A sequent $\Delta' : A'$ is a suprasequent of $\Delta : A$, iff there is a function $s$ which replaces each sentential letter from $\Delta : A$ by a (possibly complex) formula, so that $s(\Delta : A) = \Delta' : A'$.

$^2$ See [12, p. 706]. In this quote, I have adapted Tennant’s logical notation to mine in order to preserve overall coherence.

$^3$ Also Batens has showed that it is possible to isolate EFQ from CL (see [3]). His approach was based on the goal directed proof procedure from Batens and Provijn [5]. In a sense, one could say that $\mathbb{AAL}_{ns}$ is one of the possible representations of the logic behind the proof procedure from [3].

$^5$ For the proof theory, I refer to [11].
Definition 5. A sequent $\Delta : A$ is an entailment iff $\Delta : A$ has a perfectly valid suprasequent.

These definitions show us the relevance-criteria incorporated in CR. First of all, an inconsistent premise set is treated as if it is a consistent one. This is done by looking for a consistent interpretation of an inconsistent premise set. Let me illustrate this by means of an example:

Example 1. There is a relevant classical proof of $q$ from $\{p, \sim p, p \lor \sim p \lor q\}$, because $\{p, \sim p, p \lor \sim p \lor q\} : q$ is an entailment. The latter is the case, because $\{r, \sim p, p \lor \sim r \lor q\} : q$ is a perfectly valid suprasequent of $\{p, \sim p, p \lor \sim p \lor q\} : q$.

Secondly, as only the premises which are needed for deriving a consequence from a premise set are considered relevant for that consequence, entailments always have minimal premise sets. This is not problematic for Tennant, as he tried to capture what a relevant classical proof is, but it certainly is not to be considered as relevant classical deduction. The following example will show us why:

Example 2. Because $\{p, \sim p \lor q\} : q$ is an entailment, $\{\sim p, p, \sim p \lor q\} : q$ is not an entailment, even though $\{p \land \sim p, p \lor q\} : q$, and $\{p, \sim p \land (p \lor q)\} : q$ are entailments in CR.

Does this mean that there is no relevant proof of $q$ from $\{\sim p, p, \sim p \lor q\}$? In a sense it does, as this set contains more elements than we really need in order to derive $q$. But of course this should not mean that $q$ cannot relevantly be deduced from the premise set $\Gamma = \{\sim p, p, \sim p \lor q\}$, as there is a relevant proof from a subset of $\Gamma$. Nevertheless, it is obvious that CR can be used in order to capture relevant classical deduction:

Definition 6. A sequent $\Delta : A$ expresses that there is a relevant classical deduction of $A$ from $\Delta$, whenever there is a $\Delta' \subseteq \Delta$ for which $\Delta' : A$ is an entailment.

However, a major drawback of this semantical characterization of relevant classical deduction, which I will call deductive Compassionate Relevantism (dCR), is that there is no proof theory accompanying it. There is only a proof theory (stated in sequent calculus) for CR, or in other words, for deciding whether or not a sequent is an entailment.

In Sections 3 and 4, it will be shown that this drawback can be overcome by using the ambiguity-adaptive logic AALns to characterize relevant classical deduction.

3. The ambiguity-adaptive logic AALns

In this section, I will present the ambiguity-adaptive logic AALns, for which it will be shown in Section 4 that it isolates EFQ from CL, by proving its equivalence with dCR.

In Section 3.1, I will describe the language schema of AALns. In Section 3.2, I will present the paraconsistent logic AmbL. In Section 3.3, I will present the adaptive logic AALns.

3.1. The language schema of ambiguity logic

Let $\mathcal{L}$ be the language of Propositional Classical Logic (CL), with $\mathcal{S}$ and $\mathcal{W}$ respectively the sets of sentential letters and well-formed formulas.

The first step towards ambiguity logic is the construction of the language $\mathcal{L}^\sharp$. In order to get $\mathcal{L}^\sharp$, we change the language $\mathcal{L}$ of CL in the following way:

- $\mathcal{S}^\sharp = \{A^i \mid A \in \mathcal{S}, \text{ and } i \in \mathbb{N}\}$.
- $\mathcal{W}^\sharp$ is defined in $\mathcal{L}^\sharp$ in the same way $\mathcal{W}$ is defined in $\mathcal{L}$.

In view of what is to come, also the following definition is very useful:

---

8 An $A^i \in \mathcal{S}^\sharp$ is called an indexed letter.
Definition 7. \( A^\mathcal{I} \in \mathcal{I}(A) \) iff

1. \( A^\mathcal{I} \in W^\mathcal{I} \), and
2. when we drop the indices from \( A^\mathcal{I} \), we get \( A \in W \).

3.1.1. Ambiguous premise sets

For our purposes, we want our premise set to be maximally ambiguous. Therefore, we will define the set of maximally ambiguous interpretations \( \mathcal{I}(\Gamma) \) of \( \Gamma \):

Definition 8. \( \Gamma^{\text{max}} \in \mathcal{I}(\Gamma) \) iff

1. \( \Gamma^{\text{max}} \subset W^{\mathcal{I}} \), and
2. every indexed letter occurs maximally once in \( \Gamma^{\text{max}} \), and
3. when we drop the indices from \( \Gamma^{\text{max}} \), we get \( \Gamma \subset W \).

Because, in our approach, all \( \Gamma^{\text{max}} \in \mathcal{I}(\Gamma) \) will lead to the same consequences in \( \text{AAL}_{\text{ns}} \), it is better to pick out one member to represent them all. We will denote that member by \( \Gamma^{\mathcal{I}} \).

3.2. The paraconsistent logic \( \text{AmbL} \)

The logic \( \text{AmbL} \) is defined as follows:

Definition 9. \( \Gamma \vdash_{\text{AmbL}} A \) iff there is at least one \( A^\mathcal{I} \in \mathcal{I}(A) \), for which \( \Gamma^\mathcal{I} \vdash_{\text{CL}} A^\mathcal{I} \).

It is easily seen that \( \text{AmbL} \) is a paraconsistent logic. As all indexed letters occur only once in \( \Gamma^\mathcal{I} \), it is necessarily consistent, which makes it impossible in \( \text{CL} \) to derive any unwanted consequence (= derived by means of \( \text{EFQ} \)) from it, even when \( \Gamma \) is inconsistent.

Example 3. In \( \text{CL} \), we can derive \( q \) from the premise set \( \Gamma = \{p, \sim p\} \), but we cannot derive any \( q^\mathcal{I} \in \mathcal{I}(q) \) from \( \Gamma^{\mathcal{I}} = \{p^1, \sim p^2\} \).

It is also immediately clear that \( \text{AmbL} \) will not allow for a lot of genuine consequences to be derivable.

Example 4. In \( \text{CL} \), we can derive \( q \) from the (consistent) premise set \( \Gamma = \{p, \sim p \vee q\} \), but we cannot derive \( q^1 \) from \( \Gamma^{\mathcal{I}} = \{p^1, \sim p^2 \vee q^1\} \).

This problem can be overcome by interpreting the premise set \( \Gamma^{\mathcal{I}} \) as unambiguous as possible, which means that we will interpret as identical, all indexed letters which only differ from each other with regard to their index, as long as there is no reason to do otherwise. This should allow us to derive all and only the genuine consequences of a premise set.

3.3. Interpreting a premise set as unambiguous as possible

In order to interpret \( \Gamma^{\mathcal{I}} \) as unambiguous as possible, I will make use of the general framework of ambiguity-adaptive logics. Ambiguity-adaptive logics are a branch of adaptive logics, first proposed by Guido Vanackere in [13]. As a consequence, the ambiguity-adaptive logic \( \text{AAL}_{\text{ns}} \) proposed below, is a standard ambiguity-adaptive logic, which differs from those proposed by Guido Vanackere in [14,15] in that it makes use of a different adaptive strategy, the Normal Selections Strategy.

In short, the ambiguity-adaptive logic \( \text{AAL}_{\text{ns}} \) can be characterized as follows:

Definition 10. \( \Gamma \vdash_{\text{AAL}_{\text{ns}}} A \) iff there is at least one \( A^\mathcal{I} \in \mathcal{I}(A) \), for which \( \Gamma^\mathcal{I} \vdash_{\text{CL}_{\text{ns}}} A^\mathcal{I} \).
From this definition follows that it is actually the adaptive logic $\text{CL}^{\text{ns}}$, which interprets an ambiguous premise set $\Gamma^I$ as unambiguous as possible. In order to avoid confusion, in the following, I will always use “$\text{CL}^{\text{ns}}$” to refer to the adaptive logic $\text{CL}^{\text{ns}}$ as applied to a maximally ambiguous premise set $\Gamma^I$. Moreover, as it is $\text{CL}^{\text{ns}}$ which does the actual work, it will be $\text{CL}^{\text{ns}}$ which I will describe in the rest of this section. To be more precise, in Section 3.3.1, I will show how $\text{CL}^{\text{ns}}$ fits the standard format of adaptive logics. In Sections 3.3.2 and 3.3.3, the proof theory and the semantics of $\text{CL}^{\text{ns}}$ will be given. Finally, in Section 3.3.4, the predicative version of $\text{CL}^{\text{ns}}$ will be introduced.

3.3.1. The standard format

Following the standard format for adaptive logics as explicated in [2], $\text{CL}^{\text{ns}}$ will be characterized by three components: a lower limit logic, a set of abnormalities and an adaptive strategy.

The lower limit logic (LLL) is the stable part of an adaptive logic (AL), since all LLL-consequences of a premise set are AL-consequences of that premise set as well:

**Theorem 1.** If $\Gamma \vdash_{\text{LLL}} A$, then also $\Gamma \vdash_{\text{AL}} A$.

The LLL of $\text{CL}^{\text{ns}}$ is the logic $\text{CL}$, as applied to an ambiguous premise set $\Gamma^I$. This means that the logic $\text{AmbL}$ (see Section 3.2) is the LLL of the logic $\text{AAL}^{\text{ns}}$.

The set of abnormalities of $\text{CL}^{\text{ns}}$ is the set $\Omega$:

**Definition 11.** $\Omega = \{ \neg(A^i \equiv A^j) \mid A^i, A^j \in S^I \}$.

Below, $\text{Dab}(\Delta)$ will always refer to a disjunction of members of a finite $\Delta \subseteq \Omega$. Such a disjunction of abnormalities is also called a Dab-formula. A Dab-formula which has been derived from a premise set by means of the LLL is called a Dab-consequence of that premise set.

In an adaptive logic, the set of abnormalities is the set of formulas which we suppose to be false until or unless it is impossible to do otherwise. In fact, this comes down to the following:

**Conjecture 1.** If $\Gamma \vdash_{\text{LLL}} A \lor \text{Dab}(\Delta)$ then $\Gamma \vdash_{\text{AL}} A$, unless .

How the “unless”-clause should be interpreted, depends on the adaptive strategy of the adaptive logic under consideration. For $\text{CL}^{\text{ns}}$, this is the Normal Selections Strategy (NS-strategy), which means that we should interpret the “unless”-clause as follows:

**Theorem 2.** If $\Gamma \vdash_{\text{LLL}} A \lor \text{Dab}(\Delta)$ then $\Gamma \vdash_{\text{AL}} A$, unless $\Gamma \vdash_{\text{LLL}} \text{Dab}(\Delta)$.

To make this more concrete, consider again Example 4 from Section 3.2. Although it is obvious that $\Gamma^I \not\vdash_{\text{CL}} q^1$, it is also obvious that $\Gamma^I \vdash_{\text{CL}} q^1 \lor \neg(p^1 \equiv p^2)$, so that $q^1$ is derivable from $\Gamma^I$ by $\text{CL}^{\text{ns}}$, unless $\Gamma^I \vdash_{\text{CL}} \neg(p^1 \equiv p^2)$. As the latter is not the case, we can conclude that $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} q^1$, and also that $\Gamma^I \vdash_{\text{AAL}^{\text{ns}}} q$.

3.3.2. The proof theory of $\text{CL}^{\text{ns}}$

The proof theory of $\text{CL}^{\text{ns}}$, and adaptive proof theory in general, has a few characteristic features. I will first describe these features, before I will give the specific deduction rules and the marking rule of $\text{CL}^{\text{ns}}$.

First of all, the lines in a $\text{CL}^{\text{ns}}$-proof consist of five elements: (i) a line number, (ii) a formula, (iii) the line numbers of the formulas from which the formula is derived, (iv) the rule by which the formula is derived, and (v) an adaptive condition. The latter is a set of abnormalities, and should be considered as the proof theoretical “unless”-clause. This means that the formula of a line with a non-empty condition $\Delta$, is considered as derived only if $\text{Dab}(\Delta)$ is not (yet) derived on a line in the proof.

This first characteristic feature of $\text{CL}^{\text{ns}}$-proofs immediately leads to the second one: the dynamic nature of the proofs. For proofs to be dynamic means that formulas, derived at a certain stage of the proof, can become underived.
again at a later stage of that proof. It is easy to see why this should be the case: if the Dab-formula, connected with the adaptive condition of a formula A derived at stage s of the proof, is derived at a later stage s + n, then that formula A should be removed from the proof at stage s + n, as it can not be considered as a consequence of the premises anymore. That a formula should be considered as removed from the proof, will be made clear by marking it.

3.3.2.1. Deduction rules Let us have a look at the deduction rules, which are presented in a generic format. There are three of them. The first one (PREM) enables us to introduce elements from the premise set into the proof. The second one is an unconditional rule (RU) which means that it doesn’t add a new element to the adaptive condition of the formulas it enables us to derive. Finally, the third one is a conditional rule (RC) which means that it does add a new element to the adaptive condition of the formulas it enables us to derive.

**PREM** If $A^I \in \Gamma^I$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $A^I$, (iii) –, (iv) PREM, and (v) $\emptyset$.

**RU** If $A^I, \ldots, A^I_n \vdash_{\text{CL}} B^I$ and each of $A^I_1, \ldots, A^I_n$ occurs in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $B^I$, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \cdots \cup \Delta_n$.

**RC** If $A^I, \ldots, A^I_n \vdash_{\text{CL}} B^I \lor \text{Dab(\Theta)}$ and each of $A^I_1, \ldots, A^I_n$ occurs in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $B^I$, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \cdots \cup \Delta_n \cup \Theta$.

3.3.2.2. Marking rule Consider the following marking criterion which acts upon the conditions of lines derived in a proof.

**Definition 12.** $Ab_s(\Gamma^I) = \{\Theta | \text{Dab(\Theta)} \text{ is a Dab-consequence of } \Gamma^I \text{ at stage } s \text{ of the proof}\}$.

**Definition 13.** Marking for $\text{CL}^{ns}$: line $i$ is marked at stage $s$ of the proof iff, where $\Delta$ is its condition, there is a $\Theta$ for which: $\Theta \subseteq \Delta$ and $\Theta \in Ab_s(\Gamma^I)$.

3.3.2.3. Final derivability In order to complete the proof theory, we also need the following (standard) definitions considering Final Derivability:

**Definition 14.** $A^I$ is finally derived from $\Gamma^I$ on line $i$ of a proof at stage $s$ iff (i) $A^I$ is the second element of line $i$, (ii) line $i$ is not marked at stage $s$, and (iii) any extension of the proof will not result in a marking of line $i$.

**Definition 15.** $\Gamma^I \vdash_{\text{CL}^{ns}} A^I$ ($A^I$ is finally $\text{CL}^{ns}$-derivable from $\Gamma^I$) iff $A^I$ is finally derived on a line of a proof from $\Gamma^I$.

3.3.2.4. Some examples First, consider the $\text{CL}^{ns}$-proof based on the premise set $\Gamma^I = \{p^1, q^1, \sim p^2 \lor r^1, \sim q^2 \lor s^1, \sim p^3 \lor \sim q^3\}$.

<table>
<thead>
<tr>
<th>Line</th>
<th>Formula</th>
<th>Step</th>
<th>Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p^1$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>$q^1$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3</td>
<td>$\sim p^2 \lor r^1$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>4</td>
<td>$\sim q^2 \lor s^1$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>$r^1 \lor (p^1 \equiv p^2)$</td>
<td>1, 3, RU</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6</td>
<td>$r^1$</td>
<td>5; RC</td>
<td>${\sim (p^1 \equiv p^2)}$</td>
</tr>
</tbody>
</table>

---

10 Adding a line to a proof is equivalent with moving on to a next stage of that proof.
in the premise set \( \Gamma \)

made in the deduction process of specific occurrences of sentential letters of CLns

3.3.3. The semantics of CLns and the "abuse" of the premises.

understanding of the difference between relevant and irrelevant consequences of a premise set, and between the "use"

formula, are based solely on the inconsistency of the premise set

rule to the indices, an adaptive

logics.

consequences of a premise set, whether or not that premise set is inconsistent.

It is easily seen that \( q^1 \) will not get marked by any extension of the proof. This example shows us that \( q \) is AALns-derived form \( \{p, \sim p, \sim p \lor p \lor q\} \), which is a clear indication that AALns is able to derive all relevant classical

Moreover, the CLns-proof theory also gives us an intuitive grasp on how the logic AALns is able to isolate EFQ from CL. The key lays with the interplay between the indices and the conditional rule RC. By coupling the conditional rule to the indices, an adaptive CLns-proof can keep track of the use (sometimes called negation-elimination) that was made in the deduction process of specific occurrences of sentential letters of \( \Gamma \). And, as all indices occur only once in the premise set \( \Gamma^\mathcal{F} \), the derivation of a Dab-formula shows us that derivations which have made use of that Dab-formula, are based solely on the inconsistency of the premise set \( \Gamma \). Consequently, the indices provide us with a clear understanding of the difference between relevant and irrelevant consequences of a premise set, and between the “use” and the “abuse” of the premises.

3.3.3. The semantics of CLns

As for all adaptive logics, the semantics of CLns works according to the following two principles:

(1) Select (one or more) elements (the selected sets) from the set of all possible subsets of the set of all LLL-models of a premise set. How this is done differs according to the chosen adaptive strategy. For the NS-strategy, a subset of the set of all possible subsets of LLL-models of a premise set gets selected when it contains only minimal abnormal LLL-models (= models for which there are no other models that verify only a subset of the abnormalities they verify) which all verify exactly the same abnormalities.

(2) State that a formula follows from a premise set by means of the adaptive logic, if it is verified by all models from at least one selected set.

The definitions below will show us how the semantics of CLns fits into this general semantical framework of adaptive logics.

First, let \( M \models A^\mathcal{F} \) (resp. \( M \models \Gamma^\mathcal{F} \)) denote that a model \( M \) verifies the formula \( A^\mathcal{F} \) (resp. all members of \( \Gamma^\mathcal{F} \)). Now:

Definition 16. For every CL-model \( M \): \( Ab(M) = \{ A^\mathcal{F} \in \Omega \mid M \models A^\mathcal{F} \} \).
Definition 17. A \( \text{CL} \)-model \( M \) of \( \Gamma^T \) is a minimal abnormal model iff there is no \( \text{CL} \)-model \( M' \) of \( \Gamma^T \) for which \( Ab(M') \subset Ab(M) \).

Definition 18. \( \Phi(\Gamma^T) = \{ Ab(M) \mid M \text{ is a minimal abnormal } \text{CL}-model \text{ of } \Gamma^T \} \).

Definition 19. A set \( \Sigma \) of \( \text{CL} \)-models of \( \Gamma^T \) is selected iff, for some \( \varphi \in \Phi(\Gamma^T) \), \( \Sigma = \{ M \mid M \models \Gamma^T; Ab(M) = \varphi \} \).

Definition 20. \( \Gamma^T \models_{\text{CLns}} A^T \) iff \( A^T \) is verified by all members of a selected set of \( \text{CL} \)-models of \( \Gamma^T \).

3.3.3.1. Soundness and completeness  Soundness and completeness for adaptive logics based on the NS-strategy have been proven in [4]. As a consequence:

Theorem 3. \( \Gamma^T \models_{\text{CLns}} A^T \) iff \( \Gamma^T \models_{\text{AALns}} A \).

Theorem 4. \( \Gamma \vdash_{\text{AALns}} A \) iff \( \Gamma \models_{\text{AALns}} A \).

3.3.4. The predicative version of \( \text{CLns} \)  Until now, I have only considered the propositional fragment of \( \text{CLns} \). In this section, however, I will show how predicative \( \text{CLns} \) can be obtained.

In fact, upgrading propositional \( \text{CLns} \) to predicative \( \text{CLns} \) is quite simple. The only elements of propositional \( \text{CLns} \) that change fundamentally, are (1) the language schema, and (2) the set of abnormalities \( \Omega \). The rest remains basically the same, and will not be discussed here.

3.3.4.1. The predicative language schema  Let \( \mathcal{L} \) be the language of predicative Classical Logic, with \( S, \mathcal{P}^r, \mathcal{C}, \mathcal{V} \) and \( \mathcal{W} \) respectively the sets of sentential letters, predicates of rank \( r \), individual constants, individual variables and well-formed formulas. The language \( \mathcal{L}^T \) is obtained from \( \mathcal{L} \) in the following way:

- \( S^T = \{ \{ A^i \mid A \in S, \text{ and } i \in \mathbb{N} \} \} \).
- \( \mathcal{P}^r = \{ \{ \pi^i \mid \pi \in \mathcal{P}^r, \text{ and } i \in \mathbb{N} \} \} \).
- \( \mathcal{C}^T = \{ \{ \beta^i \mid \beta \in \mathcal{C}, \text{ and } i \in \mathbb{N} \} \} \).
- \( \mathcal{W} \) is defined in \( \mathcal{L}^T \) in the same way \( \mathcal{W} \) is defined in \( \mathcal{L} \).

To conclude, two remarks about individual variables. First of all, they do not get an index. Second, they do not range over individual constants (elements of \( \mathcal{C} \)), but over indexed individual constants (elements of \( \mathcal{C}^T \)).

3.3.4.2. The set of abnormalities  The set of abnormalities \( \Omega \) gets extended in the following way:

Definition 21. \( \Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), with

- \( \Omega_1 = \{ \sim(A^i \equiv A^j) \mid A^i, A^j \in S^T \} \),
- \( \Omega_2 = \{ \sim(\forall \alpha_1 \ldots \forall \alpha_n)(\pi^i \alpha_1 \ldots \alpha_n \equiv \pi^j \alpha_1 \ldots \alpha_n) \mid \pi^i, \pi^j \in \mathcal{P}^r \} \),
- \( \Omega_3 = \{ \sim(\beta^i = \beta^j) \mid \beta^i, \beta^j \in \mathcal{C}^T \} \).

4. Relevant classical deduction: \( \text{AALns} \)

In this section, I will show that \( \text{AALns} \) avoids \( \text{EFQ} \) without limiting the deductive strength of the classical derivation rules. In other words, I will show that \( \text{AALns} \) really captures relevant classical deduction.

In order to do so, I will have to show that (1) whenever the premise set is consistent, \( \text{AALns} \) and \( \text{CL} \) lead to the same consequence set (because all classical consequences can be derived relevantly from a consistent premise set), and (2) whenever the premise set is inconsistent, \( \text{AALns} \) leads to all and only those classical consequences for which there is a relevant classical derivation.
4.1. The consistent case

It is quite easy to see that \textsc{AAL}^{ns} and \textsc{CL} will lead to the same consequence set whenever the premise set \( \Gamma \) is consistent. The \textit{Derivability Adjustment Theorem (DAT)} for \textsc{AAL}^{ns} will show us why:

**Theorem 5.** \( \Gamma \vdash_{\text{CL}} A \text{ iff, for at least one } A^I \in \mathcal{I}(A), \Gamma^I \vdash_{\text{CL}} A^I \lor \text{Dab}(\Delta). \)

**Proof.** See Appendix A.1.

In fact, \textsc{DAT} tells us that whenever a formula \( A \) is derivable from a premise set \( \Gamma \) by means of \textsc{CL}, then there is always an \( A^I \in \mathcal{I}(A) \) derivable from \( \Gamma^I \) by means of \textsc{CL}^{ns}, unless the accompanying Dab-formula is derivable (remember the standard format of adaptive logics in Section 3.3.1). But, it is easily seen that a Dab-formula is only derivable from \( \Gamma^I \) when \( \Gamma \) is inconsistent, as \( \sim(p^I \equiv p^J) \) in \( \mathcal{W}^I \) is equivalent to \( p \land \sim p \) in \( \mathcal{W} \). As a consequence, the following theorem is valid:

**Theorem 6.** When \( \Gamma \) is consistent: \( \text{Cn}_{\text{AAL}^{ns}}(\Gamma) = \text{Cn}_{\text{CL}}(\Gamma) \).

**Proof.** See Appendix A.2.

4.2. The inconsistent case

That \textsc{AAL}^{ns} leads to all and only the relevant classical consequences of a premise set, has become clear intuitively by discussing some examples in Section 3.3.2. However, the following theorem proves that our intuitions have not fooled us:

**Theorem 7.** \( \Gamma \vdash_{\text{AAL}^{ns}} A \text{ iff there is a } \Gamma' \subseteq \Gamma \text{ for which } \Gamma' : A \text{ is an entailment.} \)

**Proof.** See Appendix A.3.

**Theorem 7** shows us that \textsc{AAL}^{ns} is equivalent to \textsc{dCR}, the deductive version of Tennant’s \textsc{CR}. From this it follows that \textsc{AAL}^{ns} allows for all and only those classical consequences of a premise set that do not only depend on the inconsistency of the premises in order to be derivable. In other words, the only classical “derivation rule” that is invalidated in \textsc{AAL}^{ns} is \textsc{EFQ}.

5. Conclusion and further research

In this paper, I have shown that it is possible to isolate \textsc{EFQ} from \textsc{CL} by means of the ambiguity-adaptive logic \textsc{AAL}^{ns}. This is done by proving its equivalence to \textsc{dCR}, the deductive version of Tennant’s \textsc{CR}.

As a consequence, \textsc{AAL}^{ns} does not limit the “relevant use” that can be made of the classical derivation rules. It does not, as all other paraconsistent logics do, limit the deductive strength of \textsc{CL} (see Section 1). Shortly put, \textsc{AAL}^{ns} is the paraconsistent version of \textsc{CL}, and as such it should be considered as a better candidate than most paraconsistent logics for replacing \textsc{CL} as the logic underlying human reasoning.

Two lines of further research look interesting to me. First of all, I want to find out whether the “trick with the indices” can be applied to other logics that validate \textsc{EFQ} as well, f.e. intuitionist logic. Secondly, as some people might object to the use of indices (because of the fact that it is basically a “trick”), it is probably a good idea to search for a logic which does not need them.\footnote{Unpublished papers in the reference section (and many others) are available from the internet address \url{http://logica.ugent.be/centrum/writings/}.}
Acknowledgement

Research for this paper was supported by subventions from Ghent University and from the Fund for Scientific Research—Flanders, and indirectly by the Flemish Minister responsible for Science and Technology (contract BIL.01/80). I am indebted to Bert Leuridan, Dagmar Provijn, Peter Verdée and Diderik Batens for helpful comments on a former draft.

Appendix A. Metatheoretical proofs

A.1. Proof of Theorem 5 from Section 4

Proof. We suppose $A^I \in \mathcal{I}(A)$. Now, the proof has two directions:

1. Suppose: $\Gamma \vdash_{\text{CL}} A$.
   Consider a $\text{CL}$-proof $\Phi$ of $A$ from $\Gamma'$ ($\Gamma' \subseteq \Gamma$). From $\Gamma \vdash_{\text{CL}} A$ follows that $\Gamma' \cup \{ A^j \equiv A^i \mid A \in S, \text{ and } i, j \in \mathbb{N} \} \vdash_{\text{CL}} A^I$, for at least one $A^I \in \mathcal{I}(A)$ (because the set added to $\Gamma'$ neutralizes the effect of the indices). Now, by the deduction theorem and the metatheoretical characterization of $\text{CL}$: $\Gamma^I \vdash_{\text{CL}} A^I \lor \text{Dab}(\Delta)$.

2. Suppose: $\Gamma^I \vdash_{\text{CL}} A^I \lor \text{Dab}(\Delta)$.
   Consider a $\text{CL}$-proof $\Phi^I$ of $A^I \lor \text{Dab}(\Delta)$ from $\Gamma'^I$ ($\Gamma'^I \subseteq \Gamma^I$). If you replace all formulas in $\Phi^I$ with their index-less counterparts, the result will be a proof $\Phi$ for $\Gamma \vdash_{\text{CL}} A \lor \text{Dab}(\Delta)$, and because $\text{Dab}(\Delta)$ is inconsistent ($\neg (A \equiv A) \vdash_{\text{CL}} A \land \neg A$): $\Gamma \vdash_{\text{CL}} A$. □

A.2. Proof of Theorem 6 from Section 4

Proof. Obvious from Lemma 1 below. □

Lemma 1. When $\Gamma$ is consistent: $\Gamma \vdash_{\text{CL}} A$ iff, for at least one $A^I \in \mathcal{I}(A)$, $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} A^I$.

Proof. We suppose $\Gamma$ to be consistent. Now, the proof has two directions:

1. Suppose: $\Gamma \vdash_{\text{CL}} A$.
   From $\Gamma \vdash_{\text{CL}} A$, together with Theorem 5, follows that there is at least one $A^I \in \mathcal{I}(A)$, for which $\Gamma^I \vdash_{\text{CL}} A^I \lor \text{Dab}(\Delta)$. As $\Gamma$ is consistent, and as only inconsistencies in $\Gamma$ will make $\text{Dab}$-formulas derivable from $\Gamma^I$ ($p^I \lor \neg p^I \vdash_{\text{CL}} \neg (p^I \equiv p^I)$), $\text{Dab}(\Delta)$ is not derivable, which means that there is at least one $A^I \in \mathcal{I}(A)$, for which $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} A^I$.

2. Suppose: $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} A^I$.
   From $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} A^I$ follows that $\Gamma^I \vdash_{\text{CL}} A^I \lor \text{Dab}(\Delta)$ (see Lemma 2 in [4]), which leads to $\Gamma \vdash_{\text{CL}} A$ (= obvious from the second part of the proof of Theorem 5). □

A.3. Proof of Theorem 7 from Section 4

Proof. Obvious from Lemma 2 below. □

Lemma 2. $\Gamma^I \vdash_{\text{CL}^{\text{ns}}} A^I$ for at least one $A^I \in \mathcal{I}(A)$, iff there is a $\Gamma' \subseteq \Gamma$ for which $\Gamma' : A$ is an entailment.

Proof. The proof has two directions:

1. Suppose there is a $\Gamma' \subseteq \Gamma$ for which $\Gamma' : A$ is an entailment.
   (a) $\Gamma' : A$ is an entailment, which means that there is a perfectly valid suprasequent $s(\Gamma' : A)$ of $\Gamma' : A$. We know that $s(\Gamma'')$ will be consistent, otherwise $s(\Gamma' : A)$ will not be perfectly valid (it would have a valid subsequent: $s(\Gamma' : \emptyset)$).
   (b) It is possible for $s$ to map $\Gamma'$ to $\Gamma''^I$. In this case, all sentential letters are mapped to a different formula, so that it is impossible for $s(\Gamma'')$ to be inconsistent. However, $s(A) (= A^I)$ will not necessarily be derivable.
From (a) and (b) follows that s will have to map some sentential letters on the same formulas, in order to make s(A) derivable from s(Γ'), so that it must be the case that Γ' ⊢ s(A) | A ∈ S and s has mapped those occurrences of A from Γ' which are represented in Γ' by A i and A j, onto the same formula in s(Γ') (as (I)). As s(Γ') is consistent (see (a)), it will be the case that Γ' ⊬ s(Γ') for at least one A i ∈ I(A). By the deduction theorem and the metatheoretical characterization of CL follows that Γ' ⊬ CL A i ∨ Dab(Δ) for Δ = {~A i | A ∈ S} and s has mapped those occurrences of A from Γ' which are represented in Γ' by A i and A j, onto the same formula in s(Γ') (as (I)). As s(Γ') is consistent (see (a)), it will be the case that Γ' ⊬ CL A i for at least one A i ∈ I(A). Now, as CL ns is monotonic (see Section 3.3), it is also the case that Γ' ⊢ CL ns A i for at least one A i ∈ I(A).

(2) Suppose Γ' ⊢ CL ns A i.

From Γ' ⊢ CL A i ∨ Dab(Δ), and for which Γ' ⊢ CL Dab(Δ). Suppose this to be the case for A i and Δ, as all sentential letters occur only once in Γ', inconsistencies are not derivable from Γ', so that there is a Γ' ⊢ CL Dab(Δ) for which Γ' : A i ∈ I(Δ). Now, consider a function s, so that if ~A i ∈ Δ, then s(A i) = A i. For all other indexed letters: s(A i) = A i. This will give us s(Γ') ⊢ A i with a consistent s(Γ') (as Γ' ⊬ CL Dab(Δ)), from which follows that s(Γ') : A i is a perfectly valid suprasequent of Γ' : A with Γ' ⊆ Γ.

References