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# A partial regularity result for an anisotropic smoothing functional for image restoration in BV space

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**Abstract**

Here we examine the partial regularity of minimizers of a functional used for image restoration in BV space. This functional is a combination of a regularized $p$-Laplacian for the part of the image with small gradient and a total variation functional for the part with large gradient. This model was originally introduced in Chambolle and Lions using the Laplacian. Due to the singular nature of the $p$-Laplacian we study a regularized $p$-Laplacian. We show that where the gradient is small, the regularized $p$-Laplacian smooths the image $u$, in the sense that $u \in C^{1,\alpha}$ for some $0 < \alpha < 1$. This functional thus anisotropically smooths the image where the gradient is small and preserves edges via total variation where the gradient is large.


*Keywords:* Bounded variation; Image processing; Partial regularity; Partial differential equation

## 1. Introduction

Over the past decade, PDE and variational method based diffusion models have grown significantly to tackle problems of image restoration and reconstruction. The challenge of these problems is to construct a model that can effectively remove unwanted noise while at the same time retaining significant features of the image. Thus we want to recover an image $u$ from given noisy image $I$, where $I = u + \text{noise}$.

Total variation (TV) based regularization was first proposed by Rudin, Osher, and Fatemi in [21] in order to remove noise while retain important features, such as edges. This has been studied extensively in [1,2,5,6,20,24,25] and [11]. The definition of the total variation of a function $u \in L^1(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, is

$$TV(u) = \sup\left\{ \int_\Omega u \operatorname{div}(\varphi)\, dx \colon \varphi \in C_0^1\left(\Omega, \mathbb{R}^n\right),\ |\varphi| \leqslant 1 \right\},$$

and is denoted by $\int |Du|$. Differentiability, or even continuity, of $u$ is not required for $TV(u)$ to be finite. Thus images with discontinuities (i.e. edges) are allowed as solutions in the space of functions of bounded variation ($BV$) on $\Omega$,

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which is the space of functions $u \in L^1(\Omega)$ with $TV(u) < \infty$. In addition, the diffusion resulting from minimizing the TV norm is strictly orthogonal to the gradient of the image, and tangental to the edges. This is important for preserving edges while smoothing the image. However, TV-based regularization sometimes causes a staircasing effect [7,11,12]. Consequently, the restored image can be blocky and even contain artifacts, such a false edges.

A number of alternative total variation techniques have been proposed, such as adaptive total variation (see [23]) where minimizing $\int |Du|$ is replaced by

$$\min \int \alpha(x)|Du|.$$

The control function $\alpha$ is used to reduce the diffusion where there is likely an edge.

Another model was proposed by Chambolle and Lions [11], which uses a combination of TV diffusion where there are likely edges (where $|\nabla u| > \epsilon$) and isotropic diffusion in more homogeneous regions (where $|\nabla u| \leqslant \epsilon$). This minimization problem is

$$\min_{u \in BV(\Omega)} \frac{1}{2\epsilon} \int_{|\nabla u| \leqslant \epsilon} |\nabla u|^2 + \int_{|\nabla u| > \epsilon} |\nabla u| - \frac{\epsilon}{2} + \frac{\lambda}{2} \int_{\Omega} (u - I),$$

where $I$ is the noise image and $\lambda > 0$ is a parameter.

A partial regularity result was obtained in [13] for this model which shows that if $\mathcal{L}^n(\{|\nabla u| < \epsilon\}) > 0$, then there exists a nonempty open set $\widetilde{\Omega} \subset \Omega$ on which $u$ is $C^{1,\alpha}$ and $|\nabla u| < \epsilon$. Thus we do have smoothing where $|\nabla u| < \epsilon$. In this paper we consider the problem where we replace the isotropic part in the above problem with an anisotropic term $\int_{|\nabla u| \leqslant \epsilon} |\nabla u|^p$ for $1 < p < 2$, thus resulting in a term that gives rise to anisotropic diffusion instead of isotropic in the region where $|\nabla u| \leqslant \epsilon$. The objective here is to discuss the partial regularity of this problem as for the isotropic case as in [13] for the case $p = 2$. We refer the reader to [19] for other theoretical and numerical results regarding more general functionals for use in image restoration, including the use of the $p$-Laplacian.

Improving on this, different models have been proposed using a variable exponent $p(x)$ such as Blomgren, Chan, Mullet, and Wong [7] where they propose the minimization problem

$$\min \int_{\Omega} |\nabla u|^{p(|\nabla u|)} \, dx,$$

where $\lim_{s \to 0} p(s) = 2$ and $\lim_{s \to \infty} p(s) = 1$ with $p$ monotonically decreasing; and in Chen, Levine, Rao [9] they propose a model similar the one above by Chambolle and Lions with exponent $p(x)$. Numerical results in [9] show promise for this method.

## 2. The minimization problem

For simplicity we consider the problem with $\epsilon = 1$ and $\lambda = 1$ in the above

$$\min_{u \in BV(\Omega) \cap L^2(\Omega)} \left\{ \int_{\Omega} \varphi_0(Du) + \frac{1}{2} \int_{\Omega} (u - I)^2 \, dx \right\},$$

where $\varphi_0$ is the following $C^1$ convex function defined on $\mathbb{R}^n$:

$$\varphi_0(x) = \begin{cases} \frac{1}{p}|x|^p & \text{if } |x| \leqslant 1, \\ |x| - \frac{1}{q} & \text{if } |x| > 1, \end{cases}$$

for $1 < p < 2$, $1/p + 1/q = 1$, $I \in L^\infty(\Omega) \cap BV(\Omega)$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. As in [4] or [16] we may define the above functional on $BV(\Omega)$ using

$$\int_{\Omega} \varphi_0(Du) \triangleq \int_{\Omega} \varphi_0(\nabla u) \, dx + \int_{\Omega} \left|D^s u\right|.$$

However, due to the singular nature of $\varphi_0$ we instead consider a regularized version of the above problem for proving partial regularity, namely

$$\min_{u\in BV(\Omega)\cap L^2(\Omega)}\left\{\int_\Omega \varphi_\epsilon(Du)+\frac{1}{2}\int_\Omega (u-I)^2\,dx\right\} \tag{2.1}$$

for the $C^1(\mathbb{R}^n)$ function $\varphi_\epsilon$, $\epsilon>0$, defined by

$$\varphi_\epsilon(x)=\begin{cases}\frac{1}{p}(|x|^2+\epsilon)^{p/2} & \text{if } |x|\leqslant 1,\\ (1+\epsilon)^{(p/2)-1}|x|-\frac{1}{p}(1+\epsilon)^{(p/2)-1}(p-1-\epsilon) & \text{if } |x|>1,\end{cases}$$

where $\Omega$, $I$, and $p$ are as before. As above,

$$\int_\Omega \varphi_\epsilon(Du)\triangleq\int_\Omega \varphi_\epsilon(\nabla u)\,dx+\int_\Omega \left|D^s u\right|.$$

We note that for $\epsilon>0$, $\varphi_\epsilon\in C^1(\mathbb{R}^n)$ and $C^2$ is on the interior of the unit ball $B_1(0)$. First we show that $\int\varphi_\epsilon(Du)$ is lower semicontinuous in $L^1(\Omega)$ for any $\epsilon\geqslant 0$. The following lemma is actually a special case of Lemma 2.3 in [10], however this proof can be generalized to any continuous function $\varphi$ of linear growth.

**Lemma 1.** *For any $\epsilon\geqslant 0$, the functional $\int\varphi_\epsilon(Du)$ is lower semicontinuous in $L^1(\Omega)$.*

**Proof.** Let $\mathcal{V}=\{\phi\in C_0^1(\Omega,\mathbb{R}^n)\colon\ |\phi(x)|\leqslant 1 \text{ for all } x\in\Omega\}$, where $|\cdot|$ is the usual vector norm in $\mathbb{R}^n$. Without loss of generality we can adjust $\varphi_\epsilon$ if necessary so that $\varphi_\epsilon(0)=0$ and $\varphi_\epsilon(x)=|x|-K$ where $|x|\geqslant 1$ for appropriate constant $K$. From [18] we have for each $x\in\mathbb{R}^n$, $\varphi_\epsilon(x)=\sup_{\{|y|\leqslant 1\}}\{x\cdot y-\varphi_\epsilon^*(y)\}$ where $\varphi_\epsilon^*$ is the convex conjugate of $\varphi_\epsilon$ defined by $\varphi_\epsilon^*(y)=\sup_{x\in\mathbb{R}^n}\{x\cdot y-\varphi_\epsilon(x)\}$. For the special case of $\epsilon=0$ we in fact have $\varphi_0(x)=\sup_{\{|y|\leqslant 1\}}\{x\cdot y-|y|^p/q\}$. The linear growth property of $\varphi_\epsilon$ actually gives finite values for $\varphi_\epsilon^*(y)$ only when $|y|\leqslant 1$. In addition we see that $\varphi_\epsilon^*(y)=\sup_{\{x\in\mathbb{R}^n,\ |x|\leqslant 1\}}\{x\cdot y-\varphi_\epsilon(x)\}$. We also note that for $\epsilon>0$, $\varphi_\epsilon^*(y)=x^*(y)\cdot y-\varphi_\epsilon(x^*(y))$ where $x^*(y)$ is a continuous function of $y$. From [8] we have for any $g\in L^1(\Omega)^n$,

$$\int_\Omega \varphi_\epsilon(g)\,dx=\sup_{\phi\in\mathcal{V}}\left\{\int_\Omega g\cdot\phi-\varphi_\epsilon^*(\phi)\,dx\right\}. \tag{2.2}$$

Now define the following functional on $BV(\Omega)$:

$$\begin{aligned}J(u)&=\sup_{\phi\in\mathcal{V}}\left\{-\int_\Omega u\operatorname{div}\phi+\varphi_\epsilon^*(\phi)\,dx\right\}\\&=\sup_{\phi\in\mathcal{V}}\left\{\int_\Omega \nabla u\cdot\phi-\varphi_\epsilon^*(\phi)\,dx+\int_\Omega \phi\cdot D^s u\right\}\end{aligned}$$

where the last equality follows from integration by parts. From the above discussion we easily see that for every $\phi\in\mathcal{V}$,

$$\int_\Omega \nabla u\cdot\phi-\varphi_\epsilon^*(\phi)\,dx+\int_\Omega \phi\cdot D^s u\leqslant\int_\Omega \varphi_\epsilon(Du),$$

giving $J(u)\leqslant\int_\Omega\varphi_\epsilon(Du)$.

For the reverse inequality we follow, for example, [10], noting the continuity of $\varphi_\epsilon^*$. Fix $\epsilon>0$. For any $u\in BV(\Omega)$ there exists an open set $\mathcal{O}_\epsilon$ such that support $(D^s u)\subset\mathcal{O}_\epsilon$ and $|\mathcal{O}_\epsilon|\leqslant\epsilon$. We can also find $\phi_\epsilon\in C_1^0(\Omega,\mathbb{R}^n)$ with $|\phi_1|\leqslant 1$ and

$$\int_\Omega D^s u\cdot\phi_1\geqslant\int_\Omega \left|D^s u\right|-\epsilon$$

from the definition of the TV norm. By (2.2) there exists $\phi_2\in C_1^0(\Omega,\mathbb{R}^n)$ with $|\phi_2|\leqslant 1$ such that

$$\int_\Omega \nabla u\cdot\phi_2-\varphi_\epsilon^*(\phi_2)\,dx\geqslant\int_\Omega \varphi_\epsilon(\nabla u)\,dx-\epsilon.$$

Now define

$$\phi = \begin{cases} \phi_1 & \text{on } \mathcal{O}_\epsilon, \\ \phi_2 & \text{on } \Omega \setminus \mathcal{O}_\epsilon. \end{cases}$$

Let $\eta_\alpha$ be the standard mollifier on $\mathbb{R}^n$ and let $\phi_\alpha = \eta_\alpha * \phi$. Note that $\phi_\alpha$ has compact support for sufficiently small $\alpha$. Then

$$J(u) \geqslant \int_\Omega \nabla u \cdot \phi_\alpha - \varphi_\epsilon^*(\phi_\alpha)\,dx + \int_\Omega \phi_\alpha \cdot D^s u.$$

Letting $\alpha \to 0$ in the above inequality we then have

$$\begin{aligned} J(u) &\geqslant \int_\Omega \nabla u \cdot \phi - \varphi_\epsilon^*(\phi)\,dx + \int_\Omega \phi \cdot D^s u \\ &\geqslant \int_\Omega \nabla u \cdot \phi_2 - \varphi_\epsilon^*(\phi_2)\,dx + \int_\Omega \phi_1 \cdot D^s u - \mu(\epsilon) \\ &\geqslant \int_\Omega \varphi_\epsilon(\nabla u)\,dx + \int_\Omega \left|D^s u\right| - \mu(\epsilon) - 2\epsilon \end{aligned}$$

where

$$\mu(\epsilon) = \int_{\mathcal{O}_\epsilon} |\nabla u|\,dx + \left\|\varphi_\epsilon^*\right\|_{L^\infty(B_1(0))} \mathcal{L}^n(\mathcal{O}_\epsilon)$$

and $\mathcal{L}^n$ denotes Lebesgue on $\mathbb{R}^n$. Clearly $\mu(\epsilon) \to 0$ as $\epsilon \to 0$. The reverse inequality is now proved and so

$$J(u) = \int_\Omega \varphi_\epsilon(Du).$$

Lower semicontinuity now easily follows as in Giusti [15]. □

We also have the following approximation lemma:

**Lemma 2.** *Let $u \in BV(\Omega) \cap L^2(\Omega)$. Then for any $\epsilon \geqslant 0$ there is a sequence of functions $\{u_n\} \subset BV(\Omega) \cap L^2(\Omega) \cap C^\infty(\Omega)$ such that*

$$u_n \to u \quad \text{in } L^2(\Omega) \quad \text{and}$$
$$\int_\Omega \varphi_\epsilon(Du_n)\,dx \to \int_\Omega \varphi_\epsilon(Du).$$

**Proof.** Fix $\epsilon \geqslant 0$ and for simplicity write $\varphi_\epsilon$ as $\varphi$. Consider the function $\varphi_n(x) \triangleq \varphi(x) + |x|/n$. Then from [4] there exists $u_n \in BV(\Omega) \cap C^\infty(\Omega)$ such that

(1) $\|u_n - u\|_{L^2(\Omega)} \leqslant 1/n$.
(2) $|\int_\Omega \sqrt{1 + (Du_n)^2}\,dx - \int_\Omega \sqrt{1 + (Du)^2}| \leqslant 1/n$.
(3) $|\int_\Omega \varphi_n(Du_n)\,dx - \int_\Omega \varphi_n(Du)| \leqslant 1/n$. □

**Proof.** The proposition there is actually stated for functions in BV, but an adjustment of the proof on which this proposition is based [3] gives us estimate 1 for $u \in BV(\Omega) \cap L^2(\Omega)$. We then have

$$\left|\int_\Omega \varphi(Du_n)\,dx - \int_\Omega \varphi(Du)\right| \leqslant \left|\int_\Omega \varphi_n(Du_n)\,dx - \int_\Omega \varphi_n(Du)\right| + 1/n\left(\int_\Omega |Du_n|\,dx + \int_\Omega |Du|\right)$$
$$\leqslant 1/n + 1/n\left(\int_\Omega |Du_n|\,dx + \int_\Omega |Du|\right).$$

From (2), $\int_\Omega |Du_n|\,dx$ is bounded. Letting $n \to 0$ proves the theorem. □

These lemmas now imply

**Theorem 1.** *For any $\epsilon \geqslant 0$, there exists a solution $u \in L^\infty(\Omega)$ to problem* (2.1). *In fact, we have $\|u\|_{L^\infty(\Omega)} \leqslant \|I\|_{L^\infty(\Omega)}$.*

**Proof.** The proof of existence and uniqueness is standard using lower semicontinuity and strict convexity. The $L^\infty$ bound follows as in Lemma 2.1 in [13] using both of the above lemmas and by considering the approximating functional

$$\min_{u\in W^{1,1+\delta}(\Omega)\cap L^2(\Omega)}\left\{\int_\Omega \varphi_\epsilon^\delta(Du) + \frac{1}{2}\int_\Omega (u-I)^2\,dx\right\}$$

where

$$\varphi_\epsilon^\delta(x) = \begin{cases} \frac{1}{p}(|x|^2+\epsilon)^{p/2} & \text{if } |x| \leqslant 1, \\ \frac{(1+\epsilon)^{(p/2)-1}}{1+\delta}|x|^{1+\delta} + \frac{1}{p}(1+\epsilon)^{p/2} - \frac{(1+\epsilon)^{(p/2)-1}}{1+\delta} & \text{if } |x| > 1 \end{cases}$$

is in $C^1(\mathbb{R}^n)$ and $\varphi_\epsilon(x) \leqslant \varphi_\epsilon^\delta(x)$ for all $x \in \mathbb{R}^n$. □

## 3. Main theorem

Now we state the main theorem of this paper.

**Theorem 2.** *If $u$ solves* (2.1) *for $\epsilon > 0$ and $\mathcal{L}^n(\{|\nabla u| < 1\}) > 0$, then there exists a nonempty open region $\widetilde{\Omega}$ on which $u$ is $C^{1,\alpha}$, $|\nabla u| < 1$, and $u$ solves*

$$-\mathrm{div}\big(\varphi_P(\nabla u)\big) = I - u \quad \text{on } \widetilde{\Omega},$$

*in addition, we have $|\nabla u| \geqslant 1$ a.e. on $\Omega \setminus \widetilde{\Omega}$.*

Without loss of generality we consider the case where $\epsilon = 1$ and we let $\varphi = \varphi_1$. To prove the above theorems, we follow the general procedure to that in [17] for proving partial regularity for weak solutions $u \in L^2([0,\infty], BV(\Omega))$ to the time evolution problem

$$\frac{\partial u}{\partial t} = \mathrm{div}_x\big(\varphi_P(\nabla u)\big)$$

on $\Omega \subset \mathbb{R}^1$ or $\mathbb{R}^2$ where $\varphi$ is a convex linear growth function satisfying local ellipticity and continuity assumptions. The essential part of this theorem will be a decay result. There is, however, no restriction on $n$ for this result.

## 4. Proof of the main theorem

First we prove some lemmas. We have from [17]

**Lemma 3.** *Let $u \in BV(E)$ for open region $E \Subset \Omega$ with smooth boundary. Then there exist constants $c_1, c_2 < 1/2$ such that if $\overline{p} \in B_1(0)$ and $h \in C^1(E)$ with $\sup_E |\nabla h - \overline{p}| \leqslant c_1\sigma$, then for any vector $p_1 \in B_{c_1\sigma}(\overline{p})$ we have*

$$\int_E \varphi(Du) - \int_E \varphi(\nabla h)\,dx - \int_E \varphi_{ll}(p_1)\cdot\nabla h\cdot D(u-h)$$
$$+\int_E \big(\varphi_{ll}(p_1)\cdot p_1 - \varphi_l(p_1)\big)\cdot D(u-h) + \sup_E \omega\big(|\nabla h - p_1|^2\big)\int_E |\nabla h - p_1|^2\,dx$$
$$\geqslant c_2\bigg(\int_{E\cap\{Du\notin B_\sigma(\overline{p})\}} |Du-\overline{p}| + \int_{E\cap\{Du\in B_\sigma(\overline{p})\}} \big|D(u-h)\big|^2\bigg),$$

*where* $\omega:\mathbb{R}\to\mathbb{R}$ *is a nondecreasing, nonnegative function with* $\lim_{t\to 0}\omega(t)=0$.

Recall that

$$\int_{E\cap\{Du\notin B_\sigma(\overline{p})\}} |Du-\overline{p}| + \int_{E\cap\{Du\in B_\sigma(\overline{p})\}} \big|D(u-h)\big|^2$$

means

$$\int_{E\cap\{\nabla u\notin B_\sigma(\overline{p})\}} |\nabla u-\overline{p}| + \int_E \big|D^s u\big| + \int_{E\cap\{\nabla u\in B_\sigma(\overline{p})\}} \big|\nabla(u-h)\big|^2.$$

Throughout the rest of the paper, $u$ will be the solution to (2.1).

**Lemma 4.** *Let* $E\Subset\Omega$ *be an open elliptical region. Now suppose* $h\in C^1(\overline{E})$ *satisfies* $\sup_E|\nabla h-\overline{p}|\leqslant c_1\sigma$ *and*

$$-\varphi_{l_i l_j}(p_1)\frac{\partial^2 h}{\partial x_i\,\partial x_j} = I - h \quad \text{on } E \tag{4.1}$$

*for some* $p_1\in B_{c_1\sigma}(\overline{p})$ *and smooth* $v_\beta$. *Then we have*

$$\int_{E\cap\{Du\notin B_\sigma(\overline{p})\}} |Du-\overline{p}| + \int_{E\cap\{Du\in B_\sigma(\overline{p})\}} \big|D(u-h)\big|^2$$
$$\leqslant c_3\bigg(\int_{\partial E} |u-v_\beta|\,d\mathcal{H}^{n-1} + \sup_E\omega\big(|\nabla h-p_1|^2\big)\int_E|\nabla h-p_1|^2\,dx\bigg).$$

**Proof.** Such a solution $h$ exists by the linear theory. From Lemma 2.5 in [13] we have

$$\int_E\varphi(Du) - \int_E\varphi(\nabla h) \leqslant 1/2\int_E (h-I)^2\,dx - 1/2\int_E (u-I)^2\,dx + \int_{\partial E}|h-u|\,d\mathcal{H}^{n-1}$$

where $h,u$ are understood in the sense of trace in $BV$ on $\partial E$ for the last integral. Using this, integrating by parts, and using Young's inequality for $(u-h)(I-h) = -(u-I)(h-I) + (I-h)^2$ the lemma is proved. □

Now we define the "energy" by

$$\Phi(r,p,a) \triangleq \frac{1}{|B_r|}\bigg(\int_{B_r(a)\cap\{Du\notin B_\sigma(\overline{p})\}} |Du-\overline{p}| + \int_{B_r(a)\cap\{Du\in B_\sigma(\overline{p})\}} |Du-p_1|^2\bigg).$$

Our next goal is to prove

**Theorem 3.** *There exist constants* $\epsilon_1,\epsilon_2$ *depending on* $\varphi$ *and* $\overline{p}$ *such that if*

$$\Phi(r_0,p_1,x_0)\leqslant\epsilon_1$$

*for some* $r_0, x_0\in\Omega$ *and some* $p_1\in B_{\epsilon_2}(\overline{p})$, *then there exists* $p\in B_\sigma(\overline{p})$ *such that* $\lim_{r\to\infty}\Phi(r,p_1,x_0)=0$. *Furthermore,* $p = Du(x_0)$.

From now on, we drop the "$a$" in $B_r(a)$ and $\Phi$ unless noted otherwise. In order to prove Theorem 3, we will obtain estimates for $\int_{\partial E} |u - v_\beta| \, d\mathcal{H}^{n-1}$, $|\nabla h - p_1|$, and $\int_E |\nabla h - p_1|^2 \, dx$ for a suitable Lipschitz function $v$. These will then be used to prove a decay estimate for $\Phi$ on a smaller ball and different $p$, whose difference can be estimated. This decay estimate will be used to prove the theorem.

Now fix $a$. From the proof of Theorem 2.8 in [13] we have

**Lemma 5.** *If $B_{r/2} \Subset \Omega$ and for suitable Lipschitz function $v$, we have*

$$\sup_{B_{r/2}} |\nabla v - p_1| \leqslant \Phi(r, p_1)^{2\delta}$$

*with $\delta$ such that $(1 - 4\delta)\frac{n+1}{n} = 1 + \frac{1}{2n}$, that is, $\delta = \frac{1}{8(n+1)}$. Additionally,*

$$\mathcal{L}^n\big(B_{r/2} \cap \{u \neq v\}\big) \leqslant c_4 r^n \Phi(r, p_1)^{1-4\delta}.$$

Using this lemma obtain an $L^\infty$ estimate for $u - v$.

**Lemma 6.** *If $|p_1 - \overline{p}| \leqslant c_1\sigma$, then there exists $\epsilon > 0$ such that $\Phi(r, p_1) < \epsilon$ implies $\sup_{B_{r/2}} |\nabla v - p_1| \leqslant \Phi(r, p_1)^{2\delta}$, the $v$ from the previous lemma with $\delta = \frac{1}{8(n+1)}$ and with the following estimate for $r$ sufficiently small:*

$$\|u - v\|_{L^\infty(B_{r/4})} \leqslant c_5\big(\mathcal{L}^n\big(B_{r/2} \cap \{u \neq v\}\big)\big)^{1/n}.$$

**Proof.** This follows from the previous lemma and Lemma 2.7 in [13] where we use the bound $\sup_{B_{r/2}} |\nabla v| \leqslant |\overline{p}| + c_1\sigma + \Phi(r, p_1)^{2\delta}$ in that proof instead of $\sup_{B_{r/2}} |\nabla v| \leqslant 1$ used there. □

We can now estimate $|\nabla h - p_1|$ on $E$ for any $\tilde{r} \leqslant r/2$ for the solution $h$ to (4.1). Let $v_\beta$ be a smoothing of $v$ defined by $v_\beta = \eta_{r\beta} * v$ where $\eta_\epsilon$ is the standard mollifier on $\mathbb{R}^n$. Then from the linear theory [13,14], taking $\beta = \Phi(r, p_1)$, we have $\|h\|_{L^\infty(E)} \leqslant c_6$ and

$$\sup_E |\nabla h - p_1| \leqslant c_7\big(\Phi(r, p_1)^\delta + r\|h - I\|_{L^\infty(E)}\big) \leqslant c_8\big(\Phi(r, p_1)^\delta + r\big).$$

Let $T$ be an appropriate transformation such that if $\tilde{h} = h \circ T^{-1}$, $\tilde{u} = u \circ T^{-1}$, $\widetilde{v_\beta} = v_\beta \circ T^{-1}$, $\widetilde{I} = I \circ T^{-1}$, $E$ is an ellipsoid centered at $a$, $T(E) = B'$ is a ball of radius $\tilde{r}$ centered at $T(a)$, and

$$-\Delta\tilde{h} = \widetilde{I} - \tilde{h} \quad \text{on } B',$$
$$\tilde{h} = \widetilde{v_\beta} \quad \text{on } \partial B'.$$

We note that the Jacobian of $T$, which depends only on the eigenvalues of the matrix $[\varphi_{l_i l_j}(p)]$ for $p = p_1$ is bounded from above and away from 0. Furthermore these bounds can be made independent of $a$ and $p$ due the uniform ellipticity. We also have from [13],

$$\sup_{\widetilde{B}} \frac{|\nabla\tilde{h}(x) - \nabla\tilde{h}(y)|}{|x - y|^{1/2}} \leqslant \frac{c_9}{\tilde{r}^{n+1/2}} \int_{\partial B'} |\widetilde{v_\beta}| \, d\mathcal{H}^{n-1} + \tilde{r}\|\widetilde{I} - \tilde{h}\|_{L^\infty}$$

where $\widetilde{B} \subset B'$ is concentric with $B'$ and with radius $\tilde{r}/2$. Now choose $E \subset B_{r/4}$ centered at $a$ with $\operatorname{diam}(E) = r/8$ and such that both

$$\int_{\partial E} |u - v| \, d\mathcal{H}^{n-1} \leqslant \frac{c_{10}}{r} \int_{B_{r/4}} |u - v| \, dx \tag{4.2}$$

and

$$\int_{\partial E} \big|u - \bar{u}_{B_{r/4}} - p_1 \cdot (x - a)\big| \, d\mathcal{H}^{n-1} \leqslant \frac{c_{10}}{r} \int_{B_{r/4}} \big|u - \bar{u}_{B_{r/4}} - p_1 \cdot (x - a)\big| \, dx \tag{4.3}$$

hold, where $\bar{u}_{B_{r/4}}$ denotes the average of $u$ over $B_{r/4}$.

**Lemma 7.** *Let $v$, $v_\beta$ be as above. Then*

$$\int_{\partial E} |u - v_\beta| \, d\mathcal{H}^{n-1} \leqslant c_{11} r^n \big(\Phi(r, p_1)^{1+2\delta} + \Phi(r, p_1)^{1+1/(2n)}\big).$$

**Proof.** By the properties of $v_\beta$ we have $\sup_E |v_\beta - v| \leqslant r\Phi(r, p_1)^{1+2\delta}$, and combining the results of the last two lemmas with estimate (4.2) we obtain

$$\int_{\partial E} |u - v| \, d\mathcal{H}^{n-1} \leqslant c_{11} r^n \Phi(r, p_1)^{1+1/(2n)}.$$

From these two estimates the lemma follows. □

**Lemma 8.** *If $h$ solves* (4.1) *and $|p_1 - \overline{p}| < c_1\sigma$, then*

$$\int_E |\nabla h - p_1|^2 \, dx \leqslant c_{20}\big(r^{n+1} \|h - I\|_{L^\infty(E)} + r^n \Phi(r, p_1)\big).$$

**Proof.** Multiply (4.1) by $h - v_\beta$, integrate by parts, use the fact that (4.1) is a linear equation with constant coefficients, and then use Young's inequality to arrive at

$$\int_E |\nabla h - p_1|^2 \, dx \leqslant c_{12} \int_E (h - I)(h - v_\beta) \, dx + c_{13} \int_E |\nabla v_\beta - p_1|^2 \, dx.$$

By the uniform bounds of $\nabla h$ and $\nabla v_\beta$ we see that $\|h - v_\beta\|_{L^\infty(E)} \leqslant c_{14}(\mathrm{diam}(E)) \leqslant c_{15} r$. For the other part of the estimate we use from [22]

$$\int_{B_{r/4}} |\nabla v_\beta - p_1| \, dx \leqslant c_{16} \int_{B_{r/2}} |\nabla v - p_1| \, dx.$$

Finally, to estimate $\int_{B_{r/2}} |\nabla v - p_1| \, dx$ the construction of $v$ gives

$$\int_{B_{r/2}} |\nabla v - p_1| \, dx \leqslant c_{17} \int_{B_{r/2} \cap \{Du \in B_\sigma(\overline{p})\}} |Du - p_1|^2 + c_{18} \Phi(r, p_1) \leqslant c_{19} \Phi(r, p_1).$$

By combining the above two estimates, the lemma is proved. □

We now arrive at our decay estimate.

**Theorem 4.** *There exist positive constants $\epsilon$, $c_{20}$, $\kappa$ depending only on $n$, $\Omega$, $u$ such that if $\Phi(r, p_1) \leqslant \epsilon$ and $r \leqslant c_{20}$, then there exists $p_2 \in \mathbb{R}^n$ such that $\Phi(\kappa r, p_2) \leqslant \frac{1}{2}\Phi(r, p_1) + c_{31} r$ and $|p_2 - p_2| \leqslant c_{35} \Phi(r, p_1)^{1/2} + c_{34} r$.*

**Proof.** Using Lemma 4 and the estimates obtained in Lemmas 7 and 8 we have

$$\begin{aligned} &\int_{E \cap \{Du \notin B_\sigma(\overline{p})\}} |Du - \overline{p}| + \int_{E \cap \{Du \in B_\sigma(\overline{p})\}} \big|D(u - h)\big|^2 \\ &\quad \leqslant c_{21}\big(r^n \Phi(r, p_1)^{1+1/(2n)} + \omega\big(c_8 \Phi(r, p_1)^\delta + r\big)\big)^2 \big(r^{n+1} \|h - I\|_{L^\infty(E)} + r^n \Phi(r, p_1)\big). \end{aligned}$$

Letting $p_2 = \nabla h(a)$, we now estimate $\sup|\nabla h(x) - \nabla h(a)|$ over a ball $B_{\kappa r} \subset \widetilde{E} \subset E$. Since $\tilde{h} - \bar{u}_{B_{r/4}} - p_1 \cdot (x - a)$ also satisfies (4.1) we see that

$$\sup_{\widetilde{B}} \frac{|\nabla \tilde{h}(x) - \nabla \tilde{h}(y)|}{|x - y|^{1/2}} \leqslant \frac{c_{22}}{\tilde{r}^{n+1/2}} \int_{\partial B'} \big|\widetilde{v_\beta} - \bar{u}_{B_{r/4}} - p_1 \cdot (x - a)\big| \, d\mathcal{H}^{n-1} + \tilde{r}^{1/2} \|\widetilde{I} - \tilde{h}\|_{L^\infty(B')}. \tag{4.4}$$

Recall the ball $\widetilde{B}$ is concentric with $B'$ and has radius $\tilde{r}/2$ and center $T(a)$. Changing back to the original variables, using (4.3), and Poincaré's inequality, the right side of (4.4) is estimated as

$$\begin{aligned}
&\leqslant \frac{c_{23}}{r^{n+1/2}}\int_{\partial E}\big|v_\beta-\bar{u}_{B_{r/4}}-p_1\cdot(x-a)\big|\,d\mathcal{H}^{n-1}+c_{23}r^{1/2}\|h-I\|_{L^\infty(E)}\\
&\leqslant \frac{c_{24}}{r^{n+1/2}}\int_{\partial E}|v_\beta-u|\,d\mathcal{H}^{n-1}+\frac{c_{24}}{r^{n+3/2}}\int_{B_{r/4}}\big|u-\bar{u}_{B_{r/4}}-p_1\cdot(x-a)\big|\,d\mathcal{H}^{n-1}+c_{23}r^{1/2}\|h-I\|_{L^\infty(E)}\\
&\leqslant \frac{c_{25}}{r^{n+1/2}}\int_{\partial E}|v_\beta-u|\,d\mathcal{H}^{n-1}+\frac{c_{25}}{r^{n+1/2}}\int_{B_{r/4}}|Du-p_1|+c_{25}r^{1/2}\|h-I\|_{L^\infty(E)}.
\end{aligned}$$

Now let $\widetilde{E}=T^{-1}(\widetilde{B})$ and restrict $\kappa$ as necessary so that $B_{\kappa r}\subset\widetilde{E}\subset E$. Then we can see that after changing variables in the left side of (4.4) and using Lemma 7 we obtain

$$\begin{aligned}
\sup_{B_{\kappa r}}\big|\nabla h(x)-\nabla h(y)\big| &\leqslant c_{26}\bigg(\frac{\kappa^{1/2}}{r^n}\int_{\partial E}|v_\beta-u|\,d\mathcal{H}^{n-1}+\frac{\kappa^{1/2}}{r^n}\int_{B_{r/4}}|Du-p_1|+\kappa^{1/2}r\|h-I\|_{L^\infty(E)}\bigg)\\
&\leqslant c_{27}\kappa^{1/2}\bigg(\Phi(r,p_1)^{1+2\delta}+\Phi(r,p_1)^{1+1/(2n)}+\frac{1}{r^n}\int_{B_{r/4}}|Du-p_1|+r\|h-I\|_{L^\infty(E)}\bigg).
\end{aligned}$$

We thus obtain

$$\sup_{B_{\kappa r}}\big|\nabla h(x)-\nabla h(y)\big|\leqslant c_{28}\kappa^{1/2}\big(\Phi(r,p_1)^{1+2\delta}+\Phi(r,p_1)^{1+1/(2n)}+\Phi(r,p_1)^{1/2}+r\|h-I\|_{L^\infty(E)}\big).$$

This is our desired estimate for $\sup_{B_{\kappa r}}|\nabla h(x)-\nabla h(y)|$.

Now using the inequality $|\nabla u-\nabla h|^2\geqslant 1/2|\nabla u-p_1|^2-|\nabla h-p_2|^2$ we arrive at

$$\begin{aligned}
&\int_{E\cap\{Du\notin B_\sigma(\overline{p})\}}|Du-\overline{p}|+\int_{E\cap\{Du\in B_\sigma(\overline{p})\}}|Du-p_2|^2\\
&\quad\leqslant c_{29}\big(r^n\Phi(r,p_1)^{1+1/(2n)}+\omega\big(c_8\big(\Phi(r,p_1)^\delta+r\big)\big)^2r^{n+1}\|h-I\|_{L^\infty(E)}\big)\\
&\qquad+c_{29}\bigg(\omega\big(c_8\big(\Phi(r,p_1)^\delta+r\big)\big)^2r^n\Phi(r,p_1)+\int_{B_{\kappa r}}|\nabla h-p_2|^2\,dx\bigg).
\end{aligned}\tag{4.5}$$

For the rest of the proof we denote $\Phi(r,p_1)$ by $\Phi$. Using the estimate for $|\nabla h(x)-\nabla h(y)|$, recalling that $p_2=\nabla h(a)$, and dividing the above inequality (4.5) by $\kappa^nr^n$ we have

$$\begin{aligned}
\Phi(\kappa r,p_2)&\leqslant c_{30}\kappa^{-n}\Phi^{1+1/(2n)}+c_{30}\kappa^{-n}\omega\big(c_8\big(\Phi^\delta+r\big)\big)^2+c_{30}\kappa\Phi\\
&\quad+c_{30}\big(\omega\big(c_8\big(\Phi^\delta+r\big)\big)^2\kappa^{-n}+\kappa r\big)r\|h-I\|_{L^\infty(E)}.
\end{aligned}$$

Now restrict $\kappa$ again so that $\kappa\leqslant\frac{1}{4c_{30}}$. Then restrict $\Phi$ and $r$ so that $c_{30}\kappa^{-n}\Phi^{1/(2n)}+c_{30}\kappa^{-n}\omega(c_8(\Phi^\delta+r))^2\leqslant 1/4$. This proves the decay estimate for $\Phi$. Finally, we derive the estimate for $|p_2-p_1|$. From the linear theory [14] as applied to $\tilde{h}$ we have

$$\begin{aligned}
|p_2-p_1|=\big|\nabla h(a)-p_1\big|&\leqslant c_{32}\frac{1}{|E|}\int_{\partial E}\big|v_\beta-\bar{u}_{B'}-p_1\cdot(x-a)\big|\,d\mathcal{H}^{n-1}+c_{33}\tilde{r}\|\widetilde{I}-\tilde{h}\|_{L^\infty(B')}\\
&\leqslant c_{32}\frac{1}{|E|}\int_{\partial E}|v_\beta-u|\,d\mathcal{H}^{n-1}+c_{32}\frac{1}{|E|}\int_{\partial E}\big|u-\bar{u}_E-p_1\cdot(x-a)\big|\,d\mathcal{H}^{n-1}+c_{34}r\|h-I\|_{L^\infty(E)}.
\end{aligned}$$

Then using the boundary estimates, Poincaré's inequality, and Hölder's inequality, we get $|p_2-p_1|\leqslant c_{35}\Phi(r,p_1)^{1/2}+c_{34}r\|h-I\|_{L^\infty(E)}$. □

By using this decay estimate iteratively, we then have Theorem 3. We actually have Theorem 3 holding for all $x \in B_{r/2}(x_0)$ if $\Phi(r_0, p_{1,}x_0)$ is sufficiently small by noting that $\Phi(r/2, p_{1,}x) \leqslant 2^n \Phi(r, p_{1,}x_0)$ for all $x \in B_{r/2}(x_0)$. Theorem 2 then follows [4,13,17].

## 5. Questions for further study

In addition to the above result on partial regularity and the more difficult problem of the $p$-Laplacian, one may also hope to show that the above open set $\widetilde{\Omega}$ where the solution is smooth is also dense in $\Omega$, or even that its complement has dimension $n-1$. For the case of $n=2$ in image restoration, this last case would correspond to 1-dimensional edges of objects.

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