Two-Minor Stable Matrices and the Open Leontief Model

Gerard Sierksma and Albert E. Steenge
Institute of Econometrics
University of Groningen
Groningen, The Netherlands

Submitted by Charles R. Johnson

ABSTRACT

Two-minor stable matrices are square matrices such that all (2,2) minors with no elements on the main diagonal are zero. The effects of taxes and subsidies on the competitive prices of output in Leontief's input-output model are examined by e.g. Metzler, Allen and Atsumi. We reexamine these studies and give results under more "normal" assumptions. Using two-minor stable matrices, the question is answered under what conditions the prices of the nontaxed and nonsubsidized goods do not change.

1. INTRODUCTION

The static input-output Leontief model $(I-T)x=d$ will be considered. It consists of n industries producing n commodities. The vector $x=(x_1,\dots,x_n)'$ is the annual output of the n industries, and $d=(d_1,\dots,d_n)'$ the final demand vector. T will invariably be used for a nonnegative irreducible (n,n) matrix with $n\in\mathbb{N}$ (the set of natural numbers $\geqslant 1$). For any $n\in\mathbb{N}$ we write $\langle n\rangle=\{1,2,\dots,n\}$, and use the following notation for any $x=(x_1,\dots,x_n)'\in\mathbb{R}^n$:

$$x\geqslant 0 \quad \text{if} \quad x_i\geqslant 0 \quad \text{for each } i\in\langle n\rangle;$$

$$x>0 \quad \text{if} \quad x\geqslant 0 \text{ and } x\neq 0;$$

$$x\gg 0 \quad \text{if} \quad x_i>0 \quad \text{for each } i\in\langle n\rangle.$$

Similar notation is used for matrices. The symbol r will denote the Perron-Frobenius root of T. Throughout this paper $r<1$.

We also consider the "dual" equation system $(I-T')p=v$, with p the price vector and v the vector of the value added of the commodities.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

The two systems $(I-T)x=d$ and $(I-T')p=v$ will describe the situation of the equilibrium attained after imposing a sales tax t_i (>0) on commodity i and a subsidy s_j (>0) on commodity j, such that $t_i x_i = s_j x_j$, i.e., the amount of the tax is equal to the amount of the subsidy $(i \neq j)$. For a more extensive description see Allen [1], Atsumi [2], Metzler [4], and Sierksma [7]. We shall take d and v both >0, which means that the systems are "open." It then follows from a well-known theorem (see e.g. Berman and Plemmons [3] and Seneta [6]) that $x \gg 0$, $p \gg 0$, and $\mathrm{Adj}(I-T) \gg 0$ and is nonsingular.

2. THE CHANGE OF THE PRICE VECTOR

Let $\Delta v = (t, -s, 0, \ldots, 0)'$ the change of the vector of the value added, and let Δp be the change of the price vector such that

$$(I-T')\,\Delta p = \Delta v,$$

t being a sales tax on commodity 1 and s a subsidy on commodity 2. The following theorem shows that the price p_1 of the taxed commodity rises or does not change and that the price p_2 of the subsidized commodity decreases or remains constant. Note that Atsumi proves $\Delta p_1 > 0$, $\Delta p_2 < 0$ but needs the stronger condition $d \gg 0$. Below we examine the cases $\Delta p_1 = 0$ or $\Delta p_2 = 0$.

Theorem 1. *If Δp_1 and Δp_2 are the changes of the prices after imposing a sales tax on commodity 1 and a subsidy on commodity 2, respectively, then*

$$\Delta p_1 \geqslant 0 \quad \textit{and} \quad \Delta p_2 \leqslant 0.$$

Proof. Let X be the diagonal matrix with x on the main diagonal. As $x \gg 0$, it follows that X is nonsingular. Taking $T^\circ = X^{-1}TX$, it follows that $(I-T^{\circ\prime})(X\,\Delta p) = X\,\Delta v$ and that $(I-T^\circ)1 = X^{-1}d > 0$ with $1 = (1, \ldots, 1)'$. Metzler's theorem (see e.g. [7, Theorem 12]) then says $0 < c_{ij}^\circ \leqslant c_{ii}^\circ$ with $\{c_{ji}^\circ\} = \mathrm{Adj}(I-T^\circ)$. We then find $x_1\,\Delta p_1 = (x_1 t/|I-T|)(c_{11}^\circ - c_{12}^\circ) \geqslant 0$ and $x_2\,\Delta p_2 = (x_2 s/|I-T|)(c_{21}^\circ - c_{22}^\circ) \leqslant 0$, so that in fact $\Delta p_1 \geqslant 0$ and $\Delta p_2 \leqslant 0$. ■

Throughout this paper we denote $\{c_{ji}\} = \mathrm{Adj}(I-T)$. The next theorem gives upper and lower bounds for the ratio s/t.

Theorem 2. $c_{21}/c_{22} \leqslant s/t \leqslant c_{11}/c_{12}$.

Proof. From $\Delta p = (I - T')^{-1}\Delta v = (1/|I - T|)\{c_{ij}\}(t, -s, 0, \ldots, 0)'$ it follows that

$$0 \leqslant \Delta p_1 = \frac{1}{|I - T|}(tc_{11} - sc_{12}) \quad \text{and} \quad 0 \geqslant \Delta p_2 = \frac{1}{|I - T|}(tc_{21} - sc_{22}).$$

Hence, $tc_{11} - sc_{12} \geqslant 0$ and $tc_{21} - sc_{22} \leqslant 0$, which gives the desired result. ■

THEOREM 3. *For any $i \in \langle n \rangle$ the following holds*:

$$\Delta p_i = 0 \quad \Leftrightarrow \quad \frac{s}{t} = \frac{c_{i1}}{c_{i2}}.$$

Proof. Direct consequence of $\Delta p_i = (1/|I - T|)(tc_{i1} - sc_{i2})$, $i \in \langle n \rangle$. ■

COROLLARY 4. *All prices, except for the taxed and subsidized goods, remain constant if and only if*

$$\frac{s}{t} = \frac{c_{31}}{c_{32}} = \cdots = \frac{c_{n1}}{c_{n2}}.$$

Proof. Direct consequence of Theorem 3. ■

Note that either $\Delta p_1 \neq 0$ or $\Delta p_2 \neq 0$ or both in case all other prices remain constant; otherwise $\mathrm{Adj}(I - T)$ would be singular. Also note that

$$\frac{c_{21}}{c_{22}} \leqslant \frac{c_{i1}}{c_{i2}} \leqslant \frac{c_{11}}{c_{22}}$$

for each $i \geqslant 3$, with one of the two inequalities *strict* for at least one i; see e.g. [7, Theorem 29].

3. TWO-MINOR STABLE MATRICES

In this section we characterize the matrices T such that for the models $(I - T)x = d$ and $(I - T')p = v$ the following holds: On imposing some tax and subsidy system on *any* two industries, the prices of the nontaxed and nonsubsidized goods have to stay the same. Below this question is described in detail.

We need the following concept. A *2-minor stable matrix* is a matrix such that all (2,2) minors with no element on the main diagonal are zero. That is, any (n,n) matrix $A=\{a_{ij}\}$ is called 2-minor stable iff

$$a_{ik}a_{jl}-a_{il}a_{jk}=0$$

for each i, j, k, l with no two indices equal in case $n \geqslant 4$, and for $n=3$ iff $a_{12}a_{23}a_{31}-a_{13}a_{32}a_{21}=0$. It is known that $c_{ii}c_{kl}-c_{ki}c_{il} \geqslant 0$ for each i,k,l with $k \neq l \neq i \neq k$; (see e.g. [7, Theorem 29] and [8, Corollary 4]); however, all off-diagonal 2-minors can be positive, negative, or zero.

Note that all matrices of class $M(x)$ are 2-minor stable; for any $x \in \mathbb{R}$ an (n,n) matrix $A=\{a_{ij}\}$ is of class $M(x)$ iff $a_{jj}-a_{ij}=x$ for each $i, j \in \langle n \rangle$ with $i \neq j$: see [7, p. 189]. In [8], (2,2) minors are studied with at most one main diagonal element not from the original main diagonal.

THEOREM 5. *Let A be an (n,n) matrix with $A \gg 0$ and $n \geqslant 3$. The following assertions are equivalent*:

(i) *A is 2-minor stable*;

(ii) $a_{\alpha i}a_{i\beta}/a_{\alpha\beta}=a_{\gamma i}a_{i\delta}/a_{\gamma\delta}$ *for each α,β,i and γ,δ,i with no two indices equal in each triple.*

Proof. For $n=3$ the theorem holds by definition. So let $n \geqslant 4$.

(i) $\Rightarrow$ (ii): Take any $\alpha,\beta,\gamma,\delta,i \in \langle n \rangle$. We distinguish three cases.

(1) $\gamma \neq \beta$. Then $a_{\alpha i}a_{i\beta}/a_{\alpha\beta}=a_{\gamma i}a_{i\beta}/a_{\gamma\beta}=a_{\gamma i}a_{i\delta}/a_{\gamma\delta}$.

(2) $\delta \neq \alpha$. Similar to (1).

(3) $\gamma=\beta$ and $\delta=\alpha$. Then $a_{\alpha i}a_{i\beta}/a_{\alpha\beta}=a_{\tau i}a_{i\beta}/a_{\tau\beta}=$ (for $\tau \neq \alpha,\beta,i$) $a_{\tau i}a_{i\alpha}/a_{\tau\alpha}=a_{\beta i}a_{i\alpha}/a_{\beta\alpha}$.

(ii) $\Rightarrow$ (i): Set $\delta=\beta$ in (ii). ∎

Let $\mathrm{Adj}(I-T)=\{c_{ji}\}$ be 2-minor stable; Theorem 5 enables us to define

$$c_{ii}^{*}=\frac{c_{ki}c_{il}}{c_{kl}}$$

for $k \neq l \neq i \neq k$ and independent on k and l. So the matrix

$$\begin{bmatrix} c_{11}^{*} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22}^{*} & \cdots & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn}^{*} \end{bmatrix}$$

has all 2-minors equal to zero. Note that, for all i, $c_{ii}^{*} \geqslant 0$.

THEOREM 6. *T is 2-minor stable iff* Adj$(I-T)$ *is.*

Proof. For $n=3$ the theorem follows by a straightforward calculation. So we may assume that $n \geqslant 4$. To show this theorem we use [5, Proposition 1], which states that for any nonsingular (n,n) matrix H and any $\alpha \subset \langle n \rangle$, rank $H(\alpha, \alpha^c) =$ rank $H^{-1}(\alpha, \alpha^c)$, where α and α^c denote respectively the rows and columns included in each submatrix, and α^c means the complement of α. For example, if $\alpha = \{1,2\}$, $\alpha^c = \{3,\dots,n\}$, and rank $H(\alpha,\alpha^c)=1$, this is equivalent to saying that $h_{1k}h_{2m} - h_{1m}h_{2k} = 0$ for all $k,m = 3,\dots,n$ iff $h^{-1}_{1k}h^{-1}_{1m} - h^{-1}_{1m}h^{-1}_{1k} = 0$ for all $k,m=3,\dots,n$, with $H = \{h_{ij}\}$ and $H^{-1} = \{h^{-1}_{ij}\}$. A similar statement holds for $\alpha = \{i,j\}$, $1 \leqslant i < j \leqslant n$. Putting all of these statements together implies that H is 2-minor stable iff H^{-1} is 2-minor stable. Since $I-T$ is nonsingular, the following chain of equivalences holds: T is 2-minor stable $\Leftrightarrow$ $I-T$ is 2-minor stable $\Leftrightarrow$ $(I-T)^{-1}$ is 2-minor stable $\Leftrightarrow$ Adj$(I-T)$ is 2-minor stable.[1] ■

THEOREM 7. *Let T be 2-minor stable. If $c_{ii} - c^*_{ii} = 0$ for some $i \in \langle n \rangle$, then $c_{jj} - c^*_{jj} > 0$ for each $j \neq i$. That is, the minors $c_{ii}c_{kl} - c_{ki}c_{il}$ are zero for at most one i.*

Proof. Without loss of generality we may assume that $c_{11} - c^*_{11} = 0$. It then follows that $\{c_{ji}\}$ is equivalent to

$$\begin{bmatrix} c_{11} & 0 & \cdots & 0 \\ 0 & c_{22}-c^*_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_{nn}-c^*_{nn} \end{bmatrix}.$$

This can be seen as follows. Take any $i \geqslant 2$, and consider the first and the ith row of $\{c_{ji}\}$. For any $k \neq 1$, i multiply the first row by c_{ki} and the ith row by c_{k1}, and then subtract these rows. Using the fact that Adj$(I-T)$ is 2-minor stable and the definition of c^*_{ii}, we find the row $(0,\dots,c_{ii},\dots,c^*_{ii},\dots,0)$. The zeros in the first row follow from using the first column and the fact that $c_{11} > 0$.

[1] This proof was given by W. W. Barrett (private communication).

As the rank of $\{c_{ji}\}$ is n, it follows that $c_{ii} - c_{ii}^* \neq 0$. As $c_{ii} - c_{ii}^* \geqslant 0$ (see above), we have in fact $c_{ii} - c_{ii}^* > 0$ for each $i \geqslant 2$. So for at most *one* $i \in \langle n \rangle$ we have $c_{ii} = c_{ii}^*$. ■

Note that the above theorem is also clear from the following: Suppose $c_{11} - c_{11}^* = c_{22} - c_{22}^* = 0$. Taking $\Delta v = (c_{32}, -c_{31}, 0, \ldots, 0)'$, it follows from Theorem 3 that $\Delta p = (1/|I - T|)\{c_{ij}\}\,\Delta v = 0$, which is impossible.

THEOREM 8. *For the systems $(I - T)x = d$ and $(I - T')p = v$ the following assertions are equivalent*:

(i) *T is 2-minor stable*;
(ii) *For each $i, j \in \langle n \rangle$ with $i \neq j$ there is a tax and subsidy system t_i, s_j such that $\Delta p_k = 0$ for each $k \neq i, j$.*

Proof. (i) ⇒ (ii): Take any i and j. Without loss of generality we may take $i = 1$, $j = 2$, and write $t_1 = t$, $s_2 = s$; hence $(I - T')\Delta p = (t, -s, 0, \ldots, 0)'$, or $\Delta p = (1/|I - T|)\{c_{ij}\}(t, -s, 0, \ldots, 0)'$. For any $k \in \langle n \rangle$ we than have $\Delta p_k = (1/|I - T|)(c_{k1}t - c_{k2}s)$. As $\mathrm{Adj}(I - T)$ is 2-minor stable, it follows that $c_{31}/c_{32} = \cdots = c_{n1}/c_{n2}$. Taking $s/t = c_{31}/c_{32}$, it follows for $k \geqslant 3$ that $\Delta p_k = (1/|I - T|)(c_{k1}t - c_{k2}s) = 0$.
(ii) ⇒ (i): Left to the reader. ■

THEOREM 9. *Let T be 2-minor stable. Then the following holds.*

(1) *If $c_{kk} - c_{kk}^* = 0$ for some $k \in \langle n \rangle$, then*

$$x = \frac{\alpha_k}{|I - T|}\begin{pmatrix} c_{k1} \\ \vdots \\ c_{kn} \end{pmatrix} \quad \text{and } d = \alpha_k e_k, \qquad \text{with} \quad \alpha_k > 0.$$

(2) *If $c_{ii} - c_{ii}^* > 0$ for each $i \in \langle n \rangle$, then*

$$x = \frac{\beta_k}{|I - T|}\chi\begin{pmatrix} c_{k1} \\ \vdots \\ c_{kk}^* \\ \vdots \\ c_{kn} \end{pmatrix} \quad \text{and} \quad d = \beta_k \begin{pmatrix} \dfrac{c_{11}}{c_{11} - c_{11}^*} \\ \vdots \\ \dfrac{c_{kk}^*}{c_{kk} - c_{kk}^*} \\ \vdots \\ \dfrac{c_{nn}}{c_{nn} - c_{nn}^*} \end{pmatrix}$$

for each $k \in \langle n \rangle$, $\beta_k > 0$, *and*

$$\chi = 1 + \sum_{i=1}^{n} \frac{c_{ii}^*}{c_{ii} - c_{ii}^*} = 1 - n + \sum_{i=1}^{n} \frac{c_{ii}}{c_{ii} - c_{ii}^*}.$$

Proof. According to Theorem 8, we can choose, for each $i, j \in \langle n \rangle$ with $i \neq j$, s_j and t_i with $x_i/x_j = s_j/t_i$ such that $\Delta p_k = 0$ for each $k \neq i, j$. Theorem 3 implies that $s_j/t_i = c_{ki}/c_{kj}$ and therefore $c_{kj}x_i - c_{ki}x_j = 0$ for each $i, j, k \in \langle n \rangle$ with $k \neq i < j \neq k$. The matrix of coefficients of $x = (x_1, \ldots, x_n)$ consists of $(n-2)\binom{n}{2}$ rows and n columns. The rows belonging to the equations $c_{\alpha j}x_i - c_{\alpha i}x_j = 0$ and $c_{\beta j}x_i - c_{\beta i}x_j = 0$, with $\alpha, \beta, i, j \in \langle n \rangle$ and no two indices equal, are linearly dependent because of the 2-minor stability. For each pair $i < j$ we choose one equation as follows: if $i = 1$ and $j = 2$ take $k = 3$, if $i = 1$ and $j > 2$ take $k = 2$; if $i \geqslant 2$ take $k = 1$. The rows of this matrix of coefficients, which we call B, are ordered lexicographically in i and j with $i < j$. For example, for $n = 4$ we have

$$B = \begin{bmatrix} c_{32} & -c_{31} & 0 & 0 \\ c_{23} & 0 & -c_{21} & 0 \\ c_{24} & 0 & 0 & -c_{21} \\ 0 & c_{13} & -c_{12} & 0 \\ 0 & c_{14} & 0 & -c_{12} \\ 0 & 0 & c_{14} & -c_{13} \end{bmatrix}.$$

Clearly, the first $n-1$ rows of B are linearly independent, and each other row is linearly dependent on two of the first $n-2$ rows. Hence, B has rank $n-1$. We then find n solutions, namely

$$x = \eta_1 \begin{bmatrix} c_{11}^* \\ c_{12} \\ \vdots \\ c_{1n} \end{bmatrix} = \cdots = \eta_n \begin{bmatrix} c_{n1} \\ \vdots \\ c_{nn-1} \\ c_{nn}^* \end{bmatrix}$$

with $\eta_1, \ldots, \eta_n > 0$.

From $x = (1/|I-T|)\{c_{ji}\}d$ and $x_i/x_j = c_{ki}/c_{kj}$ for some $k \neq i, j$ it follows that $(\sum_{\alpha=1}^{n} c_{\alpha i}d_\alpha)/(\sum_{\alpha=1}^{n} c_{\alpha j}d_\alpha) = c_{ki}/c_{kj}$; see also Atsumi [2, p. 36). Hence, $\sum_{\alpha=1}^{n}(c_{\alpha i}c_{kj} - c_{\alpha j}c_{ki})d_\alpha = 0$, and as Adj$(I-T)$ is 2-minor stable, we have

$$(c_{ii}c_{kj} - c_{ij}c_{ki})d_i - (c_{jj}c_{ki} - c_{ji}c_{kj})d_j = 0$$

for each $i, j \in \langle n \rangle$ with $i \neq j$ and $k \neq i, j$. So d satisfies an equation system $Cd = 0$ with C a $\left(\binom{n}{2}, n\right)$ matrix defined similarly to B. For example, for $n = 4$ we have

$$C = \begin{bmatrix} c_{11}c_{32} - c_{31}c_{12} & -(c_{22}c_{31} - c_{32}c_{21}) & 0 & 0 \\ c_{11}c_{23} - c_{21}c_{13} & 0 & -(c_{33}c_{21} - c_{23}c_{31}) & 0 \\ c_{11}c_{24} - c_{21}c_{14} & 0 & 0 & -(c_{44}c_{21} - c_{24}c_{41}) \\ 0 & c_{22}c_{13} - c_{12}c_{23} & c_{33}c_{12} - c_{13}c_{32} & 0 \\ 0 & c_{22}c_{14} - c_{12}c_{24} & 0 & -(c_{44}c_{12} - c_{14}c_{42}) \\ 0 & 0 & c_{33}c_{14} - c_{13}c_{34} & -(c_{44}c_{13} - c_{14}c_{43}) \end{bmatrix}.$$

Note that

$$C = B\,\mathrm{adj}(I - T).$$

On the other hand, because for each $i, \alpha, \beta \in \langle n \rangle$ with $i \neq \alpha \neq \beta \neq i$ we have

$$c_{ii}c_{\alpha\beta} - c_{\alpha i}c_{i\beta} = c_{\alpha\beta}\left(c_{ii} - \frac{c_{\alpha i}c_{i\beta}}{c_{\alpha\beta}}\right) = c_{\alpha\beta}(c_{ii} - c_{ii}^*),$$

it follows that

$$C = B\begin{bmatrix} c_{11} - c_{11}^* & 0 & \cdots & 0 \\ 0 & c_{22} - c_{22}^* & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} - c_{nn}^* \end{bmatrix}.$$

According to Theorem 6 we cannot have more than one $c_{ii} - c_{ii}^*$ equal to 0. So if $c_{kk} - c_{kk}^* = 0$ for some $k \in \langle n \rangle$, then, because e_k is a solution and $\mathrm{rank}\, C = n - 1$,

$$d = \alpha_k e_k \qquad \text{with} \qquad \alpha_k > 0.$$

From $(I - T)x = d$ it follows that $x = (1/|I - T|)\mathrm{Adj}(I - T)d$, so we must have in case $c_{kk}^* = c_{kk}$ that $\eta_k = \alpha_k/|I - T|$. This proves (1).

If $c_{ii} - c_{ii}^* > 0$ for each $i \in \langle n \rangle$, we have

$$B((c_{11} - c_{11}^*)d_1, \dots, (c_{nn} - c_{nn}^*)d_n)' = 0.$$

Therefore we find

$$d = \beta_1 \begin{bmatrix} \dfrac{c_{11}^*}{c_{11} - c_{11}^*} \\ \dfrac{c_{12}}{c_{22} - c_{22}^*} \\ \vdots \\ \dfrac{c_{1n}}{c_{nn} - c_{nn}^*} \end{bmatrix} = \cdots = \beta_n \begin{bmatrix} \dfrac{c_{n1}}{c_{11} - c_{11}^*} \\ \vdots \\ \dfrac{c_{nn}^*}{c_{nn} - c_{nn}^*} \end{bmatrix},$$

with $\beta_1, \dots, \beta_n \in 0$. Taking X as before, with $X1 = x$, it follows from $(I - T)x = d$ that $[X^{-1}\{c_{ji}\}X][X^{-1}d] = |I - T|1$, and

$$X^{-1}\{c_{ij}\}X = \begin{bmatrix} c_{11} & \dfrac{x_2}{x_1}c_{21} & \cdots & \dfrac{x_n}{x_1}c_{n1} \\ \dfrac{x_1}{x_2}c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{x_1}{x_n}c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{22}^* & \cdots & c_{nn}^* \\ c_{11}^* & c_{22} & \cdots & c_{nn}^* \\ \vdots & \vdots & \ddots & \vdots \\ c_{11}^* & c_{22}^* & \cdots & c_{nn} \end{bmatrix}.$$

Moreover,

$$X^{-1}d = \frac{\beta_k}{\eta_k}\left(\frac{1}{c_{11} - c_{11}^*}, \dots, \frac{1}{c_{nn} - c_{nn}^*}\right)',$$

with $k \in \langle n \rangle$. Taking e.g. the first row of $X^{-1}\{c_{ji}\}X$, we find that

$$\eta_k = \frac{1}{|I - T|}\left(\frac{c_{11}}{c_{11} - c_{11}^*} + \frac{c_{22}^*}{c_{22} - c_{22}^*} + \cdots + \frac{c_{nn}^*}{c_{nn} - c_{nn}^*}\right)\beta_k$$

$$= \frac{\beta_k}{|I - T|}\chi,$$

which proves (2). ■

As an illustration consider the following example:

$$T = \begin{pmatrix} 0.69 & 0.08 & 0.07 \\ 0.05 & 0.86 & 0 \\ 0.10 & 0 & 0.74 \end{pmatrix}.$$

Then

$$\mathrm{Adj}(I - T) = \begin{pmatrix} 0.0364 & 0.0208 & 0.0098 \\ 0.0130 & 0.0736 & 0.0035 \\ 0.0140 & 0.0080 & 0.0394 \end{pmatrix}$$

and $|I - T| = 0.009264$.

Clearly T is 2-minor stable; hence so is $\mathrm{Adj}(I - T)$. Moreover, $c_{11}^* = c_{11} = 0.0364$, $c_{22}^* = 0.0074$, $c_{33}^* = 0.0038$.

Let $s_2/t_1 = c_{31}/c_{32} = 2.8$. Then $\Delta p_1 = 0$, $\Delta p_2 = -20t_1 < 0$, $\Delta p_3 = 0$; and $d = \alpha(1,0,0)$, $x = \alpha(3.92, 1.40, 1.51)'$ with $\alpha > 0$. So it may happen that even within the taxed and subsidized sector, one of the prices does not change. Also note that in this case prices stay the same or go down.

REFERENCES

1 P. R. Allen, Taxes and subsides in Leontief's input-output model: Comment, *Quart. J. Econom.*, Feb. 1972, pp. 148–153.
2 H. Atsumi, Taxes and subsidies in the input-output model, *Quart. J. Econom.*, Feb. 1981, pp. 27–45.
3 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.
4 L. A. Metzler, Taxes and subsidies in Leontief's input-output model, *Quart. J. Econom.*, Aug. 1951, pp. 433–438.
5 R. D. Pope and W. W. Barrett, A result on the rank of submatrices and an economic application, to appear.
6 E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed., Springer, New York, 1980.
7 G. Sierksma, Non-negative matrices; the open Leontief model, *Linear Algebra Appl.* 26:175–201 (1979).
8 G. Sierksma and E. J. Bakker, Nearly principal minors of M-matrices, *Compositio Math.* 59 (1986) 73–79.

Received 1 June 1986; revised 16 July 1986