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On a Certain Class of Multiplicative Functions

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Let $\chi_0, \chi_1, \chi_2, \ldots$ be the sequence of all Dirichlet characters (in which the principal character χ_0 occurs only once), ordered with increasing moduli. Define, for a sequence $(\alpha_k)_{k \in N}$ of real numbers ($\alpha_k \to 0$ for $j \to \infty$, sufficiently quickly), the multiplicative function f by

$$\sum_n \frac{f(n)}{n^s} = \zeta^{-1}(s) \prod_{k=1}^{\infty} (L(s, \chi_k))^{\alpha_k}.$$

It is the aim of the paper to construct sequences (α_k) such that the following statements are equivalent

(i) $\sum_{n \leqslant x} f(n) = O(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$,

(ii) the Riemann hypothesis is true for all functions $L(s, \chi)$.

1. INTRODUCTION

The Möbius function μ is an example of a multiplicative function which may give information on the zeros of the Riemann zeta-function.

If

$$\sigma_{0,\zeta} = \sup_{\sigma} (\sigma \text{ is the real part of zero of the } \zeta\text{-function}),$$

$$\sigma_{1,\zeta} = \overline{\lim_{x \to \infty}} \frac{1}{\ln x} \ln \left(\left| \sum_{n \leqslant x} \mu(n) \right| + 1 \right),$$

then

$$\sigma_{0,\zeta} = \sigma_{1,\zeta}. \tag{1.1}$$

This principle can easily be extended to finitely many L-series. If $\chi^{(1)}, \ldots, \chi^{(l)}$ are Dirichlet characters, and $f\colon \mathbb{N} \to C$ is defined by

$$F(s) = \sum_n \frac{f(n)}{n^s} = L^{-1}(s, \chi^{(1)}) \cdot \cdots \cdot L^{-1}(s, \chi^{(l)}) \qquad (\sigma = \operatorname{Re} s > 1),$$

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then the analogue of (1.1) is true for the zeros of $L(s, \chi^{(1)}), ..., L(s, \chi^{(l)})$ and f instead of μ.

If one wants to deal with all L-functions at once, one is led to functions f given by

$$F(s) = \sum_n \frac{f(n)}{n^s} = \zeta^{-1}(s) \prod_\chi (L(s, \chi))^{\alpha_\chi}, \tag{1.2}$$

where (α_χ) is an appropriate sequence of real or complex numbers. It is the purpose of this paper to describe such sequences.

Arrange the non-principal characters χ to a sequence

$$\chi_1, \chi_2, ..., \chi_k \qquad \bmod q_k, \quad q_k \geqslant 2$$

with increasing moduli q_k. Obviously,

$$q_k \leqslant k \qquad \text{for} \quad q \geqslant 5. \tag{1.3}$$

The first lemma, which will be proved in the next section, is necessary for the statement of the main result.

Lemma 1. *Let* $(\tilde{\alpha}_j)$ $(j \in \mathbb{N})$ *be a sequence of positive real numbers, tending to zero, and assume*

$$r_k = \sum_{j > k} \tilde{\alpha}_j \ln(j+1) = O((\ln(k+1))^{-2k}) \qquad (k \to \infty). \tag{1.4}$$

Then there exists a $\beta \in (0, 1]$ *such that the numbers*

$$\alpha_j = \beta \tilde{\alpha}_j \tag{1.5}$$

have the following property. For all $c > 0$ *and all* $m_j \in N_0$ *with* $m_j \leqslant c \ln(j+1)$

$$\sum_{j \in \mathbb{N}} m_j \alpha_j \notin \mathbb{N}. \tag{1.6}$$

We can now state the

Theorem. *Let* (α_k) $(k \in N)$ *be a sequence as in Lemma* 1 *which, in addition, satisfies the condition*

$$\sum_{k \in \mathbb{N}} \alpha_k \ln(k+1) |\ln \alpha_k| < \infty. \tag{1.7}$$

Define the multiplicative function $f: \mathbb{N} \to C$ *by*

$$F(s) = \sum_n \frac{f(n)}{n^s} = \zeta^{-1}(s) \prod_{k \in \mathbb{N}} (L(s, \chi_k))^{\alpha_k} \tag{1.8}$$

($\sigma = \operatorname{Re} s > 1$, $\arg L(s, \chi_k)$ *is to be* $o(1)$ *for* $s = \sigma \to \infty$). *Let*

$$\sigma_0 = \sup_{\sigma} \ (\sigma \textit{ is the real part of a zero of } \zeta(s) \textit{ or an } L(s, \chi_k)).$$

$$\sigma_1 = \overline{\lim_{x \to \infty}} \frac{1}{\ln x} \ln \left(\left| \sum_{n \leqslant x} f(n) \right| + 1 \right),$$

then

$$\sigma_0 = \sigma_1. \tag{1.9}$$

In particular, the generalized Riemann hypothesis is true iff

$$\sum_{n \leqslant x} f(n) = O(x^{1/2 + \varepsilon}) \qquad \textit{for all } \varepsilon > 0.$$

Remarks. (A) The definition (1.8) gives

$$f(p) = -1 + \sum_k \alpha_k \chi_k(p) \qquad \text{for primes } p. \tag{1.10}$$

If $0 < \delta \leqslant 1$ and the numbers α_k are chosen such that

$$\sum_k \alpha_k \leqslant \delta,$$

then

$$|f(p)| \leqslant 1 + \delta \tag{1.11}$$

and

$$|f(p^l)| \leqslant 2\delta \qquad (l \geqslant 2) \tag{1.12}$$

(this will be shown in Section 3).

(B) It will be clear from the proof that one yields the following, slightly stronger

THEOREM′. *Let X be any set, finite or infinite, of non-principal Dirichlet characters, there exists a complex-valued multiplicative function f, satisfying* $|f(n)| \leqslant 1$ *everywhere, such that if* ϑ_0 *is the least upper bound of* ϑ*'s such that*

$$\sum_{n \leqslant x} f(n) = O(x^{\vartheta}),$$

then ϑ_0 *is the supremum of the real parts of the zeros of all* $L(s, \chi)$, $\chi \in X$.

A similar results holds if X is allowed to contain principal Dirichlet characters, except now that the bound $|f(n)| \leqslant 1$ must be replaced by $|f(n) \leqslant 1 \leqslant (1+\delta)^{\omega(n)}$ $(0<\delta<1$ any fixed constant).

If χ does not contain principal characters, the bound can be refined to $|f(n)| \leqslant \delta^{\Omega(n)}$.

The function f can be defined independent of deeper knowledge about the zeros of the L-functions.

The problem of getting an f with $|f(n)| \leqslant 1$ in case that principal characters occur seems to be of some interest.

(C) Lemma 1 and the Theorem can be generalized as follows.

Let $\tilde{\alpha}_j \in C \setminus \{0\}$, let (1.4) be fulfilled with $|\tilde{\alpha}_j|$, and assume

$$\sum_{j \in \mathbb{N}} m_j \tilde{\alpha}_j = 0 \qquad (m_j \in \mathbb{N}_0) \text{ only if } m_1 = m_2 = \cdots = 0.$$

Then, with an appropriate $\beta \in (0, 1]$, (1.6) holds for $\alpha_j = \beta \tilde{\alpha}_j$. Such a sequence can be used in (1.8).

2. Proof of Lemma 1

Put, for $N \in \mathbb{N}$,

$$B_N = \left\{ \eta \in \mathbb{R},\, \eta = \sum_{j \in N} m_j \tilde{\alpha}_j \,(m_j \in \mathbb{N}_0),\, m_j \leqslant N \ln(j+1) \right\}.$$

For $\eta \in B_N$ and $k \in \mathbb{N}$ one has

$$\left| \sum_{j>k} m_j \tilde{\alpha}_j \right| \leqslant N r_k.$$

The numbers $\sum_{j=1}^{k} m_j \tilde{\alpha}_j$ attain at most $(N \ln(k+1)+1)^k$ values. B_N can therefore be covered by an open set of measure

$$\leqslant 2 r_k N (N \ln(k+1)+1)^k.$$

Because of (1.4) this is $o(1)$ $(k \to \infty)$. Hence, B_N and

$$B(\tilde{\alpha}_j) = \left\{ \eta \in \mathbb{R},\, \eta = \sum_j m_j \tilde{\alpha}_j (m_j \in N_0),\, \exists C \forall j : m_j \leqslant C \ln(j+1) \right\} \tag{2.1}$$

are sets of measure zero.

For all $\xi \in (0, 1]$ the same arguments apply to the sequence $(\xi \tilde{\alpha}_j)$.

$$\mu(B(\xi \tilde{\alpha}_j)) = 0 \qquad \text{for} \quad 0 < \xi \leqslant 1. \tag{2.2}$$

Consider a $\xi \in (0, 1]$ for which there is a

$$g \in B(\xi\tilde{\alpha}_j) \cap \mathbb{N}. \tag{2.3}$$

Then, $g = \xi\eta$ with $\eta \in B(\tilde{\alpha}_j)$, or $(1/\xi \in B((1/g)\,\tilde{\alpha}_j)$. Owing to (2.2), $\bigcup_{g \in \mathbb{N}} B((1/g)\,\tilde{\alpha}_j)$ is a set of measure zero. Hence (2.3) can be true only for numbers ξ from a zero set. This gives the statement of the lemma.

3. Properties of the Function f

Because of absolute and uniform convergence in every half-plane $\sigma \geqslant 1 + \varepsilon$ we have

$$F(s) = \prod_p ((1 - p^{-s}) \prod_k (1 - \chi_k(p)\, p^{-s})^{-\alpha_k}). \tag{3.1}$$

This implies (1.10).

Proof of (1.12). For power series $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$ let us say that $B(z)$ is a majorant of $A(z)$ if $|a_n| \leqslant b_n$ for every n. This relation is preserved by product. Next, note that for any complex numbers a, α the binomial series for $(1 - |a|z)^{-|\alpha|}$ is a majorant of $(1 - az)^{\alpha}$, and moreover if $0 < \alpha \leqslant \alpha'$ then $(1 - z)^{-\alpha'}$ majorizes $(1 - z)^{-\alpha}$. Since the nth coefficient of $(1 - z)^{-\alpha}$ is $\alpha(\alpha + 1) \cdots (\alpha + n - 1)/n!$, it is clear that $1 + \alpha z/(1 - z)$ majorizes $(1 - z)^{-\alpha}$ for $0 < \alpha \leqslant 1$.

With $z = p^{-s}$, the Euler factor for $F(s)$ at p is $(1 - z)\,A(z)$, with

$$A(z) = \prod_k (1 - \chi_k(p) z)^{-\alpha_k}.$$

By the preceding remarks, $A(z)$ is majorized by $\prod_k (1 - z)^{-\alpha_k} = (1 - z)^{-\Sigma \alpha_k}$ and hence by $(1 - z)^{-\delta}$ and $1 + \delta z/(1 - z)$ if $0 < \delta \leqslant 1$. This means that if $A(z) = \sum a_n z^n$ we have $|a_n| \leqslant \delta$ for $n \geqslant 1$. It now follows that the lth coefficient of $(1 - z)\,A(z)$, namely $f(p^l) = a_l - a_{l-1}$, satisfies

$$|f(p)| \leqslant 1 + \delta, \qquad |f(p^l)| \leqslant 2\delta \qquad \text{if} \quad l \geqslant 2.$$

4. Proof of $\sigma_1 \leqslant \sigma_0$

Because of $f(n) = O_\varepsilon(n^\varepsilon)$ we may assume $\sigma_0 < 1$. With the simple inequality

$$L(\sigma + i\tau, \chi_k) \ll q_k \tau' \ln(q_k \tau')$$

$$(\tau' = |\tau| + 2,\ \sigma \geqslant \tfrac{1}{2}, \ll\text{-constant absolute})$$

and the theorems of Borel–Carathéodory and Hadamard one sees as usual (see Titchmarsh [2, Chap. XIV]): For every $\varepsilon > 0$ there exists a $T_0 \geqslant 2$ such that for

$$\sigma_0 + \varepsilon \leqslant \sigma \leqslant \tfrac{3}{2}, \qquad q_k \tau' \geqslant T_0$$

the inequality

$$(q_k \tau')^{-\varepsilon} \leqslant |L(\sigma + i\tau, \chi_k)| \leqslant (q_k \tau')^{\varepsilon}$$

holds. Hence, for $x \geqslant 2$ and $T = x^{1/2}$, on the vertical line

$$\sigma = \sigma_0 + \varepsilon(<1), \qquad |\tau| \leqslant T$$

and on the horizontal lines

$$s = \sigma \pm iT, \qquad \sigma_0 + \varepsilon \leqslant \sigma \leqslant 1 + \varepsilon$$

we have the bound

$$\begin{aligned} F(s) &= O_\varepsilon\left(T^\varepsilon \exp\left(\varepsilon \sum_k \alpha_k \ln(q_k(T+2))\right)\right) \\ &= O_\varepsilon(T^{\varepsilon(1 + c\sum \alpha_k \ln(k+1))}) \end{aligned} \tag{4.1}$$

with some absolute $c > 0$. By means of (1.7) and Perron's formula one sees

$$\sum_{n \leqslant x} f(n) = O_\delta(x^{\sigma_0 + \delta}) \qquad \text{for every} \quad \delta > 0.$$

This implies $\sigma_1 \leqslant \sigma_0$.

5. Proof of $\sigma_0 \leqslant \sigma_1$

All constants implied by the symbols $\ll$ and O are absolute.

If we suppose $\sigma_1 < \sigma_0$ then, by partial summation, F turns out to be regular in the half-plane

$$\sigma > \sigma_1 = \sigma_0 - \delta \qquad (\delta > 0). \tag{5.1}$$

Let $\rho^* = \sigma^* + i\tau^*$ be a zero of $\zeta(s)$ or some $L(s, \chi_k)$ with

$$\sigma^* > \sigma_1.$$

Put $\tau' = |\tau^*| + 2$,

$$R_1 = \{s = \sigma + i\tau, 0 \leqslant \sigma \leqslant 1, \tau^* - 2 \leqslant \tau \leqslant \tau^* + 2\},$$

$$R_2 = \{s, \sigma_1 < \sigma < 2, \tau^* - 1 < \tau < \tau^* + 1\}.$$

Write $\zeta(s) = L(s, \chi_0)$ and let $\rho_1^{(k)}, ..., \rho_{l_k}^{(k)}$ $(k \in \mathbb{N}_0)$ be the zeros of $L(s, \chi_k)$ in the rectangle R_1, counted according to their multiplicity. Then

$$l_0 \ll \ln \tau', \qquad l_k \ll \ln(k\tau') \quad (k \in \mathbb{N}) \tag{5.2}$$

(see Prachar [1, VII, Satz 2.3]).

Let us, for simplicity, suppose that $\tau^* > 2$. The case in which the pole of $L(s, \chi_0)$ has to be considered does not give any new difficulties.

For $s \in R_2$ and $k \in \mathbb{N}_0$ we have

$$\frac{L'}{L}(s, \chi_k) = \sum_{\nu=1}^{l_k} (s - \rho_\nu^{(k)})^{-1} + g_k(s), \tag{5.3}$$

where g_k is regular and $\ll \ln(k+1)\tau'$) in R_2 (see Prachar [1, VII, Satz 4.1]).

By integration from $2 + i\tau$ to $s = \sigma + i\tau \in R_2$ using (5.3) and (5.2), one sees

$$L(s, \chi_k) = H_k(s) \sum_{\nu=1}^{l_k} (s - \rho_\nu^{(k)}), \tag{5.4}$$

where H_k is regular and $\neq 0$ on R_2, and satisfies the inequality

$$H_k(s) = \exp(O(\ln(k+1)\tau')). \tag{5.5}$$

One further lemma is needed.

Lemma 2. *There exist numbers*

$$\bar{\sigma} \in (\sigma_1, \sigma^*), \qquad \xi \in (0, 1), \qquad \textit{and} \quad k_0 \in \mathbb{N}$$

with the following properties. Let W be the rectangular, closed path which connects the points

$$\tfrac{3}{2} + i(\tau^* + \xi),\ \bar{\sigma} + i(\tau^* + \xi),\ \bar{\sigma} + i(\tau^* - \xi), \qquad \textit{and} \qquad \tfrac{3}{2} + i(\tau^* + \xi).$$

Then

(a) *for $k \geq k_0$, $1 \leq \nu \leq l_k$, and $s \in W$ we have*

$$|s - \rho_\nu^{(k)}| \geq \alpha_k,$$

(b) *for $0 \leq k \leq k_0$ no $\rho_\nu^{(k)}$ lies on W.*

Proof of Lemma 2. The points $\rho_\nu^{(k)}$ $(\nu \leq l_k,\ k \geq k_0,\ k_0$ to be fixed later) are taken as centers of circles of radius α_k. Then, because of (5.2) and (1.7), the sum of all radii is

$$= \sum_{k \geqslant k_0} l_k \alpha_k \leqslant c \sum_{k \geqslant k_0} \alpha_k \ln(k\tau') < \varepsilon,$$

if $k_0 = k_0(\varepsilon, \tau')$ is sufficiently large. By (b) only finitely many rectangles of type W are excluded. So one can find a path W as stated in the lemma.

For $k \in \mathbb{N}_0$, let $\rho_1^{(k)}, ..., \rho_{m_k}^{(k)}$ be the zeros of $L(s, \chi_k)$ lying inside W. There is at least one k with $m_k > 0$. Put

$$F_{0,1}(s) = \prod_{\nu=1}^{m_0} (s - \rho_\nu^{(0)})^{-1}, \; F_{0,2}(s) = \prod_{\nu=m_0+1}^{l_0} (s - \rho_\nu^{(0)})^{-1}, \tag{5.6}$$

$$F_{k,1}(s) = \prod_{\nu=1}^{m_k} (s - \rho_\nu^{(k)})^{\alpha_k}, \; F_{k,2}(s) = \prod_{\nu=m_k+1}^{l_k} (s - \rho_\nu^{(k)})^{\alpha_k} \qquad (k \in \mathbb{N}) \tag{5.7}$$

$$H(s) = H_0^{-1}(s) \prod_{k \in \mathbb{N}} H_k^{\alpha_k}(s), \tag{5.8}$$

$$F_j(s) = F_{0,j}(s) \prod_{k \in \mathbb{N}} F_{k,j}(s) \qquad (j = 1, 2). \tag{5.9}$$

All functions $F_{k,1}$ are regular and $\neq 0$ on

$$G_1 = R_2 \backslash \{s, \bar{\sigma} < \sigma < 1, \tau^* - \xi < \tau < \tau^* + \xi\}, \tag{5.10}$$

and all $F_{k,2}$ are regular and $\neq 0$ on

$$G_2 = \operatorname{Inn}(W) \cup \{s, 1 < \sigma < 2, \tau^* - 1 < \tau < \tau^* + 1\}. \tag{5.11}$$

Because of (5.5) and (1.7),

$$H \text{ is regular and } \neq 0 \text{ on } R_2. \tag{5.12}$$

For $s \in G_1 \cup W$ and $k \geqslant k_0$, Lemma 2(a) gives

$$F_{k,1}(s) = \exp(O(\alpha_k \ln(k\tau') \, |\ln \alpha_k|)).$$

Lemma 2(b) and (1.7) therefore imply

$$F_1 \text{ is regular on } G_1, \text{ continuous and } \neq 0 \text{ on } G_1 \cup W. \tag{5.13}$$

Similarly,

$$F_2 \text{ is regular on } G_2, \text{ continous and } \neq 0 \text{ on } G_2 \cup W, \tag{5.14}$$

$$\prod_{k \in \mathbb{N}} \prod_{\nu=1}^{l_k} (s - \rho_\nu^{(k)})^{\alpha_k} \tag{5.15}$$

is absolutely and uniformly convergent on W.

In $\{s, 1<\sigma<2, \tau^*-1<\tau<\tau^*+1\}$ we obviously have

$$F(s)=F_1(s)\,F_2(s)\,H(s). \tag{5.16}$$

As F is regular on R_2, (5.12), ..., (5.16) yield that $F_2=F/F_1H$ and $F_1=F/F_2H$ can be extended to functions regular on R_2. By the continuity of F_1 and F_2 on W we finally get , for $s\in W$,

$$F(s)=H(s)\prod_{\nu=1}^{l_0}(s-\rho_\nu^{(0)})^{-1}\cdot\prod_{k\in\mathbb{N}}\prod_{\nu=1}^{l_k}(s-\rho_\nu^{(k)})^{\alpha_k}. \tag{5.17}$$

F is $\neq 0$ on W, the product converges uniformly.

Let N be the number of zeros of f inside W. Then

$$N=\frac{1}{2\pi i}\int_W \frac{F'}{F}(s)\,ds=-m_0+\sum_{k\in\mathbb{N}} m_k\alpha_k. \tag{5.18}$$

In the case $l_0>0$, $m_k=0$ for $k\in\mathbb{N}$, (5.18) is impossible. In the case $m_k>0$ for some $k\in\mathbb{N}$, (5.18) contradicts Lemma 1.

If W has to be chosen with $s=1$ inside one gets, with $m_0=0$,

$$N=1+\sum_{k\in\mathbb{N}} m_k\alpha_k$$

with at least one $m_k>0$. This gives again a contradiction to Lemma 1.

By this the desired inequality is proved.

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References

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2. E. C. Titchmarch, "The Theory of the Riemann Zeta-Function," Oxford, 1951.