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Helical polynomial curves and double Pythagorean hodographs II. Enumeration of low-degree curves

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ABSTRACT

A "double" Pythagorean-hodograph (DPH) curve $\mathbf{r}(t)$ is characterized by the property that $|\mathbf{r}'(t)|$ and $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ are *both* polynomials in the curve parameter t. Such curves possess rational Frenet frames and curvature/torsion functions, and encompass all helical polynomial curves as special cases. As noted by Beltran and Monterde, the Hopf map representation of spatial PH curves appears better suited to the analysis of DPH curves than the quaternion form. A categorization of all DPH curve types up to degree 7 is developed using the Hopf map form, together with algorithms for their construction, and a selection of computed examples of (both helical and non-helical) DPH curves is included, to highlight their attractive features. For helical curves, a separate constructive approach proposed by Monterde, based upon the inverse stereographic projection of rational line/circle descriptions in the complex plane, is used to classify all types up to degree 7. Criteria to distinguish between the helical and non-helical DPH curves, in the context of the general construction procedures, are also discussed.

1. Introduction

In a companion paper (Farouki et al., 2009), the algebraic structure of polynomial curves $\mathbf{r}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$ that incorporate a "double" Pythagorean hodograph property was described. Namely, the components of the first derivative $\mathbf{r}'(t)$ and the cross product $\mathbf{r}'(t) \times \mathbf{r}''(t)$ of the first and

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second derivatives of these curves satisfy the Pythagorean identities

$$|\mathbf{r}'|^2 = x'^2 + y'^2 + z'^2 \equiv \sigma^2, \tag{1}$$

$$|\mathbf{r}' \times \mathbf{r}''|^2 = (y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2 \equiv (\sigma\omega)^2, \tag{2}$$

for polynomials $\sigma(t)$ and $\omega(t)$. This structure was first implicitly recognized in the study (Farouki et al., 2004) of helical polynomial curves — its significance was explicitly emphasized by Beltran and Monterde (2007), who noted that the "DPH" curves satisfying (1) and (2) comprise the set of polynomial space curves whose Frenet frames, defined (Kreyszig, 1959) by

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \qquad \mathbf{p} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}, \qquad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|},$$

and curvature $\kappa = |\mathbf{r}'|^{-3}|\mathbf{r}' \times \mathbf{r}''|$ and torsion $\tau = |\mathbf{r}' \times \mathbf{r}''|^{-2}\,(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$ are *rational* functions (Wagner and Ravani, 1997) of the curve parameter t. Moreover, the DPH curves encompass all *helical* polynomial curves (Beltran and Monterde, 2007; Farouki et al., 2004) as a proper subset.

Choi et al. (2002) introduced two algebraic characterizations of solutions to condition (1), based on *quaternions* and the *Hopf map*, that are extremely useful in the construction and analysis of spatial PH curves. In the quaternion model, the hodograph $\mathbf{r}'(t)$ is expressed as a product of the form[2]

$$\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t), \tag{3}$$

$\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$ being a quaternion polynomial of degree m for a PH curve of degree $n = 2m + 1$. Using the Hopf map, $H : \mathbb{C}^2 \to \mathbb{R}^3$, the hodograph is constructed from two complex polynomials $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\,v(t)$, $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}\,p(t)$ of degree m according to

$$\begin{aligned}\mathbf{r}'(t) &= H(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) \\ &= (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t))).\end{aligned} \tag{4}$$

By identifying the imaginary unit i with the quaternion basis element $\mathbf{i}$, the relation between the two models may be expressed as

$$\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t). \tag{5}$$

Although the quaternion form (3) has seen more widespread use in developing practical algorithms (Farouki et al., 2002, 2008) for the spatial PH curves, Beltran and Monterde (2007) have shown that the Hopf map form (4) offers, in certain respects, a more natural context in which to study the *double* PH structure (1) and (2). In the quaternion model, condition (2) requires the components $u(t)$, $v(t)$, $p(t)$, $q(t)$ of the quaternion polynomial $\mathcal{A}(t)$ to satisfy

$$\begin{aligned} up' - u'p + vq' - v'q &= h(a^2 - b^2), \\ uq' - u'q - vp' + v'p &= 2hab, \\ \omega &= 2h(a^2 + b^2), \end{aligned} \tag{6}$$

for real polynomials $h(t)$, $a(t)$, $b(t)$ where $\gcd(a(t), b(t)) =$ constant (Kubota, 1972). Using the Hopf map model, this requirement acquires a simple expression in terms of the complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ — namely,

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = h(t)\,\mathbf{w}^2(t), \tag{7}$$

where $h(t)$ and $\mathbf{w}(t) = a(t) + \mathrm{i}\,b(t)$ are real and complex polynomials, with $\gcd(a(t), b(t)) =$ constant. As observed by Monterde (in press), the Hopf map also offers a more intuitive approach to constructing *helical* polynomial curves.

The plan for the remainder of this paper is as follows. First, the double PH condition is thoroughly analyzed in Section 2: after briefly reviewing the established forms of DPH cubics and quintics, three

[2] Quaternions are denoted by calligraphic letters, complex numbers and vectors by bold letters (the meaning should be clear from the context), and real numbers by italic letters.

distinct types of degree 7 DPH curves are enumerated, and systems of equations and constraints are developed that facilitate their construction. The focus of Section 3 is on the *helical* DPH curves, using the approach of Monterde (in press) based upon rational line/circle parameterizations in the complex plane. Section 4 presents criteria to distinguish between the helical and non-helical DPH curves of each type. Finally, Section 5 provides a comprehensive selection of examples of both the helical and non-helical degree 7 DPH curves, computed using the procedures developed in the preceding sections, while Section 6 summarizes key results of this paper and identifies some problems worthy of further consideration.

2. Classification of low-degree DPH curves

Following Beltran and Monterde (2007) and Monterde (in press), greater emphasis will be placed here on expression (7) of the double PH condition in the Hopf map model. Since the combination

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) \tag{8}$$

of the complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ plays a prominent role in the ensuing discussion, it merits a special name. We call it the *proportionality polynomial* for $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ — see Farouki et al. (2009). In Section 2.1 we review how the proportionality polynomial for all PH cubics, and all *helical* PH quintics, satisfies the DPH condition (7), and Section 2.2 then discusses the case of degree 7 DPH curves. The polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are assumed to be specified in the Bernstein form,

$$\boldsymbol{\alpha}(t) = \sum_{l=0}^{m} \boldsymbol{\alpha}_l \binom{m}{l} (1-t)^{m-l} t^l, \qquad \boldsymbol{\beta}(t) = \sum_{l=0}^{m} \boldsymbol{\beta}_l \binom{m}{l} (1-t)^{m-l} t^l.$$

2.1. Double PH cubics and quintics

Spatial PH cubics are defined by two linear complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$. In this case, since

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = \boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0$$

is just a complex constant, we must have $\deg(h(t)) = 0$ and $\deg(\mathbf{w}(t)) = 0$ to satisfy (7). We may, without loss of generality, take $h(t) = 1$ and $\mathbf{w}(t) = \mathbf{w}_0$, and the double PH condition then amounts to

$$\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0 = \mathbf{w}_0^2.$$

Clearly, this is satisfied for arbitrary complex values $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_1$ by taking either of the complex values $\sqrt{\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0}$ for $\mathbf{w}_0$. Hence *every* spatial PH cubic is a double PH curve — and is also a helical curve (Farouki and Sakkalis, 1994).

Spatial PH quintics are defined by quadratic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$. In this case, the proportionality polynomial (8) is the quadratic

$$2\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)\,(1-t)^2 + (\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0)\,2(1-t)t + 2\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1)\,t^2$$

and satisfaction of the double PH condition (7) can be achieved with either (a) $\deg(h(t)) = 0$ and $\deg(\mathbf{w}(t)) = 1$; or (b) $\deg(h(t)) = 2$ and $\deg(\mathbf{w}(t)) = 0$.

2.1.1. *The case* $\deg(h) = 0$ *and* $\deg(\mathbf{w}) = 1$

Choosing $h(t) = 1$ and a linear polynomial with Bernstein coefficients $\mathbf{w}_0$, $\mathbf{w}_1$ for $\mathbf{w}(t)$ in (7) for case (a), we obtain the equations

$$2\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) = \mathbf{w}_0^2, \qquad (\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) = \mathbf{w}_0\mathbf{w}_1, \qquad 2\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) = \mathbf{w}_1^2.$$

These equations can be satisfied for some $\mathbf{w}_0$, $\mathbf{w}_1$ if and only if the coefficients of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ satisfy

$$4\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) = (\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0)^2.$$

2.1.2. The case $\deg(h) = 2$ *and* $\deg(\mathbf{w}) = 0$

Taking a quadratic with Bernstein coefficients h_0, h_1, h_2 for $h(t)$ and $\mathbf{w}(t) = \mathbf{w}_0$ in case (b), and equating coefficients of the quadratic polynomials on the left and right in (7), yield the system of equations

$$2\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) = h_0\mathbf{w}_0^2, \qquad (\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) = h_1\mathbf{w}_0^2, \qquad 2\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) = h_2\mathbf{w}_0^2$$

which can be satisfied if and only if

$$\arg(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) = \arg(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) = \arg(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) \pmod{\pi}$$

— i.e., the complex numbers $\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0, \boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0, \boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ must be *real multiples* of each other in this case.

2.2. Double PH curves of degree 7

Spatial PH curves of degree 7 are specified by two cubic complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$. In this case, the proportionality polynomial (8) is the quartic

$$\begin{aligned} &3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)\,(1-t)^4 + \frac{3}{2}\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0)\,4(1-t)^3t \\ &\quad + \left[\frac{1}{2}\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0) + \frac{3}{2}\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1)\right] 6(1-t)^2t^2 \\ &\quad + \frac{3}{2}\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_1)\,4(1-t)t^3 + 3\,(\boldsymbol{\alpha}_2\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_2)\,t^4, \end{aligned}$$

and (7) may be satisfied with either (a) $\deg(h(t)) = 0$ and $\deg(\mathbf{w}(t)) = 2$; or (b) $\deg(h(t)) = 2$ and $\deg(\mathbf{w}(t)) = 1$; or (c) $\deg(h(t)) = 4$ and $\deg(\mathbf{w}(t)) = 0$. Note that the six complex values $\boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\alpha}_j\boldsymbol{\beta}_i$ for $0 \le i, j \le 3$ occurring in the coefficients of $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ are not independent: they must satisfy the *compatibility condition*

$$\begin{aligned} &(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)(\boldsymbol{\alpha}_2\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_2) \\ &\quad = (\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0)(\boldsymbol{\alpha}_1\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_1) - (\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0). \end{aligned} \tag{9}$$

2.2.1. The case $\deg(h) = 0$ *and* $\deg(\mathbf{w}) = 2$

If we choose $h(t) = 1$ and a quadratic with Bernstein coefficients $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ for $\mathbf{w}(t)$ in case (a), we obtain from (7) the equations

$$\begin{aligned} 3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) &= \mathbf{w}_0^2, \\ 3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) &= 2\,\mathbf{w}_0\mathbf{w}_1, \\ (\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0) + 3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) &= \frac{2}{3}\,(2\mathbf{w}_1^2 + \mathbf{w}_0\mathbf{w}_2), \\ 3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_1) &= 2\,\mathbf{w}_1\mathbf{w}_2, \\ 3\,(\boldsymbol{\alpha}_2\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_2) &= \mathbf{w}_2^2. \end{aligned} \tag{10}$$

Setting $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \mathbf{z}$, $\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0 = \frac{4}{3}\mathbf{w}_1^2 + \frac{2}{3}\mathbf{w}_0\mathbf{w}_2 - 3\mathbf{z}$, and invoking (9), we note that the values $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ and $\mathbf{z}$ must satisfy

$$\left(\frac{1}{3}\mathbf{w}_0^2\right)\left(\frac{1}{3}\mathbf{w}_2^2\right) = \left(\frac{2}{3}\mathbf{w}_0\mathbf{w}_1\right)\left(\frac{2}{3}\mathbf{w}_1\mathbf{w}_2\right) - \mathbf{z}\left(\frac{4}{3}\mathbf{w}_1^2 + \frac{2}{3}\mathbf{w}_0\mathbf{w}_2 - 3\mathbf{z}\right),$$

which reduces to the quadratic equation

$$27\,\mathbf{z}^2 - (12\mathbf{w}_1^2 + 6\mathbf{w}_0\mathbf{w}_2)\,\mathbf{z} + (4\mathbf{w}_1^2 - \mathbf{w}_0\mathbf{w}_2)\mathbf{w}_0\mathbf{w}_2 = 0$$

in $\mathbf{z}$. The solutions of this equation indicate that, in this case, $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ must be given in terms of $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ by

$$\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \frac{1}{3}\mathbf{w}_0\mathbf{w}_2 \quad \text{or} \quad \frac{1}{9}(4\mathbf{w}_1^2 - \mathbf{w}_0\mathbf{w}_2). \tag{11}$$

2.2.2. The case $\deg(h) = 2$ *and* $\deg(\mathbf{w}) = 1$

In case (b) we take $h(t)$ quadratic and $\mathbf{w}(t)$ linear with Bernstein coefficients h_0, h_1, h_2 and $\mathbf{w}_0, \mathbf{w}_1$ and thus obtain from (7) the equations

$$\begin{aligned}
3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) &= h_0\mathbf{w}_0^2, \\
3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) &= h_1\mathbf{w}_0^2 + h_0\mathbf{w}_0\mathbf{w}_1, \\
(\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0) + 3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) &= \frac{1}{3}\,(h_2\mathbf{w}_0^2 + 4h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2), \\
3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_1) &= h_2\mathbf{w}_0\mathbf{w}_1 + h_1\mathbf{w}_1^2, \\
3\,(\boldsymbol{\alpha}_2\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_2) &= h_2\mathbf{w}_1^2.
\end{aligned} \tag{12}$$

Setting $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \mathbf{z}$, $\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0 = \frac{1}{3}\,(h_2\mathbf{w}_0^2 + 4h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2) - 3\mathbf{z}$, we see from (9) that the values $h_0, h_1, h_2, \mathbf{w}_0, \mathbf{w}_1, \mathbf{z}$ must satisfy

$$\begin{aligned}
\left(\frac{1}{3}h_0\mathbf{w}_0^2\right)\left(\frac{1}{3}h_2\mathbf{w}_1^2\right) \;=\; & \frac{1}{3}(h_1\mathbf{w}_0^2 + h_0\mathbf{w}_0\mathbf{w}_1)\,\frac{1}{3}(h_2\mathbf{w}_0\mathbf{w}_1 + h_1\mathbf{w}_1^2) \\
& - \mathbf{z}\left[\frac{1}{3}(h_2\mathbf{w}_0^2 + 4h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2) - 3\,\mathbf{z}\right],
\end{aligned}$$

yielding the quadratic equation

$$27\,\mathbf{z}^2 - 3\,(h_2\mathbf{w}_0^2 + 4h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2)\,\mathbf{z} + h_1\mathbf{w}_0\mathbf{w}_1(h_2\mathbf{w}_0^2 + h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2) = 0$$

whose solutions indicate that $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ must be given in terms of h_0, h_1, h_2 and $\mathbf{w}_0, \mathbf{w}_1$ by

$$\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \frac{1}{3}h_1\mathbf{w}_0\mathbf{w}_1 \quad \text{or} \quad \frac{1}{9}(h_2\mathbf{w}_0^2 + h_1\mathbf{w}_0\mathbf{w}_1 + h_0\mathbf{w}_1^2). \tag{13}$$

2.2.3. The case $\deg(h) = 4$ *and* $\deg(\mathbf{w}) = 0$

Finally, choosing $h(t)$ as a quartic with Bernstein coefficients $h_0, \ldots, h_4$ and $\mathbf{w}(t) = \mathbf{w}_0$ in case (c), and equating coefficients of the quartic polynomials on the left and right in (7), yield the equations

$$\begin{aligned}
3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0) &= h_0\mathbf{w}_0^2, \\
3\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) &= 2\,h_1\mathbf{w}_0^2, \\
(\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0) + 3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1) &= 2\,h_2\mathbf{w}_0^2, \\
3\,(\boldsymbol{\alpha}_1\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_1) &= 2\,h_3\mathbf{w}_0^2, \\
3\,(\boldsymbol{\alpha}_2\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_2) &= h_4\mathbf{w}_0^2.
\end{aligned} \tag{14}$$

Setting $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \mathbf{z}$, $\boldsymbol{\alpha}_0\boldsymbol{\beta}_3 - \boldsymbol{\alpha}_3\boldsymbol{\beta}_0 = 2h_2\mathbf{w}_0^2 - 3\mathbf{z}$, and invoking (9), we see that the values $h_0, \ldots, h_4$, $\mathbf{w}_0$, and $\mathbf{z}$ must satisfy

$$\left(\frac{1}{3}h_0\mathbf{w}_0^2\right)\left(\frac{1}{3}h_4\mathbf{w}_0^2\right) = \left(\frac{2}{3}h_1\mathbf{w}_0^2\right)\left(\frac{2}{3}h_3\mathbf{w}_0^2\right) - \mathbf{z}\,(2h_2\mathbf{w}_0^2 - 3\mathbf{z}),$$

which reduces to the quadratic equation

$$27\,\mathbf{z}^2 - 18h_2\mathbf{w}_0^2\,\mathbf{z} + (4h_1h_3 - h_0h_4)\mathbf{w}_0^4 = 0$$

in $\mathbf{z}$. The solutions of this equation indicate that, in this case, $\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ must be given in terms of $h_0, \ldots, h_4$ and $\mathbf{w}_0$ by

$$\boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1 = \frac{1}{9}\left(3h_2 \pm \sqrt{9h_2^2 + 3h_0h_4 - 12h_1h_3}\right)\mathbf{w}_0^2. \tag{15}$$

2.3. Construction of degree 7 double PH curves

The above characterizations for degree 7 double PH curves of different types furnish algorithms for constructing examples of these curves.

First, we assign numerical values for the coefficients of $h(t)$ and $\mathbf{w}(t)$ on the right-hand side of Eqs. (10), (12), or (14). An appropriate value for $\mathbf{z} = \boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_1$ is then determined through the corresponding compatibility constraint from expression (11), (13), or (15). This assignment, together with Eqs. (10), (12), or (14), defines a system comprising six bilinear equations in the eight unknowns $\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_3$ and $\boldsymbol{\beta}_0, \ldots, \boldsymbol{\beta}_3$.

Since these equations are (by construction) consistent, and the variables are inherently *complex*, one can in principle assign two of them arbitrarily, and then solve the six equations for the remaining variables. Of course, this purely algebraic process is not suited for constructing curves with prescribed geometrical properties. We expect that it can be suitably modified to furnish more geometrically-intuitive constructions for double PH curves of different types, but the formulation of such algorithms is deferred to a future study.

Remark 1. Since Eqs. (10)–(11), (12)–(13), or (14)–(15) depend only on the combinations $\boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\alpha}_j\boldsymbol{\beta}_i$, if $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$ for $0 \le k \le 3$ is any solution, then $(\boldsymbol{\alpha}_k\mathbf{z}, \boldsymbol{\beta}_k/\mathbf{z})$ for $0 \le k \le 3$ is also a solution for each $\mathbf{z} \neq 0$. Hence, one may initially assign arbitrary complex values to any *three* of the coefficients $(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k)$ for $0 \le k \le 3$, and then determine corresponding values for the other five. This yields another freedom of initial assignment, beyond the two arising from the difference between the number of unknowns and equations.

3. Hopf map form of helical curves

A helical curve $\mathbf{r}(t)$ may be characterized by the fact that the locus traced by its unit tangent vector $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$ – i.e., the *tangent indicatrix* of $\mathbf{r}(t)$ – is a circle[3] on the unit sphere (Farouki et al., 2004). This characteristic property has been used by Monterde (in press) to give a geometrically-intuitive and quite general construction of helical polynomial curves, based on the Hopf map model, as follows.

For the hodograph defined in terms of complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ through the Hopf map construction (4), the tangent indicatrix is given by

$$\mathbf{t} = \frac{H(\boldsymbol{\alpha}, \boldsymbol{\beta})}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{(|\boldsymbol{\alpha}|^2 - |\boldsymbol{\beta}|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}), 2\,\mathrm{Im}(\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}))}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2}.$$

The final expression above defines the *normalized* Hopf map, which we denote by $\hat{H}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Note that $\hat{H}$ maps complex values $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ with $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2 = 1$ to a *unit* vector in $\mathbb{R}^3$ or, equivalently, a point on the unit sphere S^2.

As noted by Monterde (in press), the normalized Hopf map satisfies

$$\hat{H}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \hat{H}(\boldsymbol{\alpha}/\boldsymbol{\beta}, 1),$$

and hence, for the purpose of investigating the tangent indicatrix, it suffices to consider only the *ratio* $\boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ of the complex polynomials in (4). Thus, different spatial PH curves defined by integrating (4) with different choices for $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ may nevertheless exhibit identical tangent indicatrices, if

[3] Plane curves (which are trivially helical) have great-circle tangent indicatrices, while proper helical curves have small-circle tangent indicatrices. The center of the circle and its angular radius identify the helix axis $\mathbf{a}$ and pitch angle ψ — see equation (3) in Farouki et al. (2009).

they have the same ratio $\boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$. Such curves differ in the *magnitude*, but not the *direction*, of their hodograph vectors $\mathbf{r}'(t)$ at each parameter value t.

Now the ratio $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ of the polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ specifies a rational curve in the complex plane, and through the normalized Hopf map an image $\mathbf{c}(t) = \hat{H}(\mathbf{z}(t), 1)$ of this curve on the unit sphere S^2 in $\mathbb{R}^3$ is defined, with $|\mathbf{c}(t)| \equiv 1$. In fact, as observed by Monterde (in press), the map $\mathbf{z} \to \hat{H}(\mathbf{z}, 1)$ from $\mathbb{C}$ to S^2 is just the inverse of the familiar *stereographic projection*, used in complex analysis to visualize the "extended" complex plane (Needham, 1997). Drawing rays from the north pole of S^2 through each point $\mathbf{z} \in \mathbb{C}$, we associate with each $\mathbf{z}$ the point of S^2 at which such a ray pierces the sphere. In this manner, "infinitely distant" points in $\mathbb{C}$ – regardless of direction – are all mapped to the north pole of S^2, and we regard the extended complex plane as comprising all finite complex values $\mathbf{z}$ augmented by the single value ∞.

As is well known (Needham, 1997), all circles on S^2 are mapped to either lines or circles in $\mathbb{C}$ by stereographic projection, depending on whether or not the circle on S^2 passes through the north pole. Monterde (in press) thus observes that, if we are interested in helical polynomial curves, with circular tangent indicatrices on S^2, their construction can be reduced by the above arguments to identifying those pairs of complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ whose ratios $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ define rational parameterizations of lines or circles in $\mathbb{C}$.

3.1. Complex representation of lines/circles

Given complex numbers $\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1$ that satisfy $\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0 \neq 0$, consider the complex-valued function

$$\mathbf{z}(t) = \frac{\mathbf{a}_0(1-t) + \mathbf{a}_1 t}{\mathbf{b}_0(1-t) + \mathbf{b}_1 t} \tag{16}$$

of a real parameter t. This may be viewed as a mapping $t \to \mathbf{z}(t)$ of the real axis to a locus in the complex plane, as specified by a *Möbius transformation*. Form (16) defines all *lines* and *circles* in the complex plane (Needham, 1997; Schwerdtfeger, 1979).

If $\mathbf{b}_1\overline{\mathbf{b}}_0 - \overline{\mathbf{b}}_1\mathbf{b}_0 = 2\,\mathrm{i}\,\mathrm{Im}(\mathbf{b}_1\overline{\mathbf{b}}_0) \neq 0$, expression (16) defines a circle — one can easily verify that

$$\mathbf{z}_c = \frac{\mathbf{a}_1\overline{\mathbf{b}}_0 - \mathbf{a}_0\overline{\mathbf{b}}_1}{\mathbf{b}_1\overline{\mathbf{b}}_0 - \overline{\mathbf{b}}_1\mathbf{b}_0} \qquad \text{and} \qquad R = \left|\frac{\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0}{\mathbf{b}_1\overline{\mathbf{b}}_0 - \overline{\mathbf{b}}_1\mathbf{b}_0}\right|$$

identify the center and radius, so that

$$|\mathbf{z}(t) - \mathbf{z}_c|^2 \equiv R^2.$$

If $\mathbf{b}_1\overline{\mathbf{b}}_0 - \overline{\mathbf{b}}_1\mathbf{b}_0 = 0$, however, $\mathbf{z}_c$ and R become infinite, and $\mathbf{z}(t)$ degenerates to a straight line. This may be seen by noting that the derivative

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \frac{\mathbf{a}_1\mathbf{b}_0 - \mathbf{a}_0\mathbf{b}_1}{[\mathbf{b}_0(1-t) + \mathbf{b}_1 t]^2}$$

has direction specified by

$$\arg\left(\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t}\right) = \arg(\mathbf{a}_1\mathbf{b}_0 - \mathbf{a}_0\mathbf{b}_1) - 2\arg(\mathbf{b}_0(1-t) + \mathbf{b}_1 t).$$

Writing $\mathbf{b}_0 = b_0 + \mathrm{i}\,\beta_0$ and $\mathbf{b}_1 = b_1 + \mathrm{i}\,\beta_1$, we note that

$$\arg(\mathbf{b}_0(1-t) + \mathbf{b}_1 t) = \tan^{-1}\frac{\beta_0(1-t) + \beta_1 t}{b_0(1-t) + b_1 t} \pmod{\pi}.$$

Now $[\beta_0(1-t) + \beta_1 t] \,/\, [b_0(1-t) + b_1 t] =$ constant, so $\arg(\mathrm{d}\mathbf{z}/\mathrm{d}t) =$ constant $\pmod{\pi}$, if and only if $b_0\beta_1 - b_1\beta_0 = \mathrm{Im}(\mathbf{b}_1\overline{\mathbf{b}}_0) = (\mathbf{b}_1\overline{\mathbf{b}}_0 - \overline{\mathbf{b}}_1\mathbf{b}_0)/2\,\mathrm{i} = 0$.

The condition $b_0\beta_1 - b_1\beta_0 = 0$ implies that the complex coefficients in the denominator of (16) are of the form $(\mathbf{b}_0, \mathbf{b}_1) = (k_0\mathbf{w}, k_1\mathbf{w})$ for some complex value $\mathbf{w}$ and real values k_0, k_1. Writing $\mathbf{c}_0 = \mathbf{a}_0/\mathbf{w}$,

$\mathbf{c}_1 = \mathbf{a}_1/\mathbf{w}$ we see that, for a straight line, form (16) can be reduced to

$$\mathbf{z}(t) = \frac{\mathbf{c}_0(1-t) + \mathbf{c}_1 t}{k_0(1-t) + k_1 t}, \tag{17}$$

i.e., straight lines may be characterized by *real* denominators.

In order to construct different helical curve types, there are two ways to generate higher-order line/circle parameterizations from the basic form (16). We may multiply both the numerator and denominator of (16) by a complex polynomial, to obtain $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ where $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) \neq$ constant — this does not change the tangent indicatrix, but it does alter the *magnitude* of the hodograph $\mathbf{r}'(t)$ upon substituting $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ into (4). Curves defined in this manner are *monotone helical*, since they originate from faithful circle parameterizations. Alternatively, a rational transformation $t \to f(t)/g(t)$ of the curve parameter may be invoked, defined by real polynomials[4] $f(t)$, $g(t)$ of degree $\geq$ 2. This yields, in general, a parameterization $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ with $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant that is not faithful, and the resulting curves are general helices — i.e., they may reverse their sense of tangent rotation. These "multiplication" and "re-parameterization" procedures for generating helical curves may also be combined, but only for curves of degree ≥ 7.

3.2. Spatial PH cubics

The rational linear form (16) is the simplest (lowest-order) parameterization of lines and circles. By substituting the linear complex polynomials

$$\boldsymbol{\alpha}(t) = \mathbf{a}_0(1-t) + \mathbf{a}_1 t, \qquad \boldsymbol{\beta}(t) = \mathbf{b}_0(1-t) + \mathbf{b}_1 t$$

into the Hopf map specification (4) of a spatial Pythagorean hodograph and integrating, we obtain a spatial PH cubic. In this case, we have

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = \mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0,$$

which may be interpreted as being of the form (7) with $h(t) = 1$ and $\mathbf{w}^2(t) = \mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0$, i.e., $\deg(h(t)) = 0$ and $\deg(\mathbf{w}(t)) = 0$. Hence, all spatial PH cubics are helical, and are also double PH curves.

In Farouki et al. (2004) a curve was said to be *monotone helical* if its tangent indicatrix is a simply-traced circle on the unit sphere — i.e., it does not indicate any reversals in the sense of the tangent rotation. Since the rational linear form (16) is a faithful parameterization of lines/circles in the complex plane, and the inverse stereographic projection from the complex plane to the unit sphere is one-to-one, all PH cubics are monotone helical.

3.3. Helical PH quintics

To define helical PH quintics by means of the normalized Hopf map, we must use rational *quadratic* parameterizations of lines and circles in the complex plane. These must be true quadratic parameterizations, not degree-elevated versions of (16). There are two essentially distinct methods of obtaining such quadratic parameterizations from the basic form (16).

3.3.1. Quadratic re-parameterization

The first method involves introducing a non-linear (real) transformation of the parameter t. Imposing on (16) the parameter transformation defined by the rational quadratic function

$$t \;\to\; \frac{f(t)}{g(t)} = \frac{f_0(1-t)^2 + f_1 2(1-t)t + f_2 t^2}{g_0(1-t)^2 + g_1 2(1-t)t + g_2 t^2}, \tag{18}$$

[4] These polynomials are assumed to be relatively prime, i.e., $\gcd(f(t), g(t)) =$ constant.

we obtain the quadratic line/circle parameterization

$$\mathbf{z}(t) = \frac{\boldsymbol{\alpha}(t)}{\boldsymbol{\beta}(t)} = \frac{\boldsymbol{\alpha}_0(1-t)^2 + \boldsymbol{\alpha}_1 2(1-t)t + \boldsymbol{\alpha}_2 t^2}{\boldsymbol{\beta}_0(1-t)^2 + \boldsymbol{\beta}_1 2(1-t)t + \boldsymbol{\beta}_2 t^2},$$

where

$$\boldsymbol{\alpha}_i = f_i(\mathbf{a}_1 - \mathbf{a}_0) + g_i\mathbf{a}_0, \qquad \boldsymbol{\beta}_i = f_i(\mathbf{b}_1 - \mathbf{b}_0) + g_i\mathbf{b}_0, \quad i = 0, 1, 2.$$

We then find that the proportionality polynomial has the form

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = h(t)\,(\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0),$$

where $h(t)$ is the real quadratic polynomial defined by

$$h(t) = f'(t)g(t) - f(t)g'(t), \tag{19}$$

with Bernstein coefficients

$$h_0 = 2(f_1g_0 - f_0g_1), \qquad h_1 = f_2g_0 - f_0g_2, \qquad h_2 = 2(f_2g_1 - f_1g_2). \tag{20}$$

This is an instance of (7) with $\deg(h(t)) = 2$, $\deg(\mathbf{w}(t)) = 0$. The spatial PH quintics defined in this manner are thus double PH curves — as observed by Beltran and Monterde (2007), they correspond to general helical PH quintics.

Invoking relation (5) between the quaternion and Hopf map models, we see that these helical quintics may be specified by a quadratic quaternion polynomial $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$ with Bernstein coefficients of the form

$$\mathcal{A}_i = f_i\,[\mathbf{a}_1 - \mathbf{a}_0 + \mathbf{k}\,(\mathbf{b}_1 - \mathbf{b}_0)] + g_i\,[\mathbf{a}_0 + \mathbf{k}\,\mathbf{b}_0], \quad i = 0, 1, 2. \tag{21}$$

Since $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ are linearly dependent upon just two quaternions, $\mathbf{a}_1 - \mathbf{a}_0 + \mathbf{k}\,(\mathbf{b}_1 - \mathbf{b}_0)$ and $\mathbf{a}_0 + \mathbf{k}\,\mathbf{b}_0$, they reside in a two-dimensional subspace of $\mathbb{H}$. Hence, as noted in Farouki et al. (2004), these helical PH curves are characterized by the fact that $\mathcal{A}_1$ is linearly dependent on $\mathcal{A}_0$ and $\mathcal{A}_2$, i.e.,

$$\mathcal{A}_1 = \mathcal{A}_0\,c_0 + \mathcal{A}_2\,c_2$$

for appropriate values $c_0, c_2 \in \mathbb{R}$. Substituting from (21) into this relation, we find that these coefficients are given in terms of the quantities (20) by

$$c_0 = \frac{h_2}{2h_1} \quad \text{and} \quad c_2 = \frac{h_0}{2h_1}.$$

3.3.2. *Linear polynomial multiplication*

A different way of obtaining a quadratic rational parameterization from (16) is to multiply the numerator and denominator by the same (complex) linear polynomial, $\mathbf{w}(t) = \mathbf{w}_0(1-t) + \mathbf{w}_1 t$. The parameterization $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ defined in this manner is specified by

$$\begin{aligned}
\boldsymbol{\alpha}(t) &= [\mathbf{a}_0(1-t) + \mathbf{a}_1 t]\,[\mathbf{w}_0(1-t) + \mathbf{w}_1 t], \\
&= \mathbf{a}_0\mathbf{w}_0(1-t)^2 + \frac{1}{2}(\mathbf{a}_0\mathbf{w}_1 + \mathbf{a}_1\mathbf{w}_0)2(1-t)t + \mathbf{a}_1\mathbf{w}_1 t^2, \\
\boldsymbol{\beta}(t) &= [\mathbf{b}_0(1-t) + \mathbf{b}_1 t]\,[\mathbf{w}_0(1-t) + \mathbf{w}_1 t], \\
&= \mathbf{b}_0\mathbf{w}_0(1-t)^2 + \frac{1}{2}(\mathbf{b}_0\mathbf{w}_1 + \mathbf{b}_1\mathbf{w}_0)2(1-t)t + \mathbf{b}_1\mathbf{w}_1 t^2,
\end{aligned}$$

and hence we obtain

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = (\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0)\,[\mathbf{w}_0(1-t) + \mathbf{w}_1 t]^2.$$

Clearly, this corresponds to the case where $\deg(h(t)) = 0$ and $\deg(\mathbf{w}(t)) = 1$ in (7) — note that the factor $\sqrt{\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0}$ can be absorbed into $\mathbf{w}_0, \mathbf{w}_1$. As observed by Beltran and Monterde (Beltran and Monterde, 2007), this case corresponds to the monotone-helical PH quintics. The reason for this is clear from the present arguments: obviously, multiplying the numerator and denominator of (16) by the *same* complex polynomial $\mathbf{w}(t)$ does not change the faithfulness of the line/circle parameterization (i.e., the monotonicity of the tangent indicatrix). The sole effect of this multiplication is to modulate the hodograph *magnitude* $|\mathbf{r}'(t)|$ by the factor $|\mathbf{w}(t)|^2$ — the direction of $\mathbf{r}'(t)$ remains unchanged.

In Farouki et al. (2004) the monotone-helical PH quintics were characterized in terms of the quaternion model by the fact that their quaternion coefficients satisfy

$$\mathcal{A}_1 = \mathcal{A}_0\,\mathbf{c}_0 + \mathcal{A}_2\,\mathbf{c}_2, \tag{22}$$

$\mathbf{c}_0 = c_0 + \gamma_0\,\mathbf{i}$, $\mathbf{c}_2 = c_2 + \gamma_2\,\mathbf{i}$ being complex numbers (regarded as quaternions with vanishing $\mathbf{j}$ and $\mathbf{k}$ components) that satisfy

$$4\,\mathbf{c}_0\mathbf{c}_2 = 1. \tag{23}$$

We can verify that this is equivalent to the above Hopf map characterization by invoking relation (5) between the Hopf map and quaternion models.

For $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ as defined above, we obtain the quadratic quaternion polynomial $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$ with Bernstein coefficients

$$\begin{aligned}
\mathcal{A}_0 &= (\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_0, \\
\mathcal{A}_1 &= \frac{1}{2}\,[(\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_1 + (\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_0], \\
\mathcal{A}_2 &= (\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_1.
\end{aligned}$$

Bearing in mind that complex numbers have commutative products, one can then verify that

$$\mathcal{A}_0\mathbf{w}_1 = (\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_1\mathbf{w}_0 \quad \text{and} \quad \mathcal{A}_2\mathbf{w}_0 = (\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_0\mathbf{w}_1,$$

and dividing (on the right) by $\mathbf{w}_0$ and $\mathbf{w}_1$, respectively, we obtain

$$\mathcal{A}_1 = \mathcal{A}_0\left(\frac{\mathbf{w}_1}{2\mathbf{w}_0}\right) + \mathcal{A}_2\left(\frac{\mathbf{w}_0}{2\mathbf{w}_1}\right).$$

$\mathcal{A}_1$ is thus of the form (22) where $\mathbf{c}_0 = \mathbf{w}_1/2\mathbf{w}_0$, $\mathbf{c}_2 = \mathbf{w}_0/2\mathbf{w}_1$ satisfy (23).

3.3.3. *Degenerate common case*

The cases discussed in Sections 3.3.1 and 3.3.2 are not entirely disjoint. There are specific circumstances for these two cases in which $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)$ will degenerate to a common special form. Generically, the polynomial $h(t)$ in Section 3.3.1 is a "true" quadratic — i.e., its discriminant is non-zero, and it is not the square of a linear polynomial. If its coefficients satisfy $h_0h_2 = h_1^2$, however, $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)$ will be the product of a complex constant and the square of a real linear polynomial.

Likewise, the coefficients of the polynomial $\mathbf{w}_0(1-t)+\mathbf{w}_1t$ in Section 3.3.2 are generically linearly independent — i.e., $(\mathbf{w}_0, \mathbf{w}_1) \neq (\mathbf{c}w_0, \mathbf{c}w_1)$ for some complex value $\mathbf{c}$ and real values w_0, w_1. However, if the polynomial is of the form $\mathbf{c}\,[w_0(1-t) + w_1t]$, then $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ in this case is also the product of a complex constant and the square of a real linear polynomial.

3.4. *Helical PH curves of degree 7*

Helical PH curves of degree 7 may be generated through the normalized Hopf map using *cubic* parameterizations of lines and circles in the complex plane. These may be constructed in three essentially distinct ways — of which two are direct extensions of the methods employed above for helical PH quintics, and the third is a "hybrid" of these two.

3.4.1. *Cubic re-parameterization*

By analogy with the method used in Section 3.3.1, a rational cubic line/circle parameterization is defined by imposing the parameter transformation

$$t \;\to\; \frac{f(t)}{g(t)} = \frac{f_0(1-t)^3 + f_1 3(1-t)^2 t + f_2 3(1-t)t^2 + f_3 t^3}{g_0(1-t)^3 + g_1 3(1-t)^2 t + g_2 3(1-t)t^2 + g_3 t^3} \tag{24}$$

on (16). This yields

$$\mathbf{z}(t)=\frac{\boldsymbol{\alpha}(t)}{\boldsymbol{\beta}(t)}=\frac{\boldsymbol{\alpha}_0(1-t)^3+\boldsymbol{\alpha}_1 3(1-t)^2t+\boldsymbol{\alpha}_2 3(1-t)t^2+\boldsymbol{\alpha}_3 t^3}{\boldsymbol{\beta}_0(1-t)^3+\boldsymbol{\beta}_1 3(1-t)^2t+\boldsymbol{\beta}_2 3(1-t)t^2+\boldsymbol{\beta}_3 t^3},$$

where

$$\boldsymbol{\alpha}_i=f_i(\mathbf{a}_1-\mathbf{a}_0)+g_i\mathbf{a}_0,\qquad \boldsymbol{\beta}_i=f_i(\mathbf{b}_1-\mathbf{b}_0)+g_i\mathbf{b}_0,\quad i=0,1,2,3.$$

For this type of circle parameterization, we find that

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)-\boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)=h(t)\,(\mathbf{a}_0\mathbf{b}_1-\mathbf{a}_1\mathbf{b}_0),$$

where the real quartic polynomial $h(t)$ has the form (19), and its Bernstein coefficients are given by

$$\begin{aligned}
&h_0=3(f_1g_0-f_0g_1),\qquad h_1=\frac{3}{2}(f_2g_0-f_0g_2),\\
&h_2=\frac{3}{2}(f_2g_1-f_1g_2)+\frac{1}{2}(f_3g_0-f_0g_3),\\
&h_3=\frac{3}{2}(f_3g_1-f_1g_3),\qquad h_4=3(f_3g_2-f_2g_3).
\end{aligned}\tag{25}$$

This corresponds to the case $\deg(h(t))=4$ and $\deg(\mathbf{w}(t))=0$ of (7), and it defines a general helical double PH curve of degree seven.

In the quaternion model, such curves are specified by a cubic quaternion polynomial $\mathcal{A}(t)=\boldsymbol{\alpha}(t)+\mathbf{k}\,\boldsymbol{\beta}(t)$ with Bernstein coefficients

$$\mathcal{A}_i=f_i\,[\mathbf{a}_1-\mathbf{a}_0+\mathbf{k}\,(\mathbf{b}_1-\mathbf{b}_0)]+g_i\,[\mathbf{a}_0+\mathbf{k}\,\mathbf{b}_0],\quad i=0,1,2,3.\tag{26}$$

Since $\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ are linearly dependent on the two quaternions, $\mathbf{a}_1-\mathbf{a}_0+\mathbf{k}\,(\mathbf{b}_1-\mathbf{b}_0)$ and $\mathbf{a}_0+\mathbf{k}\,\mathbf{b}_0$, they reside in a two-dimensional subspace of $\mathbb{H}$. Hence, $\mathcal{A}_1$, $\mathcal{A}_2$ must be expressible in terms of $\mathcal{A}_0$, $\mathcal{A}_3$ in the form

$$\mathcal{A}_1=\mathcal{A}_0\,c_{10}+\mathcal{A}_3\,c_{13},\qquad \mathcal{A}_2=\mathcal{A}_0\,c_{20}+\mathcal{A}_3\,c_{23},\tag{27}$$

for suitable values $c_{10},c_{13},c_{20},c_{23}\in\mathbb{R}$. Substituting from (26) into the above, these coefficients can be expressed in terms of (25) and $k=f_3g_0-f_0g_3$ as

$$c_{10}=\frac{2h_3}{3k},\qquad c_{13}=\frac{h_0}{3k},\qquad c_{20}=\frac{h_4}{3k},\qquad c_{23}=\frac{2h_1}{3k}.\tag{28}$$

3.4.2. *Quadratic polynomial multiplication*

Instead of a cubic re-parameterization, we now consider the cubic line/circle parameterizations defined by multiplying the numerator and denominator of (16) by a complex quadratic polynomial. Writing

$$\begin{aligned}
\boldsymbol{\alpha}(t)&=[\mathbf{a}_0(1-t)+\mathbf{a}_1t]\,[\mathbf{w}_0(1-t)^2+\mathbf{w}_1 2(1-t)t+\mathbf{w}_2t^2],\\
\boldsymbol{\beta}(t)&=[\mathbf{b}_0(1-t)+\mathbf{b}_1t]\,[\mathbf{w}_0(1-t)^2+\mathbf{w}_1 2(1-t)t+\mathbf{w}_2t^2],
\end{aligned}\tag{29}$$

the Bernstein coefficients of the cubics $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are given by

$$\boldsymbol{\alpha}_0=\mathbf{a}_0\mathbf{w}_0,\qquad \boldsymbol{\alpha}_1=\frac{1}{3}(2\mathbf{a}_0\mathbf{w}_1+\mathbf{a}_1\mathbf{w}_0),\qquad \boldsymbol{\alpha}_2=\frac{1}{3}(2\mathbf{a}_1\mathbf{w}_1+\mathbf{a}_0\mathbf{w}_2),\qquad \boldsymbol{\alpha}_3=\mathbf{a}_1\mathbf{w}_2,$$

$$\boldsymbol{\beta}_0=\mathbf{b}_0\mathbf{w}_0,\qquad \boldsymbol{\beta}_1=\frac{1}{3}(2\mathbf{b}_0\mathbf{w}_1+\mathbf{b}_1\mathbf{w}_0),\qquad \boldsymbol{\beta}_2=\frac{1}{3}(2\mathbf{b}_1\mathbf{w}_1+\mathbf{b}_0\mathbf{w}_2),\qquad \boldsymbol{\beta}_3=\mathbf{b}_1\mathbf{w}_2.$$

One can then verify that

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t)-\boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)=(\mathbf{a}_0\mathbf{b}_1-\mathbf{a}_1\mathbf{b}_0)\,[\mathbf{w}_0(1-t)^2+\mathbf{w}_1 2(1-t)t+\mathbf{w}_2t^2]^2,$$

corresponding to $\deg(h(t))=0$ and $\deg(\mathbf{w}(t))=2$ in (7) — note that the complex constant $\sqrt{\mathbf{a}_0\mathbf{b}_1-\mathbf{a}_1\mathbf{b}_0}$ can be absorbed into $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$. As with the case of Section 3.3.2, multiplying the numerator and denominator of (16) preserves the faithfulness of the line/circle parameterization. Hence, in this case, we have a monotone-helical double PH curve of degree 7.

Comparing with the quaternion model for this case, the cubic quaternion polynomial $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$ has the Bernstein coefficients

$$\mathcal{A}_0 = (\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_0, \qquad \mathcal{A}_1 = \frac{1}{3}\,[2(\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_1 + (\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_0],$$
$$\mathcal{A}_2 = \frac{1}{3}\,[2(\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_1 + (\mathbf{a}_0 + \mathbf{k}\mathbf{b}_0)\mathbf{w}_2], \qquad \mathcal{A}_3 = (\mathbf{a}_1 + \mathbf{k}\mathbf{b}_1)\mathbf{w}_2.$$

By arguments similar to those of Section 3.3.2, we may infer that a degree 7 PH curve is monotone helical if and only if $\mathcal{A}_1, \mathcal{A}_2$ can be expressed in terms of $\mathcal{A}_0, \mathcal{A}_3$ in the form

$$\mathcal{A}_1 = \mathcal{A}_0\,\mathbf{c}_{10} + \mathcal{A}_3\,\mathbf{c}_{13}, \qquad \mathcal{A}_2 = \mathcal{A}_0\,\mathbf{c}_{20} + \mathcal{A}_3\,\mathbf{c}_{23}, \tag{30}$$

where the coefficients $\mathbf{c}_{10}, \mathbf{c}_{13}, \mathbf{c}_{20}, \mathbf{c}_{23}$ are given in terms of $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ by

$$\mathbf{c}_{10} = \frac{2\mathbf{w}_1}{3\mathbf{w}_0}, \qquad \mathbf{c}_{13} = \frac{\mathbf{w}_0}{3\mathbf{w}_2}, \qquad \mathbf{c}_{20} = \frac{\mathbf{w}_2}{3\mathbf{w}_0}, \qquad \mathbf{c}_{23} = \frac{2\mathbf{w}_1}{3\mathbf{w}_2}, \tag{31}$$

and satisfy

$$\mathbf{c}_{10} = 3\,\mathbf{c}_{20}\mathbf{c}_{23} \quad \text{and} \quad \mathbf{c}_{23} = 3\,\mathbf{c}_{10}\mathbf{c}_{13}. \tag{32}$$

3.4.3. Degenerate common case

As with the helical PH quintics, there is a common special instance between the cases in which the cubic line/circle representation is obtained purely by re-parameterization, and purely by multiplication. If the real quartic $h(t)$ in the former case is actually the square of a real quadratic, and the complex quadratic $\mathbf{w}_0(1-t)^2 + \mathbf{w}_1 2(1-t)t + \mathbf{w}_2 t^2$ in the latter case can be written as $\mathbf{c}\,[w_0(1-t)^2 + w_1 2(1-t)t + w_2 t^2]$, then $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ is in both cases a complex constant times the square of a real quadratic polynomial.

3.4.4. Re-parameterization and multiplication

A new approach becomes possible with the degree 7 helical PH curves, since cubic line/circle parameterizations $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ can be generated from (16) in a "hybrid" manner: we can combine a quadratic re-parameterization with multiplication by a complex linear polynomial. Imposing the parameter transformation (18) on (16) and multiplying the numerator and denominator by $\mathbf{w}_0(1-t) + \mathbf{w}_1 t$, we obtain the cubics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ with Bernstein coefficients

$$\begin{aligned}
\boldsymbol{\alpha}_0 &= [f_0(\mathbf{a}_1 - \mathbf{a}_0) + g_0\mathbf{a}_0]\,\mathbf{w}_0, \\
\boldsymbol{\alpha}_1 &= \frac{1}{3}\,\{[f_0(\mathbf{a}_1 - \mathbf{a}_0) + g_0\mathbf{a}_0]\,\mathbf{w}_1 + 2\,[f_1(\mathbf{a}_1 - \mathbf{a}_0) + g_1\mathbf{a}_0]\,\mathbf{w}_0\}\,, \\
\boldsymbol{\alpha}_2 &= \frac{1}{3}\,\{[f_2(\mathbf{a}_1 - \mathbf{a}_0) + g_2\mathbf{a}_0]\,\mathbf{w}_0 + 2\,[f_1(\mathbf{a}_1 - \mathbf{a}_0) + g_1\mathbf{a}_0]\,\mathbf{w}_1\}\,, \\
\boldsymbol{\alpha}_3 &= [f_2(\mathbf{a}_1 - \mathbf{a}_0) + g_2\mathbf{a}_0]\,\mathbf{w}_1, \\
\boldsymbol{\beta}_0 &= [f_0(\mathbf{b}_1 - \mathbf{b}_0) + g_0\mathbf{b}_0]\,\mathbf{w}_0, \\
\boldsymbol{\beta}_1 &= \frac{1}{3}\,\{[f_0(\mathbf{b}_1 - \mathbf{b}_0) + g_0\mathbf{b}_0]\,\mathbf{w}_1 + 2\,[f_1(\mathbf{b}_1 - \mathbf{b}_0) + g_1\mathbf{b}_0]\,\mathbf{w}_0\}\,, \\
\boldsymbol{\beta}_2 &= \frac{1}{3}\,\{[f_2(\mathbf{b}_1 - \mathbf{b}_0) + g_2\mathbf{b}_0]\,\mathbf{w}_0 + 2\,[f_1(\mathbf{b}_1 - \mathbf{b}_0) + g_1\mathbf{b}_0]\,\mathbf{w}_1\}\,, \\
\boldsymbol{\beta}_3 &= [f_2(\mathbf{b}_1 - \mathbf{b}_0) + g_2\mathbf{b}_0]\,\mathbf{w}_1,
\end{aligned} \tag{33}$$

and in this case, we find that

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = h(t)\,(\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0)\,[\mathbf{w}_0(1-t) + \mathbf{w}_1 t]^2,$$

$h(t)$ being the real quadratic polynomial with the Bernstein coefficients (20). This corresponds to the case $\deg(h(t)) = 2$ and $\deg(\mathbf{w}(t)) = 1$ of (7).

It should be noted that the *order* of the operations characterizing this case (*first* re-parameterization, *then* multiplication) is important, since it is not possible to achieve a cubic line/circle parameterization through a polynomial multiplication followed by a re-parameterization. For this case, setting

$$\mathcal{V}_0 = \mathbf{a}_1 - \mathbf{a}_0 + \mathbf{k}\,(\mathbf{b}_1 - \mathbf{b}_0) \qquad \text{and} \qquad \mathcal{V}_1 = \mathbf{a}_0 + \mathbf{k}\,\mathbf{b}_0,$$

we find that the quaternion representation is defined by the cubic polynomial $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$ with Bernstein coefficients

$$\begin{aligned}
\mathcal{A}_0 &= (f_0\mathcal{V}_0 + g_0\mathcal{V}_1)\mathbf{w}_0, \\
\mathcal{A}_1 &= \frac{1}{3}(f_0\mathcal{V}_0 + g_0\mathcal{V}_1)\mathbf{w}_1 + \frac{2}{3}(f_1\mathcal{V}_0 + g_1\mathcal{V}_1)\mathbf{w}_0, \\
\mathcal{A}_2 &= \frac{1}{3}(f_2\mathcal{V}_0 + g_2\mathcal{V}_1)\mathbf{w}_0 + \frac{2}{3}(f_1\mathcal{V}_0 + g_1\mathcal{V}_1)\mathbf{w}_1, \\
\mathcal{A}_3 &= (f_2\mathcal{V}_0 + g_2\mathcal{V}_1)\mathbf{w}_1.
\end{aligned}$$

As in the preceding cases $\mathcal{A}_1$, $\mathcal{A}_2$ can be expressed in terms of $\mathcal{A}_0$, $\mathcal{A}_3$ as

$$\mathcal{A}_1 = \mathcal{A}_0\,\mathbf{c}_{10} + \mathcal{A}_3\,\mathbf{c}_{13}, \qquad \mathcal{A}_2 = \mathcal{A}_0\,\mathbf{c}_{20} + \mathcal{A}_3\,\mathbf{c}_{23}. \tag{34}$$

In this case, the complex coefficients $\mathbf{c}_{10}$, $\mathbf{c}_{13}$ and $\mathbf{c}_{20}$, $\mathbf{c}_{23}$ are given in terms of $\mathbf{w}_0$, $\mathbf{w}_1$, $\mathbf{w}_2$ and the quantities (20) defined in Section 3.3.1 by

$$\mathbf{c}_{10} = \frac{h_1\mathbf{w}_1 + h_2\mathbf{w}_0}{3h_1\mathbf{w}_0}, \qquad \mathbf{c}_{13} = \frac{h_0\mathbf{w}_0}{3h_1\mathbf{w}_1}, \qquad \mathbf{c}_{20} = \frac{h_2\mathbf{w}_1}{3h_1\mathbf{w}_0}, \qquad \mathbf{c}_{23} = \frac{h_1\mathbf{w}_0 + h_0\mathbf{w}_1}{3h_1\mathbf{w}_1}. \tag{35}$$

Note that the coefficients $\mathbf{c}_{10}$, $\mathbf{c}_{13}$ and $\mathbf{c}_{20}$, $\mathbf{c}_{23}$ depend only on the three ratios h_0/h_1, h_2/h_1, and $\mathbf{w}_1/\mathbf{w}_0$. It can be shown that they must satisfy

$$(9\mathbf{c}_{13}\mathbf{c}_{20} - 1)^2 = 9\,(3\mathbf{c}_{13}\mathbf{c}_{10} - \mathbf{c}_{23})\,(3\mathbf{c}_{23}\mathbf{c}_{20} - \mathbf{c}_{10}). \tag{36}$$

3.5. Higher-order helical PH curves

To construct higher-order generalizations of the helical PH curves of degree 5 and 7 described in Sections 3.3.1 and 3.4.1, one may use re-parameterizations $t \to f(t)/g(t)$ of the line/circle (16) defined by polynomials $f(t)$ and $g(t)$ with $m = \deg(f, g) \geq 4$. Curves defined in this manner have the common feature that the coefficients of $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are of the form

$$\boldsymbol{\alpha}_i = f_i(\mathbf{a}_1 - \mathbf{a}_0) + g_i\mathbf{a}_0, \qquad \boldsymbol{\beta}_i = f_i(\mathbf{b}_1 - \mathbf{b}_0) + g_i\mathbf{b}_0, \quad i = 0, \ldots, m.$$

Invoking relation (5) between the quaternion and Hopf map models, we see that such curves are characterized by quaternion coefficients of the form

$$\mathcal{A}_i = f_i\,[\mathbf{a}_1 - \mathbf{a}_0 + \mathbf{k}\,(\mathbf{b}_1 - \mathbf{b}_0)] + g_i\,[\mathbf{a}_0 + \mathbf{k}\,\mathbf{b}_0], \quad i = 0, \ldots, m$$

for real values $f_0, \ldots, f_m$ and $g_0, \ldots, g_m$. The $m + 1$ quaternions $\mathcal{A}_0, \ldots, \mathcal{A}_m$ are thus linearly dependent on just two quaternions – $\mathbf{a}_1 - \mathbf{a}_0 + \mathbf{k}\,(\mathbf{b}_1 - \mathbf{b}_0)$ and $\mathbf{a}_0 + \mathbf{k}\,\mathbf{b}_0$ – and reside within a two-dimensional subspace of $\mathbb{H}$. Hence, as observed in Farouki et al. (2004) for the PH quintics ($m = 2$) and in Monterde (in press) for general m, such helical PH curves are characterized by the fact that the interior coefficients $\mathcal{A}_1, \ldots, \mathcal{A}_{m-1}$ are linearly dependent on the outer coefficients $\mathcal{A}_0$ and $\mathcal{A}_m$.

As generalizations of the degree 5 and 7 monotone-helical curves discussed in Sections 3.3.2 and 3.4.2, a line/circle parameterization $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ of degree m may be specified by multiplying $\mathbf{a}_0(1-t) + \mathbf{a}_1 t$ and $\mathbf{b}_0(1-t) + \mathbf{b}_1 t$ by a polynomial $\mathbf{w}(t)$ of degree $m - 1$ with Bernstein coefficients $\mathbf{w}_0, \ldots, \mathbf{w}_{m-1}$. The coefficients of $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are then given by $\boldsymbol{\alpha}_0 = \mathbf{a}_0\mathbf{w}_0$, $\boldsymbol{\alpha}_m = \mathbf{a}_1\mathbf{w}_{m-1}$ and $\boldsymbol{\beta}_0 = \mathbf{b}_0\mathbf{w}_0$, $\boldsymbol{\beta}_m = \mathbf{b}_1\mathbf{w}_{m-1}$ while for $k = 1, \ldots, m - 1$ we have

$$\boldsymbol{\alpha}_k = \frac{(m-k)\mathbf{a}_0\mathbf{w}_k + k\,\mathbf{a}_1\mathbf{w}_{k-1}}{m}, \qquad \boldsymbol{\beta}_k = \frac{(m-k)\mathbf{b}_0\mathbf{w}_k + k\,\mathbf{b}_1\mathbf{w}_{k-1}}{m},$$

and such curves satisfy $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = (\mathbf{a}_0\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_0)\,\mathbf{w}^2(t)$. As in Sections 3.3.2 and 3.4.2, the quaternion form of these curves is characterized by the fact that $\mathcal{A}_1, \ldots, \mathcal{A}_m$ can be written as linear combinations of $\mathcal{A}_0, \mathcal{A}_m$ with suitable complex coefficients $\mathbf{c}_{k0}, \mathbf{c}_{km}$ for $k = 1, \ldots, m-1$.

Finally, as generalizations of the degree 7 helical curves in Section 3.4.4, one may invoke any combination of (real) rational re-parameterizations and complex polynomial multiplications, specified in a particular order.

4. Non-helical double PH curves

A helical PH curve must be a double PH curve, but not all double PH curves are helical. As observed by Beltran and Monterde (2007), the lowest-order double PH curves that are non-helical have degree 7. We now seek criteria that serve to distinguish the non-helical double PH curves of degree 7 from the helical curves, for each of the three types enumerated in Section 2.2.

Assuming $\boldsymbol{\alpha}(t) \not\equiv 0$ and $\boldsymbol{\beta}(t) \not\equiv 0$ in (4), we begin with some observations concerning the possible common factors of these polynomials.

Lemma 1. *For cubic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ let $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ where $r = \deg(\boldsymbol{\gamma}(t))$ satisfies $0 \le r \le 3$, so that $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\beta}}(t)$ with $\gcd(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)) = \text{constant}$ and $\deg(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)) = 3 - r$. Condition (7) for a DPH curve then becomes*

$$\boldsymbol{\gamma}^2(t)\,[\tilde{\boldsymbol{\alpha}}(t)\tilde{\boldsymbol{\beta}}'(t) - \tilde{\boldsymbol{\alpha}}'(t)\tilde{\boldsymbol{\beta}}(t)] = h(t)\,\mathbf{w}^2(t), \tag{37}$$

and we must have $r \le 1$ for a curve satisfying this condition to be non-helical.

Proof. If $r = 3$, the cubics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ are proportional, and hence the curve degenerates (Farouki et al., 2009) to a straight line — which is trivially helical. If $r = 2$, we have $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\beta}}(t)$ with $\tilde{\boldsymbol{\alpha}}(t)$, $\tilde{\boldsymbol{\beta}}(t)$ linear and $\boldsymbol{\gamma}(t)$ quadratic, so that $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t) = \tilde{\boldsymbol{\alpha}}(t)/\tilde{\boldsymbol{\beta}}(t)$ defines a line/circle of the form (16) in the complex plane, and the double PH curve is helical (see Section 3). Thus, we must have $r \le 1$ for a non-helical DPH curve of degree 7. ■

Lemma 2. *For cubics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ let r be the degree of $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ and let $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\alpha}}(t)$, $\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\beta}}(t)$ as in Lemma 1. Then (7) cannot be satisfied with $r = 1$ if $h(t)$ is a constant or a perfect square.*

Proof. If $r = 1$, $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\alpha}}(t)$ and $\boldsymbol{\beta}(t) = \boldsymbol{\gamma}(t)\,\tilde{\boldsymbol{\beta}}(t)$ with $\boldsymbol{\gamma}(t)$ linear and $\tilde{\boldsymbol{\alpha}}(t)$, $\tilde{\boldsymbol{\beta}}(t)$ quadratic and relatively prime. In this case, $\tilde{\boldsymbol{\alpha}}(t)\tilde{\boldsymbol{\beta}}'(t) - \tilde{\boldsymbol{\alpha}}'(t)\tilde{\boldsymbol{\beta}}(t)$ is quadratic, and it must be a perfect square if $h(t)$ in (37) is a constant or a perfect square, i.e., we must have

$$\tilde{\boldsymbol{\alpha}}(t)\tilde{\boldsymbol{\beta}}'(t) - \tilde{\boldsymbol{\alpha}}'(t)\tilde{\boldsymbol{\beta}}(t) = \boldsymbol{\delta}^2(t) \tag{38}$$

for some linear polynomial $\boldsymbol{\delta}(t)$. Then, if τ is the root of $\boldsymbol{\delta}(t)$, we have

$$\tilde{\boldsymbol{\alpha}}(\tau)\tilde{\boldsymbol{\beta}}'(\tau) - \tilde{\boldsymbol{\alpha}}'(\tau)\tilde{\boldsymbol{\beta}}(\tau) = \tilde{\boldsymbol{\alpha}}(\tau)\tilde{\boldsymbol{\beta}}''(\tau) - \tilde{\boldsymbol{\alpha}}''(\tau)\tilde{\boldsymbol{\beta}}(\tau) = 0. \tag{39}$$

Now since $\gcd(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)) = \text{constant}$, $\tilde{\boldsymbol{\alpha}}(\tau)$ and $\tilde{\boldsymbol{\beta}}(\tau)$ cannot be both zero. If we assume both are non-zero, equations (39) imply that

$$\tilde{\boldsymbol{\alpha}}(\tau) : \tilde{\boldsymbol{\alpha}}'(\tau) : \tilde{\boldsymbol{\alpha}}''(\tau) = \tilde{\boldsymbol{\beta}}(\tau) : \tilde{\boldsymbol{\beta}}'(\tau) : \tilde{\boldsymbol{\beta}}''(\tau).$$

But since $\tilde{\boldsymbol{\alpha}}(t)$ and $\tilde{\boldsymbol{\beta}}(t)$ are quadratic, this implies that they are proportional — contradicting $\gcd(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)) = \text{constant}$. If we assume $\tilde{\boldsymbol{\alpha}}(\tau) = 0 \neq \tilde{\boldsymbol{\beta}}(\tau)$, equations (39) imply that $\tilde{\boldsymbol{\alpha}}'(\tau) = \tilde{\boldsymbol{\alpha}}''(\tau) = 0$, so $\tilde{\boldsymbol{\alpha}}(t) = \boldsymbol{\alpha}_0(t-\tau)^2$ for some constant $\boldsymbol{\alpha}_0 \neq 0$. Substituting in (38) and writing $\boldsymbol{\delta}(t) = \boldsymbol{\delta}_0(t-\tau)^2$ gives

$$2\,\boldsymbol{\alpha}_0\,\tilde{\boldsymbol{\beta}}(t) = [\boldsymbol{\alpha}_0\,\tilde{\boldsymbol{\beta}}'(t) - \boldsymbol{\delta}_0^2(t-\tau)^2]\,(t-\tau),$$

contradicting the assumption that $\tilde{\boldsymbol{\beta}}(\tau) \neq 0$. A similar contradiction arises if we assume $\tilde{\boldsymbol{\alpha}}(\tau) \neq 0 = \tilde{\boldsymbol{\beta}}(\tau)$. Hence, we infer that (7) cannot be satisfied by cubics $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ with $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ of degree 1 if $h(t)$ is a constant or a perfect square. ■

The following lemmas give useful alternatives to the (rational) line/circle representations discussed in Section 3.1, that will be invoked subsequently.

Lemma 3. *Let* $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ *be complex constants such that* $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \neq 0$, *and let* ϕ *be a real variable. Then if* $|\mathbf{b}_1| \neq |\mathbf{b}_2|$ *the function*

$$\mathbf{z}(\phi) = \frac{\mathbf{a}_1 e^{i\phi} + \mathbf{a}_2}{\mathbf{b}_1 e^{i\phi} + \mathbf{b}_2} \tag{40}$$

defines a circle with center and radius given by

$$\mathbf{z}_c = \frac{\mathbf{a}_1\overline{\mathbf{b}}_1 - \mathbf{a}_2\overline{\mathbf{b}}_2}{|\mathbf{b}_1|^2 - |\mathbf{b}_2|^2} \qquad \text{and} \qquad R = \left| \frac{\mathbf{a}_2\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_2}{|\mathbf{b}_1|^2 - |\mathbf{b}_2|^2} \right|,$$

while if $|\mathbf{b}_1| = |\mathbf{b}_2|$ *the function* (40) *defines a straight line.*

Proof. The condition $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \neq 0$ guarantees that the numerator and denominator of (40) are not proportional, so $\mathbf{z}(\phi)$ does not degenerate to a constant. By subtracting $\mathbf{z}_c$ from $\mathbf{z}(\phi)$ and simplifying, one obtains

$$\mathbf{z}(\phi) - \mathbf{z}_c = \frac{\mathbf{a}_2\mathbf{b}_1 - \mathbf{a}_1\mathbf{b}_2}{|\mathbf{b}_1|^2 - |\mathbf{b}_2|^2} \, \frac{\overline{\mathbf{b}}_1 e^{-i\phi} + \overline{\mathbf{b}}_2}{\mathbf{b}_1 e^{i\phi} + \mathbf{b}_2} \, e^{i\phi},$$

and since the two factors dependent on ϕ have unit magnitude, we see that $|\mathbf{z}(\phi) - \mathbf{z}_c| = R$. For the case $|\mathbf{b}_1| = |\mathbf{b}_2|$, in which $\mathbf{z}_c$ and R become infinite, we have a circle of infinite radius — i.e., a straight line. ■

Lemma 4. *If* τ_1, τ_2 *are both real or complex conjugates,*[5] *the function*

$$\mathbf{z}(t) = \frac{\mathbf{a}_1(t-\tau_1)^m + \mathbf{a}_2(t-\tau_2)^m}{\mathbf{b}_1(t-\tau_1)^m + \mathbf{b}_2(t-\tau_2)^m}$$

of the real variable t *defines, for integer* m, *a line/circle in the complex plane.*

Proof. Writing $\mathbf{f}(t) = (t-\tau_1)^m/(t-\tau_2)^m$ we have

$$\mathbf{z}(t) = \frac{\mathbf{a}_1\mathbf{f}(t) + \mathbf{a}_2}{\mathbf{b}_1\mathbf{f}(t) + \mathbf{b}_2}.$$

If τ_1, τ_2 are real, $\mathbf{f}(t)$ becomes a *real* function $f(t)$, and we may regard $\mathbf{z}(t)$ as arising from a real re-parameterization $t \to f(t)$ applied to the rational linear form (16) of a line/circle. On the other hand, if τ_1, τ_2 are complex conjugates, we have $\mathbf{f}(t) = \exp(i\,2m\arg(t-\tau_1))$, and writing $\phi = 2m\arg(t-\tau_1)$ we see that $\mathbf{z}(t)$ has the alternative line/circle form (40). ■

These lemmas simplify the identification of criteria to distinguish helical and non-helical degree 7 DPH curves, as described in the following sections.

4.1. *The case* $\deg(h) = 0$, $\deg(\mathbf{w}) = 2$

In this case $h(t)$ is a constant, h_0. To identify the non-helical DPH curves of this type, we set $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ and $r = \deg(\boldsymbol{\gamma}(t))$ as in Lemma 1.

Proposition 1. *A degree* 7 *DPH curve satisfying* (7) *with* $h(t)$ *constant and* $\mathbf{w}(t)$ *quadratic is non-helical if the roots* τ_1, τ_2 *of* $\mathbf{w}(t)$ *are neither both real nor complex conjugates, and* $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ *can be expressed in terms of them as*

$$\boldsymbol{\alpha}(t) = \mathbf{a}_1(t-\tau_1)^3 + \mathbf{a}_2(t-\tau_2)^3, \qquad \boldsymbol{\beta}(t) = \mathbf{b}_1(t-\tau_1)^3 + \mathbf{b}_2(t-\tau_2)^3, \tag{41}$$

where $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 \neq 0$.

[5] Here, the case of complex conjugates subsumes the case of a (real) double root, $\tau_1 = \tau_2$.

Proof. By Lemmas 1 and 2, we need only consider $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ of degree $r = 0$. For relatively prime cubics $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ satisfying

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = h_0\,\mathbf{w}^2(t), \tag{42}$$

let τ_1, τ_2 be the roots of the quadratic $\mathbf{w}(t)$. Then τ_1, τ_2 must be double roots of (8), and we have

$$\boldsymbol{\alpha}(\tau_i)\boldsymbol{\beta}'(\tau_i) - \boldsymbol{\alpha}'(\tau_i)\boldsymbol{\beta}(\tau_i) = \boldsymbol{\alpha}(\tau_i)\boldsymbol{\beta}''(\tau_i) - \boldsymbol{\alpha}''(\tau_i)\boldsymbol{\beta}(\tau_i) = 0 \tag{43}$$

for $i = 1, 2$. To study when these conditions can be satisfied, we note that if $\tau_1 \neq \tau_2$ we may write $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ in the Bernstein-like form

$$\boldsymbol{\alpha}(t) = \sum_{k=0}^{3} \mathbf{p}_k \binom{3}{k} (\tau_2 - t)^{3-k}(t - \tau_1)^k, \qquad \boldsymbol{\beta}(t) = \sum_{k=0}^{3} \mathbf{q}_k \binom{3}{k} (\tau_2 - t)^{3-k}(t - \tau_1)^k$$

for suitable complex coefficients $\mathbf{p}_k, \mathbf{q}_k$. We note that $(\boldsymbol{\alpha}(\tau_i), \boldsymbol{\beta}(\tau_i)) \neq (0, 0)$ for $i = 1, 2$ since $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant, and consider two possible cases:

case (a): $\boldsymbol{\alpha}(\tau_i) \neq 0$ and $\boldsymbol{\beta}(\tau_i) \neq 0$ for $i = 1, 2$. In this case relations (43) can be written as

$$\frac{\boldsymbol{\alpha}'(\tau_i)}{\boldsymbol{\alpha}(\tau_i)} = \frac{\boldsymbol{\beta}'(\tau_i)}{\boldsymbol{\beta}(\tau_i)}, \qquad \frac{\boldsymbol{\alpha}''(\tau_i)}{\boldsymbol{\alpha}(\tau_i)} = \frac{\boldsymbol{\beta}''(\tau_i)}{\boldsymbol{\beta}(\tau_i)}$$

for $i = 1, 2$. In terms of the coefficients $\mathbf{p}_k, \mathbf{q}_k$ these imply that

$$\frac{\mathbf{p}_1}{\mathbf{p}_0} = \frac{\mathbf{q}_1}{\mathbf{q}_0}, \qquad \frac{\mathbf{p}_2}{\mathbf{p}_0} = \frac{\mathbf{q}_2}{\mathbf{q}_0}, \qquad \frac{\mathbf{p}_2}{\mathbf{p}_3} = \frac{\mathbf{q}_2}{\mathbf{q}_3}, \qquad \frac{\mathbf{p}_1}{\mathbf{p}_3} = \frac{\mathbf{q}_1}{\mathbf{q}_3}.$$

Now satisfying these conditions with $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{q}_1$, $\mathbf{q}_2$ not all zero implies that $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ are proportional — contradicting $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant. We may satisfy them without contradiction, however, by taking $\mathbf{p}_1 = \mathbf{p}_2 = 0$ and $\mathbf{q}_1 = \mathbf{q}_2 = 0$, so that

$$\boldsymbol{\alpha}(t) = \mathbf{p}_0(\tau_2 - t)^3 + \mathbf{p}_3(t - \tau_1)^3, \qquad \boldsymbol{\beta}(t) = \mathbf{q}_0(\tau_2 - t)^3 + \mathbf{q}_3(t - \tau_1)^3,$$

where $\mathbf{p}_0\mathbf{q}_3 - \mathbf{p}_3\mathbf{q}_0 \neq 0$ is stipulated to ensure non-proportionality. Hence, setting $\mathbf{a}_1 = \mathbf{p}_3, \mathbf{a}_2 = -\mathbf{p}_0$ and $\mathbf{b}_1 = \mathbf{q}_3, \mathbf{b}_2 = -\mathbf{q}_0$, polynomials (41) define a non-helical DPH curve provided that τ_1, τ_2 are not both real and not complex conjugates since, by Lemma 4, $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ does not define a line/circle in the complex plane.

case (b): At least one of $\boldsymbol{\alpha}(\tau_i)$ and $\boldsymbol{\beta}(\tau_i)$ for $i = 1, 2$ is zero. Assuming that $\boldsymbol{\alpha}(\tau_1) = 0$, we have $\boldsymbol{\beta}(\tau_1) \neq 0$, i.e., $\mathbf{q}_0 \neq 0$, since $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant. Equations (43) then imply that $\boldsymbol{\alpha}'(\tau_1) = \boldsymbol{\alpha}''(\tau_1) = 0$, so $\mathbf{p}_0 = \mathbf{p}_1 = \mathbf{p}_2 = 0$ and $\boldsymbol{\alpha}(t) = \mathbf{p}_3(t - \tau_1)^3$. From (43) with $i = 2$ we then infer that $\mathbf{q}_1 = \mathbf{q}_2 = 0$, and hence $\boldsymbol{\beta}(t) = \mathbf{q}_0(\tau_2 - t)^3 + \mathbf{q}_3(t - \tau_1)^3$. Thus, $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ in this case are also of the form (41), but with $\mathbf{a}_1 = \mathbf{p}_3$, $\mathbf{a}_2 = 0$ and $\mathbf{b}_1 = \mathbf{q}_3$, $\mathbf{b}_2 = -\mathbf{q}_0$. Analogous results are obtained when $\boldsymbol{\alpha}(\tau_2) = 0 \neq \boldsymbol{\beta}(\tau_2)$ or $\boldsymbol{\alpha}(\tau_1) \neq 0 = \boldsymbol{\beta}(\tau_1)$ or $\boldsymbol{\alpha}(\tau_2) \neq 0 = \boldsymbol{\beta}(\tau_2)$ — namely, one of the coefficients $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{b}_1, \mathbf{b}_2$ in (41) vanishes. Again, the curve is non-helical, since $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ does not describe a line/circle under the stated constraints on τ_1, τ_2.

We assumed above that $\mathbf{w}(t)$ has distinct roots τ_1, τ_2. If $\tau_1 = \tau_2$, so $\mathbf{w}(t)$ has a double root and $\mathbf{w}^2(t)$ is the fourth power of a linear polynomial, one may verify that (42) can only be satisfied with $h_0 = 0$ and $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ proportional, contradicting $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant. ■

4.2. *The case* $\deg(h) = 2$, $\deg(\mathbf{w}) = 1$

By Lemma 1 we need only consider cases in which $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ is of degree $r = 1$ or $r = 0$. In the following propositions, we shall see that non-helical DPH curves can exist only in the latter case.

Proposition 2. *There are no non-helical degree 7 DPH curves satisfying (7) with $h(t)$ quadratic, $\mathbf{w}(t)$ linear, and $\boldsymbol{\gamma}(t) = \gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ of degree $r = 1$.*

Proof. By Lemma 2, we need only consider quadratics $h(t)$ with two *distinct* roots τ_1, τ_2 in this case, and from (37) it is apparent that $\tilde{\boldsymbol{\alpha}}(t)\tilde{\boldsymbol{\beta}}'(t) - \tilde{\boldsymbol{\alpha}}'(t)\tilde{\boldsymbol{\beta}}(t)$ and $\boldsymbol{\gamma}(t)$ must be proportional to $h(t)$ and $\mathbf{w}(t)$, respectively. For suitable coefficients $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2$ we may write

$$\tilde{\boldsymbol{\alpha}}(t) = \mathbf{p}_0(\tau_2 - t)^2 + \mathbf{p}_1 2(\tau_2 - t)(t - \tau_1) + \mathbf{p}_2(t - \tau_1)^2,$$
$$\tilde{\boldsymbol{\beta}}(t) = \mathbf{q}_0(\tau_2 - t)^2 + \mathbf{q}_1 2(\tau_2 - t)(t - \tau_1) + \mathbf{q}_2(t - \tau_1)^2,$$

and the fact that $\tilde{\boldsymbol{\alpha}}(t)\tilde{\boldsymbol{\beta}}'(t) - \tilde{\boldsymbol{\alpha}}'(t)\tilde{\boldsymbol{\beta}}(t)$ must vanish at τ_1 and τ_2 implies that

$$\mathbf{p}_0\mathbf{q}_1 - \mathbf{p}_1\mathbf{q}_0 = 0 \qquad \text{and} \qquad \mathbf{p}_1\mathbf{q}_2 - \mathbf{p}_2\mathbf{q}_1 = 0.$$

Now if $\mathbf{p}_1, \mathbf{q}_1$ are not both zero, these equations imply that $\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)$ must be proportional,[6] which contradicts $\gcd(\tilde{\boldsymbol{\alpha}}(t), \tilde{\boldsymbol{\beta}}(t)) =$ constant. But they are satisfied without contradiction if $\mathbf{p}_1 = \mathbf{q}_1 = 0$. Taking $\mathbf{a}_1 = \mathbf{p}_0, \mathbf{a}_2 = \mathbf{p}_2$ and $\mathbf{b}_1 = \mathbf{q}_0, \mathbf{b}_2 = \mathbf{q}_2$ and a suitable choice of constants, we then have

$$\boldsymbol{\alpha}(t) \;= \mathbf{w}(t)\,[\mathbf{a}_1(t - \tau_1)^2 + \mathbf{a}_2(t - \tau_2)^2],$$
$$\boldsymbol{\beta}(t) \;= \mathbf{w}(t)\,[\mathbf{b}_1(t - \tau_1)^2 + \mathbf{b}_2(t - \tau_2)^2],$$

where the roots τ_1, τ_2 must be both real or complex conjugates, since $h(t)$ is a real polynomial. Hence, Lemma 4 indicates that $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ describes a line/circle in the complex plane, so the DPH curve must be helical. ■

If $\deg(h) = 2$, $\deg(\mathbf{w}) = 1$ and $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ are relatively prime, it is not so easy to derive a characterization for these polynomials in terms of the roots of $h(t)$ and $\mathbf{w}(t)$, analogous to (41), that yields non-helical curves. However, we may appeal to the analysis of helical DPH curves in Section 3.4 to obtain a simple resolution of this question.

Proposition 3. *All double PH curves of degree* 7 *that satisfy* (7) *with* $h(t)$ *quadratic,* $\mathbf{w}(t)$ *linear, and* $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ *of degree* $r = 0$ *are non-helical.*

Proof. Section 3.4 enumerates all possible construction modes for degree 7 helical curves, starting from a line/circle parameterization of the form (16) in the complex plane. These include a cubic re-parameterization, multiplication with a quadratic polynomial, and a quadratic re-parameterization followed by multiplication with a linear polynomial. Of these, only the latter mode (Section 3.4.4) yields helical curves with $\deg(h) = 2$ and $\deg(\mathbf{w}) = 1$ in (7), and for such curves $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ must have the linear polynomial $\mathbf{w}(t)$ as a common factor. Hence, degree 7 DPH curves with $h(t)$ quadratic, $\mathbf{w}(t)$ linear, and $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) =$ constant are necessarily non-helical. ■

4.3. *The case* $\deg(h) = 4$, $\deg(\mathbf{w}) = 0$

In this case, we appeal to the characterization of degree 7 helical DPH curves derived in Section 3.4.1, and find that a simple quadratic expression in the Bernstein coefficients of the real quartic polynomial $h(t)$ serves to distinguish between helical and non-helical double PH curves.

Proposition 4. *A degree* 7 *double PH curve with* $\deg(h) = 4$ *and* $\deg(\mathbf{w}) = 0$ *in* (7) *is helical or non-helical according to whether or not the quantity*

$$\Delta = 9\,h_2^2 + 3\,h_0h_4 - 12\,h_1h_3, \tag{44}$$

defined in terms of the Bernstein coefficients of $h(t)$*, is non-negative.*

Proof. As noted in Section 3.4.1, a degree 7 double PH curve with $\deg(h) = 4$ and $\deg(\mathbf{w}) = 0$ in (7) is helical if and only if the real quartic polynomial $h(t)$ can be written in terms of real cubics $f(t), g(t)$ in the form (19). This is equivalent to the requirement that the Bernstein coefficients of $h(t)$ should be such as to admit real solutions of equations (25) for the Bernstein coefficients of $f(t), g(t)$. Now system (25) may be interpreted as *five* linear equations in the *six* quantities $f_ig_j - f_jg_i$ with $i \neq j$ for

[6] They are trivially proportional when one of them vanishes identically.

$0 \leq i, j \leq 3$. So we can choose one of these quantities arbitrarily. Setting $f_2 g_1 - f_1 g_2 = c$, we obtain

$$f_1 g_0 - f_0 g_1 = \frac{1}{3} h_0, \qquad f_2 g_0 - f_0 g_2 = \frac{2}{3} h_1, \qquad f_3 g_0 - f_0 g_3 = 2h_2 - 3c,$$

$$f_2 g_1 - f_1 g_2 = c, \qquad f_3 g_1 - f_1 g_3 = \frac{2}{3} h_3, \qquad f_3 g_2 - f_2 g_3 = \frac{1}{3} h_4.$$

However, these equations in f_i, g_i for $0 \leq i \leq 3$ may not be consistent. Since

$$(f_2 g_0 - f_0 g_2)(f_3 g_1 - f_1 g_3) - (f_2 g_1 - f_1 g_2)(f_3 g_0 - f_0 g_3) = (f_1 g_0 - f_0 g_1)(f_3 g_2 - f_2 g_3),$$

the values $h_0, \ldots, h_4$ and c must satisfy the consistency condition

$$\left(\frac{2}{3} h_1\right)\left(\frac{2}{3} h_3\right) - c(2h_2 - 3c) = \left(\frac{1}{3} h_0\right)\left(\frac{1}{3} h_4\right),$$

which can be reduced to a quadratic equation in c, namely

$$27\, c^2 - 18\, h_2 c + 4\, h_1 h_3 - h_0 h_4 = 0.$$

Clearly, the solutions

$$c = f_2 g_1 - f_1 g_2 = \frac{1}{9}\left(3h_2 \pm \sqrt{9h_2^2 + 3h_0 h_4 - 12 h_1 h_3}\right)$$

can be real if and only if the discriminant Δ defined by (44) is non-negative. In such cases, $h(t)$ can be expressed in the form (19) for real cubics $f(t)$, $g(t)$ and the degree 7 double PH is helical, since $\boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ corresponds to the real cubic re-parameterization (24) of the line/circle (16).

In all other cases, no real cubics $f(t)$, $g(t)$ exist, such that $h(t)$ is given by (19). Since these cases do not correspond to real cubic re-parameterizations of the line/circle (16), they define *non-helical* degree 7 double PH curves. ■

5. Computed examples

For a spatial PH curve of degree 7, we use a cubic quaternion polynomial $\mathcal{A}(t)$ in (3), specified in the Bernstein form as

$$\mathcal{A}(t) = \mathcal{A}_0\,(1-t)^3 + \mathcal{A}_1\,3\,(1-t)^2\,t + \mathcal{A}_2\,3\,(1-t)\,t^2 + \mathcal{A}_3\,t^3. \tag{45}$$

Integrating hodograph (3) then gives the Bézier form

$$\mathbf{r}(t) = \sum_{i=0}^{7} \mathbf{p}_i \binom{7}{i} (1-t)^{7-i}\, t^i$$

of the degree 7 PH curve, with control points $\mathbf{p}_i = x_i\,\mathbf{i} + y_i\,\mathbf{j} + z_i\,\mathbf{k}$ given by

$$\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{7}\,\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*, \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{14}\,(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_0^*), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{1}{35}\,(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + 3\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*), \\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{140}\,(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_3^* + 9\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_2^* + 9\,\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_3\,\mathbf{i}\,\mathcal{A}_0^*), \\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{35}\,(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_3^* + 3\,\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_3\,\mathbf{i}\,\mathcal{A}_1^*), \\
\mathbf{p}_6 &= \mathbf{p}_5 + \frac{1}{14}\,(\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_3^* + \mathcal{A}_3\,\mathbf{i}\,\mathcal{A}_2^*), \\
\mathbf{p}_7 &= \mathbf{p}_6 + \frac{1}{7}\,\mathcal{A}_3\,\mathbf{i}\,\mathcal{A}_3^*,
\end{aligned} \tag{46}$$

where we usually take $\mathbf{p}_0 = (0, 0, 0)$ as the integration constant.

5.1. Degree 7 helical DPH curves

We begin with examples that illustrate the direct construction of helical DPH curves of degree 7 from complex-plane line/circle parameterizations, through the Hopf map method of Monterde (in press) described in Section 3.

Example 1 (*Cubic Re-parameterization*). Using the complex line/circle (16) defined by $(\mathbf{a}_0, \mathbf{a}_1) = (1, 1+\mathrm{i})$ and $(\mathbf{b}_0, \mathbf{b}_1) = (1-\mathrm{i}, \mathrm{i})$ and re-parameterization function (24) specified by $(f_0, f_1, f_2, f_3) = (1, 2, 2, 1)$ and $(g_0, g_1, g_2, g_3) = (1, 2, 3, 3)$ we obtain the form $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ with $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = (1+\mathrm{i}, 2+2\,\mathrm{i}, 3+2\,\mathrm{i}, 3+\mathrm{i})$ and $(\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) = (\mathrm{i}, 2\,\mathrm{i}, 1+\mathrm{i}, 2-\mathrm{i})$. The corresponding coefficients $\mathcal{A}_l = \boldsymbol{\alpha}_l + \mathbf{k}\,\boldsymbol{\beta}_l$ of the cubic quaternion polynomial $\mathcal{A}(t)$ are then

$$(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = (1+\mathbf{i}+\mathbf{j}, 2+2\,\mathbf{i}+2\,\mathbf{j}, 3+2\,\mathbf{i}+\mathbf{j}+\mathbf{k}, 3+\mathbf{i}-\mathbf{j}+2\,\mathbf{k})$$

and from (46) we obtain the control points

$$\begin{aligned}
\mathbf{p}_0 &= (0.0000, 0.0000, 0.0000), & \mathbf{p}_1 &= (0.1429, 0.2857, -0.2857),\\
\mathbf{p}_2 &= (0.4286, 0.8571, -0.8571), & \mathbf{p}_3 &= (1.0000, 1.7714, -1.7143),\\
\mathbf{p}_4 &= (2.1000, 2.8286, -2.4857), & \mathbf{p}_5 &= (3.6143, 3.9143, -2.6571),\\
\mathbf{p}_6 &= (5.0429, 5.0571, -1.9429), & \mathbf{p}_7 &= (5.7571, 6.4857, -0.5143).
\end{aligned}$$

This helical curve has the curvature/torsion ratio $|\kappa(t)/\tau(t)| = \sqrt{5}/2$.

Example 2 (*Quadratic Polynomial Multiplication*). Using $(\mathbf{a}_0, \mathbf{a}_1) = (5\,\mathrm{i}, 1+\mathrm{i})$ and $(\mathbf{b}_0, \mathbf{b}_1) = (1-\mathrm{i}, 2+5\,\mathrm{i})$ in (16), and the complex quadratic specified by $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2) = (1, 1+\mathrm{i}, 1)$ in (29), yield the rational cubic $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ with $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = (5\,\mathrm{i}, -3+\frac{11}{3}\,\mathrm{i}, 3\,\mathrm{i}, 1+\mathrm{i})$ and $(\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) = (1-\mathrm{i}, 2+\frac{5}{3}\,\mathrm{i}, -\frac{5}{3}+\frac{13}{3}\,\mathrm{i}, 2+5\,\mathrm{i})$. For the coefficients $\mathcal{A}_l = \boldsymbol{\alpha}_l + \mathbf{k}\,\boldsymbol{\beta}_l$ of the cubic quaternion polynomial $\mathcal{A}(t)$, we then have

$$\begin{aligned}
(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \bigg(&5\,\mathbf{i}-\mathbf{j}+\mathbf{k}, -3+\frac{11}{3}\,\mathbf{i}+\frac{5}{3}\,\mathbf{j}+2\,\mathbf{k},\\
&3\,\mathbf{i}+\frac{13}{3}\,\mathbf{j}-\frac{5}{3}\,\mathbf{k}, 1+\mathbf{i}+5\,\mathbf{j}+2\,\mathbf{k}\bigg)
\end{aligned}$$

and from (46) we obtain the control points

$$\begin{aligned}
\mathbf{p}_0 &= (0.0000, 0.0000, 0.0000), & \mathbf{p}_1 &= (3.2857, -1.4286, 1.4286),\\
\mathbf{p}_2 &= (5.8571, -1.1905, 2.9524), & \mathbf{p}_3 &= (8.4000, -0.1048, 4.7619),\\
\mathbf{p}_4 &= (9.4286, 3.5810, 6.5905), & \mathbf{p}_5 &= (7.6857, 6.7238, 7.0286),\\
\mathbf{p}_6 &= (5.4952, 9.2476, 7.0286), & \mathbf{p}_7 &= (1.6381, 11.2476, 6.1714).
\end{aligned}$$

These control points define a monotone-helical curve, with curvature/torsion ratio $|\kappa(t)/\tau(t)| = \sqrt{829}/2$.

Example 3 (*Re-parameterization and Multiplication*). The values $(\mathbf{a}_0, \mathbf{a}_1) = (1+\mathrm{i}, 1)$ and $(\mathbf{b}_0, \mathbf{b}_1) = (1-\mathrm{i}, 2)$ in (16), together with $(f_0, f_1, f_2) = (1, 2, 1)$, $(g_0, g_1, g_2) = (1, 2, 2)$, and $(\mathbf{w}_0, \mathbf{w}_1) = (1, 1+\mathrm{i})$ in (33) yield $\mathbf{z}(t) = \boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ with $(\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3) = (1, \frac{5}{3}+\frac{1}{3}\,\mathrm{i}, 2+\frac{5}{3}\,\mathrm{i}, 1+3\,\mathrm{i})$ and $(\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3) = (2, \frac{10}{3}+\frac{2}{3}\,\mathrm{i}, \frac{11}{3}+\frac{7}{3}\,\mathrm{i}, 4+2\,\mathrm{i})$. The coefficients $\mathcal{A}_l = \boldsymbol{\alpha}_l + \mathbf{k}\,\boldsymbol{\beta}_l$ of the cubic quaternion polynomial $\mathcal{A}(t)$ are then

$$\begin{aligned}
(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \bigg(&1+2\,\mathbf{k}, \frac{5}{3}+\frac{1}{3}\,\mathbf{i}+\frac{2}{3}\,\mathbf{j}+\frac{10}{3}\,\mathbf{k},\\
&2+\frac{5}{3}\,\mathbf{i}+\frac{7}{3}\,\mathbf{j}+\frac{11}{3}\,\mathbf{k}, 1+3\,\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}\bigg)
\end{aligned}$$

and from (46) we obtain the control points

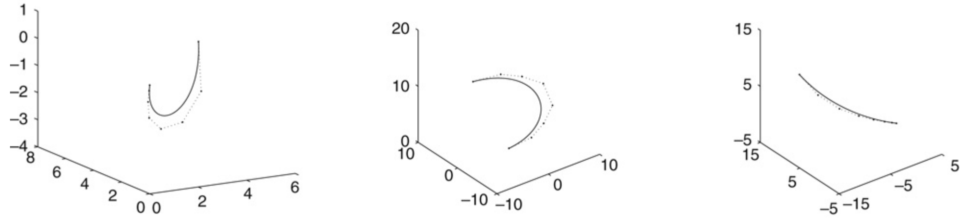

Fig. 1. The three types of degree 7 helical DPH curves (Examples 1–3).

$$
\begin{aligned}
\mathbf{p}_0 &= (0.0000, 0.0000, 0.0000), & \mathbf{p}_1 &= (-0.4286, 0.5714, 0.0000),\\
\mathbf{p}_2 &= (-1.1429, 1.5238, 0.0000), & \mathbf{p}_3 &= (-2.1905, 2.9524, 0.0571),\\
\mathbf{p}_4 &= (-3.5619, 4.9238, 0.3143), & \mathbf{p}_5 &= (-5.2857, 7.5714, 0.9810),\\
\mathbf{p}_6 &= (-7.0476, 10.7143, 2.6000), & \mathbf{p}_7 &= (-8.4762, 13.5714, 5.4571).
\end{aligned}
$$

For this curve, the curvature/torsion ratio is $|\kappa(t)/\tau(t)| = \sqrt{10}$.

Fig. 1 illustrates the three degree 7 helical DPH curves constructed in the preceding examples, together with their control polygons. The following examples will illustrate the construction of DPH curves through the methods described in Section 2.3, for various combinations of the degrees of $h(t)$ and $\mathbf{w}(t)$ in (7). These examples also illustrate the use of the criteria in Section 4 to distinguish between helical and non-helical DPH curves.

5.2. *DPH curves with* $\deg(h) = 0$, $\deg(\mathbf{w}) = 2$

Example 4 ($\deg(h) = 0$, $\deg(\mathbf{w}) = 2$, *Helical*). In equations (10) we choose the numerical values

$$
h_0 = 1, \qquad \mathbf{w}_0 = 1, \qquad \mathbf{w}_1 = 1 + \mathrm{i}, \qquad \mathbf{w}_2 = \mathrm{i}.
$$

Assigning values to $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_0$ and solving the bilinear system specified by (10) and the second expression in (11) for the other coefficients, we obtain

$$
\begin{aligned}
\boldsymbol{\alpha}_0 &= 1, & \boldsymbol{\alpha}_1 &= 2\,\mathrm{i}, & \boldsymbol{\alpha}_2 &= -4 + \frac{5}{3}\,\mathrm{i}, & \boldsymbol{\alpha}_3 &= -4 - 2\,\mathrm{i},\\
\boldsymbol{\beta}_0 &= \mathrm{i}, & \boldsymbol{\beta}_1 &= -\frac{5}{3}, & \boldsymbol{\beta}_2 &= -1 - \frac{10}{3}\mathrm{i}, & \boldsymbol{\beta}_3 &= 2 - 3\,\mathrm{i}.
\end{aligned}
$$

The resulting hodograph

$$
\begin{aligned}
x'(t) &= 14\,t^6 - 28\,t^5 + 14\,t^4 + 7\,t^2,\\
y'(t) &= -\,12\,t^6 + 28\,t^5 - 20\,t^4 + 4\,t^3 - 6\,t^2 + 2\,t,\\
z'(t) &= -\,84\,t^6 + 192\,t^5 - 136\,t^4 + 32\,t^3 - 46\,t^2 + 12\,t - 2,
\end{aligned}
$$

is non-primitive, since $\gcd(x'(t), y'(t), z'(t)) = 2\,t^4 - 4\,t^3 + 2\,t^2 + 1$. We have

$$
\begin{aligned}
&\sigma(t) = |\mathbf{r}'(t)| = (43\,t^2 - 12\,t + 2)(2\,t^4 - 4\,t^3 + 2\,t^2 + 1),\\
&|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,(2\,t^4 - 4\,t^3 + 2\,t^2 + 1)\,\sigma(t),
\end{aligned}
$$

and a constant curvature/torsion ratio, $|\kappa(t)/\tau(t)| = 1/7$. This curve has an especially simple rational Frenet frame, given by

$$\mathbf{t} = \frac{(7\,t^2, -6\,t^2+2\,t, -42\,t^2+12\,t-2)}{43\,t^2-12\,t+2},$$
$$\mathbf{p} = \frac{(-42\,t^2+14\,t, -7\,t^2-12\,t+2, -6\,t^2+2\,t)}{43\,t^2-12\,t+2},$$
$$\mathbf{b} = \frac{(-6\,t^2-12\,t+2, 42\,t^2-14\,t, -7\,t^2)}{43\,t^2-12\,t+2},$$

and the rational curvature function is just $\kappa(t) = 2/(43\,t^2-12\,t+2)\,\sigma(t)$. For this curve, we have the quaternion coefficients

$$\mathcal{A}_0 = 1+\mathbf{j}, \qquad \mathcal{A}_1 = 2\,\mathbf{i} - \frac{5}{3}\,\mathbf{k},$$
$$\mathcal{A}_2 = -4+\frac{5}{3}\,\mathbf{i} - \frac{10}{3}\,\mathbf{j} - \mathbf{k}, \qquad \mathcal{A}_3 = -4-2\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}.$$

Here $\mathcal{A}_1, \mathcal{A}_2$ can be specified in terms of $\mathcal{A}_0, \mathcal{A}_3$ using (30), with the values

$$\mathbf{c}_{10} = \frac{1}{3}(2+2\,\mathrm{i}), \qquad \mathbf{c}_{13} = -\frac{1}{3}\,\mathrm{i}, \qquad \mathbf{c}_{20} = \frac{1}{3}\,\mathrm{i}, \qquad \mathbf{c}_{23} = \frac{1}{3}\,(2-2\,\mathrm{i})$$

for coefficients (31), which evidently satisfy relations (32).

Example 5 (deg(h) = 0, deg($\mathbf{w}$) = 2, *Non-helical*). Using the numerical values of the previous example, but the first rather than the second expression in (11), we obtain

$$\boldsymbol{\alpha}_0 = 1, \qquad \boldsymbol{\alpha}_1 = 2\,\mathrm{i}, \qquad \boldsymbol{\alpha}_2 = -4+3\,\mathrm{i}, \qquad \boldsymbol{\alpha}_3 = -12-2\,\mathrm{i},$$
$$\boldsymbol{\beta}_0 = \mathrm{i}, \qquad \boldsymbol{\beta}_1 = -\frac{5}{3}, \qquad \boldsymbol{\beta}_2 = -\frac{7}{3} - \frac{10}{3}\mathrm{i}, \qquad \boldsymbol{\beta}_3 = 2 - \frac{29}{3}\,\mathrm{i}.$$

The corresponding hodograph components

$$x'(t) = \frac{86}{9}\,t^6 + \frac{44}{3}\,t^5 + 14\,t^4 + \frac{16}{3}\,t^3 + 7\,t^2,$$
$$y'(t) = -\frac{4}{3}\,t^6 - 4\,t^5 - 4\,t^4 + 4\,t^3 - 6\,t^2 + 2\,t,$$
$$z'(t) = -\frac{124}{3}\,t^6 - 80\,t^5 - 56\,t^4 - \frac{80}{3}\,t^3 - 46\,t^2 + 12\,t - 2,$$

possess no common factor, and we have

$$\sigma(t) = |\mathbf{r}'(t)| = \frac{382}{9}\,t^6 + \frac{244}{3}\,t^5 + 58\,t^4 + \frac{80}{3}\,t^3 + 47\,t^2 - 12\,t + 2,$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,(2\,t^4 - 4\,t^3 + 2\,t^2 + 1)\,\sigma(t).$$

The curvature and torsion have a non-constant ratio $\kappa(t)/\tau(t)$, namely

$$\frac{9\,(2\,t^4 - 4\,t^3 + 2\,t^2 + 1)^2}{460\,t^8 - 1840\,t^7 - 296\,t^6 - 2688\,t^5 + 1272\,t^4 + 624\,t^3 - 180\,t^2 + 144\,t + 63}.$$

In the present example, the polynomial $\mathbf{w}(t) = \mathbf{w}_0(1-t)^2 + \mathbf{w}_1 2(1-t)t + \mathbf{w}_2 t^2$ has roots $\tau_1, \tau_2 = \frac{1}{2}(1 \pm \sqrt{2} + \mathrm{i})$ that are neither both real nor complex conjugates, and one can verify that $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ can be expressed in terms of them in the form (41), with coefficients

$$\mathbf{a}_1 = \frac{1}{2}\,[\sqrt{2} - 1 + (4\sqrt{2} - 5)\,\mathrm{i}], \qquad \mathbf{a}_2 = -\frac{1}{2}\,[\sqrt{2} + 1 + (4\sqrt{2} + 5)\,\mathrm{i}],$$
$$\mathbf{b}_1 = \frac{1}{3}\,[6 - 5\sqrt{2} + (\sqrt{2} - 1)\,\mathrm{i}], \qquad \mathbf{b}_2 = \frac{1}{3}\,[6 + 5\sqrt{2} - (\sqrt{2} + 1)\,\mathrm{i}].$$

Since $\mathbf{a}_1\mathbf{b}_2 - \mathbf{a}_2\mathbf{b}_1 = -\sqrt{2}\,\mathrm{i}/3 \neq 0$ for these coefficients, the non-helical nature of the curve is consistent with Proposition 1.

5.3. *DPH curves with* $\deg(h) = 2$, $\deg(\mathbf{w}) = 1$

Example 6 ($\deg(h) = 2$, $\deg(\mathbf{w}) = 1$, *Helical*). In equations (12) we choose the numerical values

$$h_0 = 1, \qquad h_1 = 2, \qquad h_2 = 1, \qquad \mathbf{w}_0 = \mathrm{i}, \qquad \mathbf{w}_1 = 1.$$

Assigning values to $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_0$ and solving the bilinear system specified by (12) and the second expression in (13) for the other coefficients then give

$$\boldsymbol{\alpha}_0 = 1, \qquad \boldsymbol{\alpha}_1 = 1, \qquad \boldsymbol{\alpha}_2 = 2 - \frac{1}{3}\,\mathrm{i}, \qquad \boldsymbol{\alpha}_3 = 2 - 5\,\mathrm{i},$$

$$\boldsymbol{\beta}_0 = -1, \qquad \boldsymbol{\beta}_1 = -\frac{4}{3}, \qquad \boldsymbol{\beta}_2 = -\frac{8}{3} + \frac{2}{3}\mathrm{i}, \qquad \boldsymbol{\beta}_3 = -2 + 7\,\mathrm{i}.$$

The corresponding hodograph components are

$$\begin{aligned} x'(t) &= -14\,t^6 - 6\,t^5 + 3\,t^4 - 4\,t^3 - t^2 - 2\,t, \\ y'(t) &= -52\,t^6 + 4\,t^5 - 18\,t^4 + 4\,t^3 - 12\,t^2 - 2\,t - 2, \\ z'(t) &= -4\,t^6 - 4\,t^5 + 2\,t^4 - 2\,t^2. \end{aligned}$$

This hodograph is non-primitive: it has $\gcd(x'(t), y'(t), z'(t)) = 2\,t^2 - 2\,t + 1$ as the common factor of its components. For this case, we have

$$\sigma(t) = |\mathbf{r}'(t)| = (2\,t^2 - 2\,t + 1)(27\,t^4 + 26\,t^3 + 21\,t^2 + 6\,t + 2),$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,|2\,t^2 - 2\,t - 1|\,(2\,t^2 - 2\,t + 1)\,\sigma(t),$$

and the curvature/torsion ratio has the constant value $|\kappa(t)/\tau(t)| = 1/2$. In this case $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ have the common factor $\mathbf{w}(t) = t - \frac{1}{2}(1 - \mathrm{i})$, so the helical nature of the curve is consistent with Proposition 2.

For this curve, the rational Frenet frame is defined by

$$\mathbf{t} = -\frac{(t(t+1)(7t^2+3t+2), 2(13t^4+12t^3+10t^2+3t+1), 2t^2(t+1)^2)}{27t^4+26t^3+21t^2+6t+2},$$

$$\mathbf{p} = \frac{(2(11t^4+8t^3+8t^2+3t+1), -t(t+1)(7t^2+3t+2), 2t(t+1)(7t^2+3t+2))}{27t^4+26t^3+21t^2+6t+2},$$

$$\mathbf{b} = \frac{(-2t(t+1)(7t^2+3t+2), 2t^2(t+1)^2, 23t^4+18t^3+17t^2+6t+2)}{27t^4+26t^3+21t^2+6t+2},$$

and the rational curvature function is

$$\kappa(t) = \frac{|4\,t^2 - 4\,t - 2|}{(2\,t^2 - 2\,t + 1)\,(27\,t^4 + 26\,t^3 + 21\,t^2 + 6\,t + 2)^2}.$$

The Bernstein coefficients of (45) for this curve are

$$\mathcal{A}_0 = 1 - \mathbf{k}, \qquad \mathcal{A}_1 = 1 - \frac{4}{3}\,\mathbf{k},$$

$$\mathcal{A}_2 = 2 - \frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} - \frac{8}{3}\,\mathbf{k}, \qquad \mathcal{A}_3 = 2 - 5\,\mathbf{i} + 7\,\mathbf{j} - 2\,\mathbf{k},$$

and in this case $\mathcal{A}_1$, $\mathcal{A}_2$ can be expressed in terms of $\mathcal{A}_0$, $\mathcal{A}_3$ in the form (34), with the values

$$\mathbf{c}_{10} = \frac{1}{6}(1 - 2\,\mathrm{i}), \qquad \mathbf{c}_{13} = \frac{1}{6}\,\mathrm{i}, \qquad \mathbf{c}_{20} = -\frac{1}{6}\,\mathrm{i}, \qquad \mathbf{c}_{23} = \frac{1}{6}(1 + 2\,\mathrm{i})$$

of the coefficients (35). These coefficients satisfy (36), the expressions on the left and right having the common value $\frac{9}{16}$.

Example 7 ($\deg(h) = 2$, $\deg(\mathbf{w}) = 1$, *Non-helical*). Using the numerical values of the previous example, but the first rather than the second expression in (13), we obtain

$$\boldsymbol{\alpha}_0 = 1, \qquad \boldsymbol{\alpha}_1 = 1, \qquad \boldsymbol{\alpha}_2 = 2 + \mathrm{i}, \qquad \boldsymbol{\alpha}_3 = 2 - \mathrm{i},$$
$$\boldsymbol{\beta}_0 = -1, \qquad \boldsymbol{\beta}_1 = -\frac{4}{3}, \qquad \boldsymbol{\beta}_2 = -\frac{8}{3} - \frac{2}{3}\mathrm{i}, \qquad \boldsymbol{\beta}_3 = -2 + \frac{5}{3}\mathrm{i}.$$

The corresponding hodograph is primitive, with components

$$\begin{aligned} x'(t) &= -\frac{22}{9}t^6 - \frac{10}{3}t^5 + 11t^4 - 4t^3 - t^2 - 2t, \\ y'(t) &= -\frac{124}{3}t^6 + 68t^5 - 26t^4 + 4t^3 - 12t^2 - 2t - 2, \\ z'(t) &= -\frac{28}{3}t^6 + 12t^5 + 2t^4 - \frac{16}{3}t^3 - 2t^2, \end{aligned}$$

and we have

$$\sigma(t) = |\mathbf{r}'(t)| = \frac{382}{9}t^6 - \frac{206}{3}t^5 + 25t^4 - 4t^3 + 13t^2 + 2t + 2,$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,|2t^2 - 2t - 1|\,(2t^2 - 2t + 1)\,\sigma(t).$$

The curvature/torsion ratio is non-constant, namely

$$\frac{\kappa(t)}{\tau(t)} = \frac{9|2t^2 - 2t - 1|(2t^2 - 2t + 1)^2}{2\,(92t^6 - 276t^5 - 60t^4 + 228t^3 - 126t^2 + 54t + 9)}.$$

In this case we find that $\gcd(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)) = $ constant, so the non-helical nature of the curve is consistent with Proposition 3.

5.4. *DPH curves with* $\deg(h) = 4$, $\deg(\mathbf{w}) = 0$

Example 8 ($\deg(h) = 4$, $\deg(\mathbf{w}) = 0$, *Helical*). In equations (14) we choose the numerical values

$$h_0 = -1, \qquad h_1 = 2, \qquad h_2 = 3, \qquad h_3 = 4, \qquad h_4 = -5, \qquad \mathbf{w}_0 = 1.$$

Choosing complex values for $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_0$ and solving for the five remaining coefficients from the system of bilinear equations defined by (14) and (15) with the "+" sign, we obtain

$$\boldsymbol{\alpha}_0 = 1, \qquad \boldsymbol{\alpha}_1 = 1 - \mathrm{i}, \qquad \boldsymbol{\alpha}_2 = -1 + 4\,\mathrm{i}, \qquad \boldsymbol{\alpha}_3 = -1 + 9\,\mathrm{i},$$
$$\boldsymbol{\beta}_0 = -1 + \mathrm{i}, \qquad \boldsymbol{\beta}_1 = -\frac{1}{3} + 2\,\mathrm{i}, \qquad \boldsymbol{\beta}_2 = -\frac{5}{3} - 5\,\mathrm{i}, \qquad \boldsymbol{\beta}_3 = -5 - 10\,\mathrm{i}.$$

The components of the hodograph defined through (4) by the cubic complex polynomials $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t)$ are then

$$\begin{aligned} x'(t) &= -48t^6 + 216t^5 - 276t^4 + 48t^3 + 20t^2 - 2t - 1, \\ y'(t) &= -120t^6 + 600t^5 - 872t^4 + 208t^3 + 18t^2 - 2t - 2, \\ z'(t) &= -80t^6 + 384t^5 - 552t^4 + 128t^3 + 12t^2 - 2. \end{aligned}$$

This hodograph is primitive, and we have

$$\sigma(t) = |\mathbf{r}'(t)| = 152t^6 - 744t^5 + 1068t^4 - 248t^3 - 26t^2 + 2t + 3,$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,|12t^4 - 8t^3 + 12t^2 - 12t + 1|\,\sigma(t).$$

In this case, the curvature/torsion ratio has the constant value $|\kappa(t)/\tau(t)| = 1/8$. Note that, with the chosen values for the coefficients of $h(t)$, the quantity (44) is 0, so the fact that the curve is helical is consistent with Proposition 4. One can verify that for this curve, $\boldsymbol{\alpha}(t)/\boldsymbol{\beta}(t)$ may be regarded

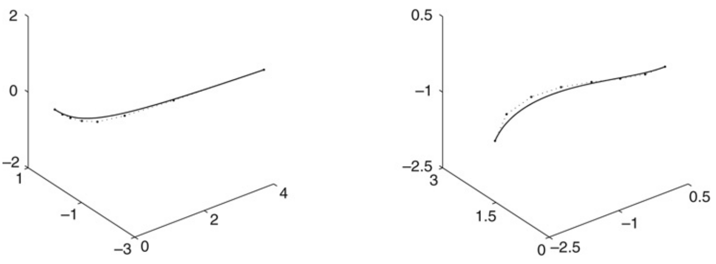

Fig. 2. The degree 7 non-helical DPH curves of Examples 9 and 10.

as arising from the re-parameterization defined by (24) with $(f_0, f_1, f_2, f_3) = (1, 2, -5, -10)$ and $(g_0, g_1, g_2, g_3) = (-1, -\frac{5}{3}, \frac{11}{3}, 7)$ applied to the line/circle form (16) with $(\mathbf{a}_0, \mathbf{a}_1) = (-3-3\,\mathrm{i}, -5-6\,\mathrm{i})$ and $(\mathbf{b}_0, \mathbf{b}_1) = (5, 9+\mathrm{i})$.

For the Bernstein coefficients $\mathcal{A}_l = \boldsymbol{\alpha}_l + \mathbf{k}\,\boldsymbol{\beta}_l$ of the quaternion polynomial (32) we obtain

$$\mathcal{A}_0 = 1 + \mathbf{j} - \mathbf{k}, \qquad \mathcal{A}_1 = 1 - \mathbf{i} + 2\,\mathbf{j} - \frac{1}{3}\,\mathbf{k},$$
$$\mathcal{A}_2 = -1 + 4\,\mathbf{i} - 5\,\mathbf{j} - \frac{5}{3}\,\mathbf{k}, \qquad \mathcal{A}_3 = -1 + 9\,\mathbf{i} - 10\,\mathbf{j} - 5\,\mathbf{k},$$

and we note that $\mathcal{A}_1$, $\mathcal{A}_2$ can be written in terms of $\mathcal{A}_0$, $\mathcal{A}_3$ in the form (27) where, with $k = 3$, the values of coefficients (28) are

$$c_{10} = \frac{8}{9}, \qquad c_{13} = -\frac{1}{9}, \qquad c_{20} = -\frac{5}{9}, \qquad c_{23} = \frac{4}{9}.$$

Example 9 ($\deg(h) = 4$, $\deg(\mathbf{w}) = 0$, *Non-helical*). Using equations (14) again, we now choose the numerical values

$$h_0 = 1, \qquad h_1 = 2, \qquad h_2 = 2, \qquad h_3 = 2, \qquad h_4 = 1, \qquad \mathbf{w}_0 = 1.$$

Choosing complex values for $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_0$ and solving for the five remaining coefficients from the system of bilinear equations defined by (14) and (15) with the "−" sign, we obtain

$$\boldsymbol{\alpha}_0 = 1, \qquad \boldsymbol{\alpha}_1 = 1, \qquad \boldsymbol{\alpha}_2 = 2 + \mathrm{i}, \qquad \boldsymbol{\alpha}_3 = 2 + 3\,\mathrm{i},$$
$$\boldsymbol{\beta}_0 = -1, \qquad \boldsymbol{\beta}_1 = -\frac{2}{3}, \qquad \boldsymbol{\beta}_2 = -\frac{2}{3} - \mathrm{i}, \qquad \boldsymbol{\beta}_3 = -2\,\mathrm{i}.$$

These give the hodograph components

$$x'(t) = 2\,t^6 - 4\,t^5 + 6\,t^4 + 3\,t^2 + 2\,t,$$
$$y'(t) = -4\,t^6 + 16\,t^5 - 28\,t^4 + 12\,t^3 - 8\,t^2 + 2\,t - 2,$$
$$z'(t) = 4\,t^6 - 12\,t^5 + 12\,t^4 + 4\,t^3.$$

This hodograph is also primitive, with

$$\sigma(t) = |\mathbf{r}'(t)| = 6\,t^6 - 20\,t^5 + 30\,t^4 - 8\,t^3 + 9\,t^2 - 2\,t + 2,$$
$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\,|2\,t^4 - 4\,t^3 + 6\,t^2 - 4\,t - 1|\,\sigma(t).$$

In this case, the curvature/torsion ratio is non-constant, namely

$$\frac{\kappa(t)}{\tau(t)} = \frac{|2\,t^4 - 4\,t^3 + 6\,t^2 - 4\,t - 1|}{2\,t\,(4\,t^3 + t^2 - 3\,t + 6)}.$$

For the specified coefficients of $h(t)$, the quantity (44) has the value $\Delta = -9$, so the non-helical nature of this curve is consistent with Proposition 4. This degree 7 non-helical DPH curve is illustrated in Fig. 2.

5.5. Example of Beltran and Monterde

Example 10 (*Beltran and Monterde*). Beltran and Monterde (2007) identify a degree 7 double PH curve $\mathbf{r}(t)$, given by

$$x(t) = \frac{1}{21}\,t^7 + \frac{1}{5}\,t^5 + t^3 - 3\,t, \qquad y(t) = -\frac{1}{2}\,t^4 + 3\,t^2, \qquad z(t) = -2\,t^3.$$

This curve has a primitive hodograph, and satisfies

$$\sigma(t) = \frac{t^6 + 3\,t^4 + 9\,t^2 + 9}{3}, \qquad |\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2\left(t^2 + 1\right)\left(t^6 + 3\,t^4 + 9\,t^2 + 9\right).$$

The curvature/torsion ratio for this curve is non-constant, namely

$$\frac{\kappa(t)}{\tau(t)} = -\,\frac{9\,(t^2 + 1)^2}{2\,t^6 + 9\,t^4 - 9}.$$

The cubic polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ have the Bernstein coefficients

$$\boldsymbol{\alpha}_0 = 0, \qquad \boldsymbol{\alpha}_1 = \frac{1}{\sqrt{3}}\,\mathrm{i}, \qquad \boldsymbol{\alpha}_2 = \frac{1}{\sqrt{3}}\,(1 + 2\,\mathrm{i}), \qquad \boldsymbol{\alpha}_3 = \frac{1}{\sqrt{3}}\,(3 + 2\,\mathrm{i}),$$
$$\boldsymbol{\beta}_0 = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \boldsymbol{\beta}_3 = \sqrt{3}\,\mathrm{i},$$

and are thus given by

$$\boldsymbol{\alpha}(t) = \sqrt{3}\left[t^2 + \left(t - \frac{1}{3}t^3\right)\mathrm{i}\right], \qquad \boldsymbol{\beta}(t) = \sqrt{3}\,\mathrm{i}.$$

In this case, the proportionality polynomial is

$$\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t) = -3\,(t + \mathrm{i})^2.$$

Hence, this curve satisfies the double PH condition (7) with $\deg(h) = 0$ and $\deg(\mathbf{w}) = 1$. This is a special (degenerate) DPH curve of degree 7 — for which $\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)$ is just quadratic, and is thus deficient in degree compared to the generic case of a quartic. It may be interpreted as a special coincidence of the cases $\deg(h) = 0$, $\deg(w) = 2$ and $\deg(h) = 2$, $\deg(w) = 1$ discussed in Sections 2.2.1 and 2.2.2 — in the former case, $w(t)$ is considered to exhibit a degree reduction from 2 to 1; in the latter case, $h(t)$ is considered to exhibit a degree reduction from 2 to 0.

For this curve, the quaternion polynomial (45) has Bernstein coefficients

$$\mathcal{A}_0 = \sqrt{3}\,\mathbf{j}, \qquad \mathcal{A}_1 = \sqrt{3}\left(\frac{1}{3}\,\mathbf{i} + \mathbf{j}\right), \qquad \mathcal{A}_2 = \sqrt{3}\left(\frac{1}{3} + \frac{2}{3}\,\mathbf{i} + \mathbf{j}\right),$$
$$\mathcal{A}_3 = \sqrt{3}\left(1 + \frac{2}{3}\,\mathbf{i} + \mathbf{j}\right).$$

Substituting into (46), the Bézier control points of this curve are found to be

$$\begin{aligned}
\mathbf{p}_0 &= (0.0000, 0.0000, 0.0000), & \mathbf{p}_1 &= (-0.4286, 0.0000, 0.0000),\\
\mathbf{p}_2 &= (-0.8571, 0.1429, 0.0000), & \mathbf{p}_3 &= (-1.2571, 0.4286, -0.0571),\\
\mathbf{p}_4 &= (-1.6000, 0.8429, -0.2286), & \mathbf{p}_5 &= (-1.8476, 1.3571, -0.5714),\\
\mathbf{p}_6 &= (-1.9429, 1.9286, -1.1429), & \mathbf{p}_7 &= (-1.7524, 2.5000, -2.0000).
\end{aligned}$$

This curve is illustrated, together with its control polygon, in Fig. 2.

6. Closure

A complete categorization of *double* Pythagorean-hodograph (DPH) curves of degrees 3, 5, and 7 has been presented, together with algorithms for their construction and a representative selection of computed examples. Such DPH curves possess the attractive distinguishing property that their Frenet frames, and curvature and torsion functions, have a *rational* dependence on the curve parameter, and they incorporate all *helical* polynomial curves.

All spatial PH cubics are DPH curves — they are also helical, and admit simple characterizations in terms of the Bézier control polygon geometry (Farouki and Sakkalis, 1994). As noted in Beltran and Monterde (2007), the DPH curves of degree 5 correspond to the *helical* spatial PH quintics, discussed in Beltran and Monterde (2007) and Farouki et al. (2004). The focus of this paper was therefore on the degree 7 DPH curves, which admit both helical and non-helical instances (Beltran and Monterde, 2007). In particular, the Hopf map formulation (7) of the DPH condition – specified in terms of complex polynomials $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ – was invoked to categorize the degree 7 DPH curves in terms of the possible combinations of the degrees for the real polynomial $h(t)$ and the complex polynomial $\mathbf{w}(t)$ in (7).

For each category of the degree 7 DPH curves, a system of equations and compatibility constraints was derived, whose solutions facilitate the construction of representative example curves. Moreover, simple criteria were formulated to distinguish between helical and non-helical degree 7 DPH curves in each category. For the helical DPH curves, a more intuitive construction — based on the approach of Monterde (in press) that uses inverse stereographic projection of a line/circle to generate a circular tangent indicatrix — was also described. Starting from lines/circles parameterized in terms of rational linear complex functions, all higher-order representations are generated by multiplying the numerator and denominator by a complex polynomial, by a (real) non-linear rational re-parameterization, or by a combination of these schemes.

Although the results of this paper have been couched primarily in terms of the Hopf map representation, they are easily translated into the language of the quaternion model (more commonly employed in practice), as described in the companion paper (Farouki et al., 2009). Finally, we note that the construction algorithms for degree 7 DPH curves described herein are mostly algebraic in character, and hence do not offer much insight into the shape properties of the resulting curves. For geometric design applications, it would be desirable to formulate more geometrically-intuitive constructions, such as Hermite interpolation — such algorithms are a fruitful topic for further research.

References

Beltran, J.V., Monterde, J., 2007. A characterization of quintic helices. J. Comput. Appl. Math. 206, 116–121.

Choi, H.I., Lee, D.S., Moon, H.P., 2002. Clifford algebra, spin representation, and rational parameterization of curves and surfaces. Adv. Comput. Math. 17, 5–48.

Farouki, R.T., al-Kandari, M., Sakkalis, T., 2002. Hermite interpolation by rotation-invariant spatial Pythagorean hodograph curves. Adv. Comput. Math. 17, 369–383.

Farouki, R.T., Giannelli, C., Manni, C., Sestini, A., 2008. Identification of spatial PH quintic Hermite interpolants with near-optimal shape measures. Comput. Aided Geom. Design 25, 274–297.

Farouki, R.T., Giannelli, C., Sestini, A., 2009. Helical polynomial curves and double Pythagorean hodographs I. Quaternion and Hopf map representations. J. Symbolic Comput. 44 (2), 161–179.

Farouki, R.T., Han, C.Y., Manni, C., Sestini, A., 2004. Characterization and construction of helical polynomial space curves. J. Comput. Appl. Math. 162, 365–392.

Farouki, R.T., Sakkalis, T., 1994. Pythagorean-hodograph space curves. Adv. Comput. Math. 2, 41–66.

Kreyszig, E., 1959. Differential Geometry. University of Toronto Press.

Kubota, K.K., 1972. Pythagorean triples in unique factorization domains. Amer. Math. Monthly 79, 503–505.

Monterde, J., 2008. A characterization of helical polynomial curves of any degree, Adv. Comput. Math. (in press).

Needham, T., 1997. Visual Complex Analysis. Oxford University Press.

Schwerdtfeger, H., 1979. Geometry of Complex Numbers. Dover Publications, New York. Reprint.

Wagner, M.G., Ravani, B., 1997. Curves with rational Frenet–Serret motion. Comput. Aided Geom. Design 15, 79–101.