Computers and Mathematics with Applications 61 (2011) 1162–1166

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A theorem on the Cesàro summability method

İbrahim Çanak *

Ege University, Department of Mathematics, İzmir 35100, Turkey

ARTICLE INFO

Article history:
Received 23 July 2010
Received in revised form 26 December 2010
Accepted 27 December 2010

Keywords:
Slow oscillation
General control modulo
Cesàro summability
Tauberian theorems
Subsequence convergence

ABSTRACT

In this paper we retrieve the backward Cesàro convergence of a real sequence $u = (u_n)$ from the Cesàro summability of the general control modulo of the oscillatory behavior of integer order m of (u_n) under certain conditions.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

Throughout this paper, the symbols $u_n = o(1)$ and $u_n = O(1)$ mean respectively that $u_n \to 0$ as $n \to \infty$ and that (u_n) is bounded for large enough n. Let the sequence of the backward differences of a sequence $u = (u_n)$ be denoted by $\Delta u = (\Delta u_n)$, where $\Delta u_n = u_n - u_{n-1}$ for $n \geq 1$, and $\Delta u_0 = u_0$ for $n = 0$. For each integer $m \geq 0$ and for all nonnegative integers n we define $\sigma_n^{(m)}(u)$ by

$$\sigma_n^{(m)}(u) = \begin{cases} \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} \sigma_k^{(m-1)}(u) = u_0 + \sum_{k=1}^{n} \frac{V_k^{(m-1)}(\Delta u)}{k}, & m \geq 1 \\ u_n, & m = 0 \end{cases}$$

where

$$V_n^{(m)}(\Delta u) = \begin{cases} \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} V_k^{(m-1)}(\Delta u), & m \geq 1 \\ \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} k\Delta u_k, & m = 0. \end{cases}$$

The identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) \tag{1.1}$$

which is well-known and will be used extensively in the proofs is known as the Kronecker identity. Since $\sigma_n^{(1)}(u) = \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0$, the Kronecker identity can be rewritten in terms of the generator sequence $V_n^{(0)}(\Delta u)$ of (u_n) as

$$u_n = V_n^{(0)}(\Delta u) + \sum_{k=1}^{n} \frac{V_k^{(0)}(\Delta u)}{k} + u_0. \tag{1.2}$$

* Tel.: +90 232 388 4000; fax: +90 232 342 6951.
E-mail addresses: ibrahim.canak@ege.edu.tr, ibrahimcanak@yahoo.com.

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.12.065

The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$ and the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of (u_n) is defined inductively in [1,2] by

$$\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u)) \tag{1.3}$$

for all nonnegative integers n. We define $(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1}u_n)$ for each integer $m \geq 1$ and for all nonnegative integers n, where $(n\Delta)_0 u_n = u_n$. It is proved in [3] that $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u)$ for each integer $m \geq 1$.

A sequence (u_n) is Cesàro summable to s if $\lim_n \sigma_n^{(1)}(u) = s$.

A sequence (u_n) is backward convergent if $\Delta u_n = o(1)$. A sequence (u_n) is backward Cesàro convergent if $\Delta\sigma_n^{(1)}(u) = o(1)$.

A sequence (u_n) is said to be subsequentially convergent [1] if there exists a finite interval $I(u)$ such that all accumulation points of (u_n) are in $I(u)$ and every point of $I(u)$ is an accumulation point of (u_n). We note that bounded backward convergent sequences are subsequentially convergent.

A sequence (u_n) is said to be slowly oscillating [4] if

$$\lim_{\lambda\to 1^+} \limsup_n \max_{n+1\leq k\leq [\lambda n]} |u_k - u_n| = 0,$$

where $[\lambda n]$ denotes the integer part of λn. It is proved in [2] that (u_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta u))$ is bounded and slowly oscillating.

A sequence (u_n) is said to be $|C, 1|_p$ summable [5] if for $p > 1$

$$\sum_{j=1}^{\infty} j^{p-1}|\Delta\sigma_j^{(1)}(u)|^p < \infty.$$

A positive sequence (u_n) is O-regularly varying [6] if $\overline{\lim}_n \frac{u_{[\lambda n]}}{u_n} < \infty$ for $\lambda > 1$. A sequence (u_n) is said to be slowly varying [7] if

$$\lim_n \frac{u([\lambda n])}{u(n)} = 1$$

for $\lambda > 1$.

Our goal is to retrieve backward Cesàro convergence of a real sequence $u = (u_n)$ from the Cesàro summability of the general control modulo of oscillatory behavior of integer order m of (u_n) under certain conditions.

We prove the following theorem.

Theorem 1.1. *Let $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ be Cesàro summable to s. If for some $p > 1$*

$$(\lambda - 1)^{p-1} \limsup_n \sum_{j=n+1}^{[\lambda n]} \frac{|\omega_j^{(m)}(u)|^p}{j} = o(1), \quad \lambda \to 1^+, \tag{1.4}$$

then (u_n) is backward Cesàro convergent.

Remark 1.2. We note that for $p > 1$,

$$\sum_{j=1}^{n} \frac{|\omega_j^{(m)}(u)|^p}{j} = \sum_{j=1}^{n} j^{p-1}|\Delta\sigma^{(1)}(\omega_j^{(m-1)}(u))|^p. \tag{1.5}$$

If

$$\sum_{j=1}^{n} j^{p-1}|\Delta\sigma^{(1)}(\omega_j^{(m-1)}(u))|^p = \log v_n, \quad p > 1 \tag{1.6}$$

for some O-regularly varying sequence (v_n), then (1.6) is equivalent to

$$V_n^{(0)}(|\Delta\omega^{(m)}(u)|, p) = \frac{1}{n}\sum_{k=1}^{n} k^p|\Delta\omega_k^{(m)}(u)|^p = O(1), \quad p > 1$$

which is a Tauberian condition for the Cesàro and Abel summability method (see [8,9,3,10]). For more detailed information about this equivalence, we refer the reader to [11].

The proof is based on the following lemmas.

Lemma 1.3 ([4]). *For* $\lambda > 1$,

$$u_n - \sigma_n^{(1)}(u) = \frac{[\lambda n]+1}{[\lambda n]-n}(\sigma_{[\lambda n]}^{(1)}(u) - \sigma_n^{(1)}(u)) - \frac{1}{[\lambda n]-n}\sum_{k=n+1}^{[\lambda n]}\sum_{j=n+1}^{k}\Delta u_j, \tag{1.7}$$

where $[\lambda n]$ *denotes the integer part of* λn.

Lemma 1.4. *Let* (u_n) *be a sequence of real numbers. If* $\omega_n^{(m)}(u) = na_n$ *for some sequence* $a = (a_n)$, *then*

$$\omega_n^{(m-j)}(u) = n\sum_{k=0}^{j}\binom{j}{k}\sigma_n^{(k)}(a) \tag{1.8}$$

for $j = 0, 1, 2, \ldots, m-1$.

Proof. Let $\omega_n^{(m)}(u) = na_n$ for some sequence $a = (a_n)$. Hence, $\sigma_n^{(1)}(\omega^{(m-1)}(u)) = S_n(a)$. Using the equality (1.3) and $\omega_n^{(m)}(u) = na_n$, we get

$$\omega_n^{(m-1)}(u) = n(a_n + \sigma_n^{(1)}(a)). \tag{1.9}$$

Similarly, the equality (1.3) and (1.9) gives

$$\omega_n^{(m-2)}(u) = n(a_n + 2\sigma_n^{(1)}(a) + \sigma_n^{(2)}(a)). \tag{1.10}$$

Continuing in this way, we have

$$\omega_n^{(m-j)}(u) = n\sum_{k=0}^{j}\binom{j}{k}\sigma_n^{(k)}(a)$$

for $j = 0, 1, 2, \ldots, m-1$. □

2. Proof of Theorem 1.1

Applying Lemma 1.3 to $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$, we have

$$\begin{aligned}|\sigma_n^{(1)}(\omega^{(m-1)}(u)) - \sigma_n^{(2)}(\omega^{(m-1)}(u))| \le{}& \frac{[\lambda n]+1}{[\lambda n]-n}|\sigma_{[\lambda n]}^{(2)}(\omega^{(m-1)}(u)) - \sigma_n^{(2)}(\omega^{(m-1)}(u))| \\ &+ \max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_j^{(1)}(\omega^{(m-1)}(u))\right|.\end{aligned} \tag{2.1}$$

Since $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ is Cesàro summable to s, the first term on the right-hand side of (2.1) is $o(1)$ and so (2.1) becomes

$$\limsup_n |\sigma_n^{(1)}(\omega^{(m-1)}(u)) - \sigma_n^{(2)}(\omega^{(m-1)}(u))| \le \limsup_n \max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_j^{(1)}(\omega^{(m-1)}(u))\right|. \tag{2.2}$$

For the second term on the right-hand side of (2.1) we have

$$\begin{aligned}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta\sigma_j^{(1)}(\omega^{(m-1)}(u))\right| &\le \sum_{j=n+1}^{[\lambda n]}|\Delta\sigma_j^{(1)}(\omega^{(m-1)}(u))| \\ &\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(m)}(u)|^p}{j^p}\right)^{\frac{1}{p}}, \quad \text{where } \frac{1}{p}+\frac{1}{q}=1 \\ &\le ([\lambda n]-n)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(m)}(u)|^p}{j^{p-1}j}\right)^{\frac{1}{p}} \\ &\le ([\lambda n]-n)^{\frac{1}{q}}\left(\frac{1}{n^{p-1}}\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(m)}(u)|^p}{j}\right)^{\frac{1}{p}} \\ &\le \frac{([\lambda n]-n)^{\frac{1}{q}}}{n^{\frac{1}{q}}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(m)}(u)|^p}{j}\right)^{\frac{1}{p}} \\ &\le (\lambda-1)^{\frac{1}{q}}\left(\sum_{j=n+1}^{[\lambda n]}\frac{|\omega_j^{(m)}(u)|^p}{j}\right)^{\frac{1}{p}}.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3) we have

$$\limsup_n |\sigma_n^{(1)}(\omega^{(m-1)}(u)) - \sigma_n^{(2)}(\omega^{(m-1)}(u))| \le (\lambda-1)^{\frac{1}{q}} \limsup_n \left(\sum_{j=n+1}^{[\lambda n]} \frac{|\omega_j^{(m)}(u)|^p}{j} \right)^{\frac{1}{p}}. \tag{2.4}$$

Letting $\lambda \to 1^+$ in (2.4) and taking (1.4) into account, we deduce that

$$\limsup_n |\sigma_n^{(1)}(\omega^{(m-1)}(u)) - \sigma_n^{(2)}(\omega^{(m-1)}(u))| \le 0. \tag{2.5}$$

From (2.5) we have $\lim_n \sigma_n^{(1)}(\omega^{(m-1)}(u)) = s$. Since $\sigma_n^{(1)}(\omega^{(m-1)}(u)) = \omega_n^{(m-1)}(\sigma^{(1)}(u))$ for any sequence $u = (u_n)$ and any integer $m \ge 1$, we have $\lim_n \omega_n^{(m-1)}(\sigma^{(1)}(u)) = s$. Taking $j = m-1$ in Lemma 1.4, we obtain

$$\omega_n^{(1)}(\sigma^{(1)}(u)) = n \sum_{k=0}^{m-1} \binom{m-1}{k} \sigma_n^{(k)}(\epsilon) \tag{2.6}$$

for some null sequence $\epsilon = (\epsilon_n)$. Dividing the equality (2.6) by n and then summing the resulting equality from $k = 0$ to n, we have

$$V_n^{(1)}(\Delta u) = \sum_{j=0}^{n} \sum_{k=0}^{m-1} \binom{m-1}{k} \sigma_j^{(k)}(\epsilon).$$

Since $\Delta V_n^{(1)}(\Delta u) = o(1)$ and $\Delta \sigma_n^{(2)}(u) = o(1)$, it follows by the Kronecker identity that $\Delta \sigma_n^{(1)}(u) = o(1)$.

Furthermore, $(\Delta \sigma_n^{(1)}(u))$ is slowly oscillating and for some slowly varying sequence (B_n), we have $\Delta \sigma_n^{(1)}(u) = O(B_n)$. Since $\Delta \sigma_n^{(1)}(u) = O(B_n), n \to \infty$, it follows that there exists a finite interval I such that for every $r \in I$, there is a subsequence $(\frac{\Delta \sigma_{n(r)}^{(1)}(u)}{B(n(r))})$ such that $\lim_{n(r)} \frac{\Delta \sigma_{n(r)}^{(1)}(u)}{B(n(r))} = r$ (see [1,12]).

As a corollary we have the following result.

Corollary 2.1. *Let $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ be Cesàro summable to s. If for some $p > 1$, $(\sigma_n^{(1)}(\omega^{(m-1)}(u)))$ is $|C,1|_p$ summable, then (u_n) is backward Cesàro convergent.*

Example. Let (a_n) be a bounded sequence such that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty. \tag{2.7}$$

Consider a sequence (u_n) defined by

$$u_n = \sum_{k=2}^{n} \frac{a_k}{k} + \sum_{k=2}^{n} \frac{1}{k} \left(\sum_{j=1}^{k-1} \frac{a_j}{j} \right) \tag{2.8}$$

for $n \ge 2$ and $u_1 = u_0 = 0$.

Put $m = 1$ in Theorem 1.1. For the sequence (u_n), we have

$$\omega_n^{(1)}(u) = (n\Delta)_1 V_n^{(0)}(\Delta u) = a_n.$$

The condition (2.7) implies that $(\sigma_n^{(1)}(\omega^{(0)}(u))) = (V_n^{(0)}(\Delta u))$ is Cesàro summable. It follows by the boundedness of (a_n) that

$$\sum_{j=n+1}^{[\lambda n]} \frac{|\omega_j^{(1)}(u)|^p}{j} = \sum_{j=n+1}^{[\lambda n]} \frac{|a_j|^p}{j} \le C \sum_{j=n+1}^{[\lambda n]} \frac{1}{j} \le C \frac{[\lambda n] - n}{n} \tag{2.9}$$

for some positive constant C. Taking the lim sup in (2.9) gives

$$\limsup_n \sum_{j=n+1}^{[\lambda n]} \frac{|\omega_n^{(1)}(u)|^p}{j} \le C(\lambda - 1), \tag{2.10}$$

which shows that (1.4) holds for $m = 1$. So, the sequence (u_n) is backward Cesàro convergent by Theorem 1.1.

Note that one can easily construct sequences like in the example above for the case $m \ge 2$ in Theorem 1.1.

References

[1] F. Dik, Tauberian theorems for convergence and subsequential convergence of sequences with controlled oscillatory behavior, Math. Morav. 5 (2001) 19–56.
[2] M. Dik, Tauberian theorems for sequences with moderately oscillatory control modulo, Math. Morav. 5 (2001) 57–94.
[3] İ. Çanak, Ü. Totur, A Tauberian theorem with a generalized one-sided condition, Abstr. Appl. Anal. (2007) (2007) Article ID 45852.
[4] Č.V. Stanojević, Analysis of divergence: control and management of divergent process, in: İ. Çanak (Ed.), Graduate Research Seminar Lecture Notes, University of Missouri-Rolla, Fall, 1998.
[5] T.M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. Lond. Math. Soc., III. Ser. 7 (1957) 113–141.
[6] V.G. Avakumović, Sur une extensions de la condition de convergence des theorems inverses de sommabilite, C. R. Acad. Sci. Paris 200 (1935) 1515–1517.
[7] J. Karamata, Sur un mode de croissance régulière des fonctions, Mathematica 4 (1930) 38–53.
[8] İ. Çanak, M. Albayrak, A note on a Tauberian theorem for (A, i) limitable method, Int. J. Pure Appl. Math. 35 (3) (2007) 421–424.
[9] I. Çanak, Ü. Totur, M. Dik, Subsequential convergence conditions, J. Inequal. Appl. (2007) (2007) Article ID 87414.
[10] İ. Çanak, Ü. Totur, Tauberian theorems for Abel limitability method, Cent. Eur. J. Math. 6 (2) (2008) 301–306.
[11] V.B. Stanojević, Fourier and trigonometric transforms with complex coefficients regularly varying in mean, in: William O. Bray, et al. (Eds.), Fourier Analysis: Analytic and Geometric Aspects. Proceedings of the 6th International Workshop on Analysis and its Applications, IWAA, held at the University of Maine, Orono, USA, June 15–21, 1992, in: Lect. Notes Pure Appl. Math., vol. 157, Marcel Dekker, New York, 1994, pp. 423–432.
[12] F. Dik, M. Dik, İ. Çanak, Applications of subsequential Tauberian theory to classical Tauberian theory, Appl. Math. Lett. 20 (2007) 946–950.