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Advances in Mathematics 229 (2012) 2656–2668

ADVANCES IN Mathematics

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Differentiating the Weyl generic dimension formula with applications to support varieties

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Received 23 February 2011; accepted 13 January 2012

Available online 6 February 2012

Communicated by Henning Krause

Abstract

The authors compute the support varieties of all irreducible modules for the small quantum group $u_\zeta(\mathfrak{g})$, where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra, and ζ is a primitive ℓ-th root of unity with ℓ larger than the Coxeter number of $\mathfrak{g}$. The calculation employs the prior calculations and techniques of Ostrik and of Nakano, Parshall, and Vella, as well as deep results involving the validity of the Lusztig character formula for quantum groups and the positivity of parabolic Kazhdan–Lusztig polynomials for the affine Weyl group. Analogous support variety calculations are provided for the first Frobenius kernel G_1 of a reductive algebraic group scheme G defined over the prime field $\mathbb{F}_p$.

MSC: primary 17B55, 20G; secondary 17B50

Keywords: Quantum groups; Support varieties; Cohomology; Frobenius kernels; Restricted Lie algebra

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[1] Research of the first author was partially supported by NSF VIGRE grant DMS-0738586.

[2] Research of the second author was partially supported by NSF grants DMS-0654169 and DMS-1002135.

[3] Research of the third author was partially supported by NSF grants DMS-0701116 and DMS-1001900.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2012.01.007

1. Introduction

Let $u_\zeta(\mathfrak{g})$ be the small quantum group associated to the finite-dimensional simple complex Lie algebra $\mathfrak{g}$, with parameter $\zeta \in \mathbb{C}$ a primitive ℓ-th root of unity. Assume that ℓ is odd, not divisible by 3 in case $\mathfrak{g}$ is of type G_2, and that ℓ is greater than the Coxeter number h of $\mathfrak{g}$. In 1993, Ginzburg and Kumar [11] proved that the cohomology algebra $R := \mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C})$ is isomorphic to the coordinate algebra $\mathbb{C}[\mathcal{N}]$ of the closed affine subvariety $\mathcal{N} = \mathcal{N}(\mathfrak{g})$ of nilpotent elements in $\mathfrak{g}$. Given a finite-dimensional $u_\zeta(\mathfrak{g})$-module M, the annihilator in R of the Ext-group $\mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{g})}(M, M)$ defines a closed subvariety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ of $\mathcal{N}$, called the support variety of M. Support varieties provide a method for understanding the interplay between the underlying geometry and the overall representation theory. For the small quantum group, the support varieties for the restriction to $u_\zeta(\mathfrak{g})$ of Weyl modules for the Lusztig quantum group $U_\zeta(\mathfrak{g})$ were calculated by Ostrik [22], and in subsequent work by Bendel, Nakano, Parshall, and Pillen [3].

The main result of the present paper provides an explicit determination of the support variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L)$ when L is an arbitrary irreducible $u_\zeta(\mathfrak{g})$-module. To describe this result, it can be assumed that L is the restriction to $u_\zeta(\mathfrak{g})$ of an irreducible $U_\zeta(\mathfrak{g})$-module $L_\zeta(\lambda)$ whose highest weight λ is ℓ-restricted. Let $W_\ell = \ell Q \rtimes W$ be the affine Weyl group, which acts via the dot action on the Euclidean space $\mathbb{E}$ spanned by Φ. In the root system Φ of $\mathfrak{g}$, let $\Phi_\lambda = \{\alpha \in \Phi\colon \langle \lambda + \rho, \alpha^\vee \rangle \equiv 0 \pmod{\ell}\}$ be the set of roots α such that, for some $n \in \mathbb{Z}$, λ is fixed by the affine reflection $s_{\alpha,n}$. Here we are using standard notation, most of it explained at the end of this introduction. Then there exists $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$, the subroot system generated by a subset J of a fixed set Π of simple roots in Φ. Let $\mathfrak{b}$ be the negative Borel subalgebra of $\mathfrak{g}$ defined by Π, let $\mathfrak{p}_J \supset \mathfrak{b}$ be the standard parabolic subalgebra corresponding to J, and let $\mathfrak{u}_J$ be the nilradical of $\mathfrak{p}_J$. Then Theorem 3.3 establishes that

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}\big(L_\zeta(\lambda)\big) = G \cdot \mathfrak{u}_J,$$

where G is the simple, simply-connected complex algebraic group acting on its Lie algebra $\mathfrak{g}$ by the adjoint action. In other words, the support variety of $L_\zeta(\lambda)$ is the closure in $\mathcal{N}$ of a certain explicitly described Richardson orbit in $\mathcal{N}$. Observe that results of either [22] or [3] imply that $G \cdot \mathfrak{u}_J = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda))$, the support variety of the quantum Weyl module $\Delta_\zeta(\lambda)$ of highest weight λ.

As a guide to the reader, we provide a brief outline here of the proof of our main result. A first step consists of showing that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) \subseteq \mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda))$. If the dominant weight λ lies in the bottom ℓ-alcove, then $L_\zeta(\lambda) = \Delta_\zeta(\lambda)$, so the equality follows in this case. Otherwise, $L_\zeta(\lambda)$ is the head of $\Delta_\zeta(\lambda)$, and the other composition factors of $\Delta_\zeta(\lambda)$ have the form $L_\zeta(\mu)$ with $\mu < \lambda$ and μ linked to λ under the dot action of W_ℓ. Given an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finite-dimensional $u_\zeta(\mathfrak{g})$-modules and $i \in \{1, 2, 3\}$, the support variety of M_i is contained in the union of the support varieties of the other two modules. Now an easy induction shows for arbitrary dominant λ that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) \subseteq \mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda))$.

The more difficult step entails showing the reverse containment. Interestingly, this half of the argument is "analytic" in style. Since support varieties are closed subvarieties of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$, it is sufficient to prove that

$$\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}\big(L_\zeta(\lambda)\big) \geqslant \dim \mathcal{V}_{u_\zeta(\mathfrak{g})}\big(\Delta_\zeta(\lambda)\big).$$

Given a finite-dimensional $u_\zeta(\mathfrak{g})$-module V, it is well known (see, e.g., [7]) that the complexity of V can be defined as the rate of growth of various sequences $\{a_n\}_{n=0}^{\infty}$, with one choice being to take $a_n = \dim \operatorname{Ext}^n_{u_\zeta(\mathfrak{g})}(V, V)$. The complexity has a geometric interpretation as $\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(V)$. In addition, the complexity of V is bounded *below* by the order of a pole at ζ of a certain naturally defined rational function. This rational function is easily described in terms of the so-called generic dimension $\dim_t V$ of V, defined in (2.2.2) below. As developed in [21, §3] (influenced by earlier work of [22]), this leads to the inequality $c_{u_\zeta(\mathfrak{g})}(V) \geqslant |\Phi| - 2s + 2$, where s is the multiplicity of ζ as a root of the Laurent polynomial $\dim_t V$.

For $V = L_\zeta(\lambda)$, the multiplicity s is determined by repeatedly differentiating a variant $f(t)$ of the generic dimension $\dim_t L_\zeta(\lambda)$ and then substituting $t = \zeta$. In this way, we are able to determine the smallest s for which $f^{(s)}(\zeta) \neq 0$. Because $\dim_t L_\zeta(\lambda)$ is easily determined from the character of $L_\zeta(\lambda)$, a first step, carried out in Proposition 3.1, amounts to rewriting the Lusztig character formula for $L_\zeta(\lambda)$ in terms of certain parabolic Kazhdan–Lusztig polynomials $P^{I,-1}_{y,w}$; these polynomials were introduced by Deodhar [6], though our notation follows Kashiwara–Tanisaki [18]. Because the coefficients of $P^{I,-1}_{y,w}$ have interpretations as multiplicities of composition factors in certain Hodge modules, these coefficients are all non-negative [19]. Finally, making use of a subtle combinatorial result, given in Theorem 2.4, involving the alcove geometry of W_ℓ, we determine that $s = |\Phi^+_\lambda|$. This gives

$$\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}\bigl(L_\zeta(\lambda)\bigr) = c_{u_\zeta(\mathfrak{g})}\bigl(L_\zeta(\lambda)\bigr) \geqslant |\Phi| - |\Phi_\lambda| = \dim G \cdot \mathfrak{u}_J = \dim \mathcal{V}_{u_\zeta(\mathfrak{g})}\bigl(\Delta_\zeta(\lambda)\bigr),$$

as required.

Section 4 shifts attention to a simple, simply-connected algebraic group G defined and split over a prime field $\mathbb{F}_p$, $p > 0$. We show that the above calculation of support varieties for irreducible modules holds when $u_\zeta(\mathfrak{g})$ is replaced by the restricted enveloping algebra $u(\mathfrak{g})$ (or, equivalently, the first Frobenius kernel G_1) associated to G; cf. Theorem 4.1. Thus, $\mathfrak{g}$ is now the Lie algebra of G and the support varieties are closed subvarieties of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$. The calculation assumes that $p > h$, so that early results of Friedlander and Parshall [9] and Andersen and Jantzen [1] can be used. It is also assumed that *the modular Lusztig character formula holds for G for all restricted dominant weights*. This last assumption is known to hold for p sufficiently large, depending on the root system Φ of $\mathfrak{g}$; see Andersen, Jantzen, and Soergel [2] and the subsequent work of Fiebig [8], which provides explicit, albeit large, sufficient bounds on the size of p. Some instances for which the Lusztig character formula is known to hold are listed in Section 4. At present, it is still expected that Lusztig's conjecture holds if $p \geqslant h$. For the $p = h$ version of the above support variety result, see Remark 4.2.

The results presented in this paper signify the first complete non-trivial calculation of the support varieties of irreducible modules for a large class of important algebras. Some evidence for our calculations exists already in the literature. The cases in which the highest weight of L is regular or lies on a single wall (i.e., the subregular case) were established in [23] (inspired by earlier work [15] of Jantzen for algebraic groups). In fact, the results in [23] on the generic dimension motivate the results of Section 2. For irreducible G-modules having *regular* highest weights, the calculation given in Section 4 was already shown in [21] and was attributed there to Jantzen. It has been known for some time (cf. [5]) that the validity of the Lusztig character formula completely determines, through parity considerations, the groups $\operatorname{Ext}^\bullet_A(L, L)$, for $A \in \{G_1, u_\zeta(\mathfrak{g})\}$ and L, L' irreducible A-modules with regular highest weights. In fact, the dimensions of these cohomology groups are given in terms of Kazhdan–Lusztig polynomials. Conversely, essentially no results are known for these Ext-groups when L, L' have singular highest weights. Thus, from this viewpoint, it seems remarkable that the support varieties for *all* irreducible modules can be determined explicitly in the quantum case or in the modular case assuming the validity of the Lusztig character formula.

1.1. Some preliminary notation and conventions

All of the following notation is standard.

(1) $\mathfrak{g}$: finite-dimensional simple complex Lie algebra.
(2) $\mathfrak{h}$: Cartan subalgebra of $\mathfrak{g}$.
(3) Φ: corresponding irreducible root system.
(4) $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, Φ^+, α_0: simple roots, positive roots, and maximal short root.
(5) $\mathfrak{b}$: Borel subalgebra of $\mathfrak{g}$ consisting of the span of negative root vectors and $\mathfrak{h}$, and opposite to $\mathfrak{b}^+$ (span of positive root vectors and $\mathfrak{h}$).
(6) $\mathbb{E}$: Euclidean space spanned by Φ.
(7) $Q = \mathbb{Z}\Phi$: root lattice in $\mathbb{E}$.
(8) $\langle\cdot,\cdot\rangle$: W-invariant inner product on $\mathbb{E}$, normalized so that $\langle\alpha,\alpha\rangle = 2$ if $\alpha \in \Phi$ is a short root.
(9) ρ: Weyl weight defined by $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.
(10) $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$: coroot of $\alpha \in \Phi$.
(11) $h = \langle\rho, \alpha_0^\vee\rangle + 1$: Coxeter number of Φ.
(12) $X = \mathbb{Z}\varpi_1 \oplus \cdots \oplus \mathbb{Z}\varpi_n$: weight lattice in $\mathbb{E}$, where the fundamental dominant weights ϖ_i are defined by $\langle\varpi_i, \alpha_j^\vee\rangle = \delta_{ij}$, $1 \leqslant i, j \leqslant n$.
(13) $X^+ = \mathbb{N}\varpi_1 + \cdots + \mathbb{N}\varpi_n$: cone of dominant weights.
(14) $X_\ell^+ = \{\lambda \in X^+ \colon \langle\lambda, \alpha^\vee\rangle < \ell \text{ for all } \alpha \in \Phi\}$: the set of ℓ-restricted dominant weights.
(15) $s_\beta : \mathbb{E} \to \mathbb{E}$ ($\beta \in \Phi$): reflection across the hyperplane H_β of vectors orthogonal to β.
(16) $W \subset \mathbb{O}(\mathbb{E})$: Weyl group of Φ, generated by the orthogonal reflections $\{s_{\alpha_1}, \ldots, s_{\alpha_n}\}$.
(17) $W_\ell = \ell Q \rtimes W$: affine Weyl group, generated by the affine reflections $s_{\alpha,r} : \mathbb{E} \to \mathbb{E}$, defined for $\alpha \in \Phi$ and $r \in \mathbb{Z}$ by $s_{\alpha,r}(x) = x - [\langle x, \alpha^\vee\rangle - rl]\alpha$. For $\theta \in Q$, let $t_{\ell\theta} : \mathbb{E} \to \mathbb{E}$ be the translation operator in W_ℓ given by $x \mapsto x + \ell\theta$. The affine Weyl group W_ℓ is a Coxeter group with fundamental system $S_\ell = \{s_{\alpha_1}, \ldots, s_{\alpha_n}\} \cup \{s_{\alpha_0,-1}\}$.
(18) $l : W_\ell \to \mathbb{N}$: usual length function on W_ℓ.
(19) $l|_W$: length function on the parabolic subgroup W of W_ℓ.
(20) $\Phi_{\lambda,\ell} = \{\alpha \in \Phi \colon \langle\lambda + \rho, \alpha^\vee\rangle \equiv 0 \pmod{\ell}\}$ for $\lambda \in X$. When ℓ is clear from context, denote $\Phi_{\lambda,\ell}$ simply by Φ_λ. Set $\Phi_\lambda^+ = \Phi^+ \cap \Phi_\lambda$.
(21) $C^- = \{\lambda \in \mathbb{E} \colon -\ell < \langle\lambda + \rho, \alpha^\vee\rangle < 0 \text{ for all } \alpha \in \Phi^+\}$. The closure $\overline{C^-}$ is a fundamental domain for the dot action of W_ℓ on $\mathbb{E}$ (which is defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$). Let $C_{\mathbb{Z}}^- = C^- \cap X$ and $\overline{C}_{\mathbb{Z}}^- = \overline{C^-} \cap X$.
(22) $\chi(\lambda) = \sum_{w\in W}(-1)^{l(w)}e(w(\lambda + \rho))/\sum_{w\in W}(-1)^{l(w)}e(w\rho) \in \mathbb{Z}[X]$ for $\lambda \in X$: This is Weyl's character formula if $\lambda \in X^+$.
(23) $\Psi_\ell(t) \in \mathbb{Z}[t]$: cyclotomic polynomial for a primitive ℓ-th root of unity $\zeta \in \mathbb{C}$.
(24) $d_\alpha = \langle\alpha,\alpha\rangle/2 = \langle\alpha,\alpha\rangle/\langle\alpha_0,\alpha_0\rangle \in \{1, 2, 3\}$ for $\alpha \in \Phi$.

Throughout this paper we will assume that *ℓ is an odd positive integer, $l > h$, and $(\ell, r) = 1$ for each bad prime r of Φ*. In Section 4 we also assume that $\ell = p$ is a prime integer. The assumption that $(l, r) = 1$ whenever r is bad for Φ guarantees that Φ_λ is a closed subroot system of Φ. In fact, one then has $(d_\alpha, \ell) = 1$ for all $\alpha \in \Phi$, so that, since $d_\alpha\alpha^\vee = \alpha$, we have $\langle\lambda + \rho, \alpha\rangle =$

$d_\alpha\langle\lambda+\rho,\alpha^\vee\rangle\in\mathbb{Z}$ for all $\alpha\in\Phi$, and hence $\Phi_\lambda=\{\alpha\in\Phi\colon\ \langle\lambda+\rho,\alpha\rangle\equiv 0 \pmod{\ell}\}$. The fact that ℓ is not divisible by a bad prime means, by definition, that $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ has no ℓ-torsion for any closed subsystem Φ' of Φ. Then $\mathbb{Q}\Phi_\lambda\cap\Phi=\Phi_\lambda$, so by [4, VI.1.7 Proposition 24], there is a set of simple roots for Φ_λ contained in a set of simple roots for Φ. Thus, there exists $w\in W$ and a subset $J\subseteq\Pi$ such that $\Phi_\lambda=w(\Phi_J)$, where $\Phi_J=\mathbb{Z}J\cap\Phi$. The argument of this paragraph largely reiterates that given in [21, §6.2] for the special case when $\ell=p$ is a good prime.

2. Differentiating the generic dimension

For $\theta\in Q$, write $\theta=\sum_{i=1}^n m_i\alpha_i$ ($m_i\in\mathbb{Z}$). The *height* of θ is defined by $\mathrm{ht}(\theta)=\sum_{i=1}^n m_i$. A *weighted* height on X will be defined and used later. We require the following elementary result.

Lemma 2.1. *For $\beta\in\Phi^+$, $l(s_\beta)<2\,\mathrm{ht}(\beta)$.*

Proof. The result is true if $\mathrm{ht}(\beta)=1$, i.e., if $\beta\in\Pi$. So assume that $\mathrm{ht}(\beta)>1$, and that the result is true for positive roots of smaller height. Choose $\alpha\in\Pi$ so that $s_\alpha(\beta)=\gamma\in\Phi^+$ with $\mathrm{ht}(\gamma)<\mathrm{ht}(\beta)$. Then $s_\beta=s_\alpha s_\gamma s_\alpha$. Since $\langle\gamma,\alpha\rangle<0$, it is easily verified that $l(s_\beta)=l(s_\gamma)+2$. Then

$$l(s_\beta)=l(s_\gamma)+2<2\,\mathrm{ht}(\gamma)+2\leqslant 2\,\mathrm{ht}(\beta),$$

as required. □

Fix $\lambda^-\in\overline{C}_{\mathbb{Z}}^-$ throughout this section. An element $w\in W_\ell$ is called dominant for λ^- provided that $w\cdot\lambda^-\in X^+$. Let $w\in W_\ell$ be dominant for λ^-, and write $w=t_{\ell\theta}x$ with $\theta\in Q$, $x\in W$. Since $w\cdot\lambda^-=x(\lambda^-+\rho)+\ell\theta\in\rho+X^+$, and

$$\left|\left\langle x\left(\lambda^-+\rho\right),\alpha^\vee\right\rangle\right|=\left|\left\langle\lambda^-+\rho,x^{-1}\alpha^\vee\right\rangle\right|\leqslant\ell\quad\text{for all }\alpha\in\Phi,$$

it follows that $\theta\in X^+$. In addition, w is called *minimal* dominant for λ^- if it has minimal length among all $y\in W_\ell$ such that $y\cdot\lambda^-=w\cdot\lambda^-$.

Let $w=t_{\ell\theta}x\in W_\ell$ with $\theta\in Q$, $x\in W$. If $\theta\in X^+$, then it follows that

$$\ell(w)=\ell(x)+2\,\mathrm{ht}(\theta).\tag{2.1.1}$$

This result is proved in [14, Proposition 1.23]. A routine adjustment must be made in the formula given there, since a different set of fundamental reflections for W_ℓ is used.

Lemma 2.2. *Let $w=t_{\ell\theta}x$ be minimal dominant for $\lambda^-\in\overline{C}_{\mathbb{Z}}^-$. Given $\alpha\in\Phi_{\lambda^-}^+$, $x\alpha\in-\Phi^+$ if and only if $\langle\lambda^-+\rho,\alpha^\vee\rangle=-\ell$.*

Proof. Suppose that $\alpha\in\Phi_{\lambda^-}^+$. Then $\langle\lambda^-+\rho,\alpha^\vee\rangle\in\{0,-\ell\}$. If this value is 0, then $xs_\alpha\cdot\lambda^-=x\cdot\lambda^-$. By hypothesis on w, this means that

$$2\,\mathrm{ht}(\theta)+l(x)=l(w)<l(ws_\alpha)=2\,\mathrm{ht}(\theta)+l(xs_\alpha),$$

so that $l(xs_\alpha) > l(x)$, and hence $x\alpha > 0$. Thus, if $x\alpha < 0$, then necessarily $\langle \lambda^- + \rho, \alpha^\vee \rangle = -\ell$. Conversely, assume that $\langle \lambda^- + \rho, \alpha^\vee \rangle = -\ell$. We will show that $x\alpha < 0$. Suppose otherwise, viz., $x\alpha > 0$. We have $s_{\alpha,-1} \cdot \lambda^- = \lambda^-$. Also,

$$ws_{\alpha,-1} = t_{\ell\theta} x t_{-\ell\alpha} s_\alpha = t_{\ell\theta - \ell x\alpha} x s_\alpha.$$

Since $ws_{\alpha,-1} \cdot \lambda^- = w \cdot \lambda^-$ is dominant, $\theta - x\alpha$ must be dominant. Then, by (2.1.1), $ws_{\alpha,-1}$ has length equal to $l(xs_\alpha) + 2\,\mathrm{ht}(\theta - x\alpha)$. But,

$$\begin{aligned} l(ws_{\alpha,-1}) &= l(xs_\alpha) + 2\,\mathrm{ht}(\theta - x\alpha) = l(s_{x\alpha} x) + 2\,\mathrm{ht}(\theta) - 2\,\mathrm{ht}(x\alpha) \\ &\leqslant l(x) + l(s_{x\alpha}) + 2\,\mathrm{ht}(\theta) - 2\,\mathrm{ht}(x\alpha) < l(w) \end{aligned}$$

since Lemma 2.1 guarantees that $l(s_{x\alpha}) < 2\,\mathrm{ht}(x\alpha)$ when $x\alpha \in \Phi^+$. This inequality contradicts the minimality of w, so we must conclude that if $\langle \lambda^- + \rho, \alpha^\vee \rangle = -\ell$, then $x\alpha < 0$. □

Following [23], we will use a weighted height function $\mathrm{wht}: X \to \mathbb{Z}[\frac{1}{2}]$. For $\alpha \in \Phi$, recall $d_\alpha = \langle \alpha, \alpha \rangle / 2 = \langle \alpha, \alpha \rangle / \langle \alpha_0, \alpha_0 \rangle \in \{1, 2, 3\}$. Given $\lambda = \sum_{\alpha \in \Pi} r_\alpha \alpha \in X$ ($r_\alpha \in \mathbb{Q}$), put

$$\mathrm{wht}(\lambda) := \sum_{\alpha \in \Pi} r_\alpha d_\alpha = \frac{2\langle \lambda, \rho \rangle}{\langle \alpha_0, \alpha_0 \rangle} = \frac{1}{2} \sum_{\alpha \in \Phi^+} d_\alpha \langle \lambda, \alpha^\vee \rangle. \tag{2.2.1}$$

See [23, Lemma 1.1] for the verification that these quantities are all equal. Given a finite-dimensional X-graded vector space $V = \bigoplus_{\lambda \in X} V_\lambda$, its *generic dimension* is the Laurent polynomial

$$\dim_t V := \sum_{\lambda \in X} (\dim V_\lambda) t^{-2\,\mathrm{wht}(\lambda)} \in \mathbb{Z}[t, t^{-1}]. \tag{2.2.2}$$

We also put $\mathrm{ch}(V) = \sum_{\lambda \in X} (\dim V_\lambda) e(\lambda)$ for the character of V.

For $\lambda \in X$, set

$$D_\lambda(t) = \prod_{\alpha \in \Phi^+} \left(t^{d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} \right) \in \mathbb{Z}[t, t^{-1}]. \tag{2.2.3}$$

Lemma 2.3. *(See [23, Theorem 1.3].) Suppose that V is a finite-dimensional X-graded vector space such that* $\mathrm{ch}(V) = \chi(\lambda)$ *for some $\lambda \in X^+$. Then*

$$\dim_t V = D_\lambda(t) / D_0(t). \tag{2.3.1}$$

We call (2.3.1) the Weyl generic dimension formula. Its value at $t = 1$ gives Weyl's classical dimension formula for the irreducible $\mathfrak{g}$-module of highest weight λ.

Let $\lambda^- \in \overline{C}_{\mathbb{Z}}^-$ as before, and let $w = t_{\ell\theta} x$, $\theta \in X^+ \cap Q$, $x \in W$, be minimal dominant for λ^-. Set $\lambda = w \cdot \lambda^-$, and set $s = |\Phi_\lambda^+|$. For $\alpha \in \Phi^+$, $2d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle$ is divisible by ℓ if and only if $\alpha \in \Phi_\lambda^+$. Also, by our assumptions, ℓ does not divide any of the $2d_\alpha \langle \rho, \alpha^\vee \rangle$. It follows that the cyclotomic polynomial $\Psi_\ell(t)$ occurs as a factor of $t^{d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle}$ in $\mathbb{Z}[t, t^{-1}]$ if and only if $\alpha \in \Phi_\lambda^+$, hence that $\Psi_\ell(t)$ occurs as a factor of $D_\lambda(t)$ in $\mathbb{Z}[t, t^{-1}]$ precisely s times. In particular, $\Psi_\ell(t)$ is relatively prime to $D_0(t)$.

For the main result of this paper, we need to calculate the value at $t = \zeta$ of the Laurent polynomial

$$D_\lambda^{(s)}(t) := \frac{d^s}{dt^s} D_\lambda(t),$$

obtained by differentiating $D_\lambda(t)$ s times.

Theorem 2.4. *Fix $\lambda^- \in \overline{C}_{\mathbb{Z}}^-$. Let $w = t_{\ell\theta} x \in W_\ell$ be minimal dominant for λ^-, and put $\lambda = w \cdot \lambda^-$. Set $s = |\Phi_\lambda^+|$, and set*

$$a_{\lambda^-} = \left|\left\{\alpha \in \Phi_{\lambda^-}^+ \colon \left\langle \lambda^- + \rho, \alpha^\vee \right\rangle = -\ell\right\}\right|.$$

Then

$$0 \neq D_\lambda^{(s)}(\zeta) = (-1)^{l(w)-(a_{\lambda^-})}(s!)\left(\prod_{\alpha \in \Phi_\lambda^+} 2d_\alpha \left\langle \lambda + \rho, \alpha^\vee \right\rangle \zeta^{-1}\right)$$
$$\times \left(\prod_{\alpha \in \Phi^+ \backslash \Phi_{\lambda^-}^+} \zeta^{d_\alpha \langle \lambda^- + \rho, \alpha^\vee \rangle} - \zeta^{-d_\alpha \langle \lambda^- + \rho, \alpha^\vee \rangle}\right).$$

Proof. Write $f_\alpha(t) = t^{d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} - t^{-d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle}$, so that $D_\lambda(t) = \prod_{\alpha \in \Phi^+} f_\alpha(t)$. If $(d^i/dt^i) f_\alpha(t)$ is denoted by $f_\alpha^{(i)}(t)$, then $D_\lambda^{(s)}(t)$ is a sum of terms

$$\left[s!/\left(\prod i_\alpha!\right)\right] \cdot \prod f_\alpha^{(i_\alpha)}(t) \tag{2.4.1}$$

over distinct sequences $(i_\alpha)_{\alpha \in \Phi^+}$ of non-negative integers i_α summing to s. Since $\Psi_\ell(t)$ divides $f_\alpha(t)$ precisely when $\alpha \in \Phi_\lambda^+$ (and then divides it with multiplicity one), the only terms in (2.4.1) that do not vanish upon the substitution $t = \zeta$ are those in which $i_\alpha = 1$ for all $\alpha \in \Phi_\lambda^+$ (and thus $i_\alpha = 0$ for all $\alpha \in \Phi^+ \backslash \Phi_\lambda^+$). However,

$$f_\alpha^{(1)}(t) = f_\alpha'(t) = d_\alpha \left\langle \lambda + \rho, \alpha^\vee \right\rangle \left(t^{d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle - 1} + t^{-d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle - 1}\right),$$

so that, for $\alpha \in \Phi_\lambda^+$,

$$f_\alpha'(\zeta) = 2d_\alpha \left\langle \lambda + \rho, \alpha^\vee \right\rangle \zeta^{-1}.$$

Furthermore, each $\langle \lambda + \rho, \alpha^\vee \rangle$ is a positive integer because $\lambda \in X^+$.

Next, for $\alpha \in \Phi^+ \backslash \Phi_\lambda^+$,

$$\zeta^{d_\alpha \langle \lambda + \rho, \alpha^\vee \rangle} = \zeta^{d_\alpha \langle x(\lambda^- + \rho), \alpha^\vee \rangle} \zeta^{d_\alpha \ell \langle \theta, \alpha^\vee \rangle} = \zeta^{d_\alpha \langle \lambda^- + \rho, x^{-1}\alpha^\vee \rangle}.$$

Observe that $\alpha \notin \Phi_\lambda$ implies that $x^{-1}\alpha \notin \Phi_{\lambda^-}$. If $x^{-1}\alpha < 0$, then we write the (non-zero) quantity

$$\zeta^{d_\alpha\langle\lambda^-+\rho, x^{-1}\alpha^\vee\rangle} - \zeta^{-d_\alpha\langle\lambda^-+\rho, x^{-1}\alpha^\vee\rangle}$$

as

$$-\left(\zeta^{d_\alpha\langle\lambda^-+\rho, -x^{-1}\alpha^\vee\rangle} - \zeta^{-d_\alpha\langle\lambda^-+\rho, -x^{-1}\alpha^\vee\rangle}\right).$$

By Lemma 2.2, there are $l(w) - (a_{\lambda^-})$ such sign changes. The theorem now follows. □

In the next section, it will be convenient to write

$$E_{\lambda^-}(\zeta) = \prod_{\alpha\in\Phi^+\setminus\Phi^+_{\lambda^-}} \left(\zeta^{d_\alpha\langle\lambda^-+\rho,\alpha^\vee\rangle} - \zeta^{-d_\alpha\langle\lambda^-+\rho,\alpha^\vee\rangle}\right). \tag{2.4.2}$$

3. Support varieties for quantum irreducible modules

We continue to assume that the positive integer ℓ satisfies the conditions (with respect to the root system Φ) stated in the last paragraph of Section 1. Let $\zeta \in \mathbb{C}$ be a primitive ℓ-th root of 1. Let $U_\zeta(\mathfrak{g})$ be the Lusztig quantum enveloping algebra associated to $\mathfrak{g}$ at ζ, and let $u_\zeta(\mathfrak{g})$ be the corresponding small quantum group. Then $u_\zeta(\mathfrak{g})$ is the Hopf-algebraic kernel of the Frobenius morphism on $U_\zeta(\mathfrak{g})$.

For $\lambda \in X^+$, let $L_\zeta(\lambda)$ be the irreducible type-1 integrable $U_\zeta(\mathfrak{g})$-module of highest weight λ. Similarly, let $\Delta_\zeta(\lambda)$ and $\nabla_\zeta(\lambda)$ denote the standard and costandard (i.e., Weyl and induced) modules of highest weight λ. Then $\Delta_\zeta(\lambda)$ (resp. $\nabla_\zeta(\lambda)$) has head (resp. socle) $L_\zeta(\lambda)$, with all other composition factors $L_\zeta(\mu)$ satisfying $\mu < \lambda$ in the usual partial ordering on X^+. Furthermore, if $L_\zeta(\mu)$ is a composition factor of $\Delta_\zeta(\lambda)$ (resp. $\nabla_\zeta(\lambda)$), then μ is linked to λ (i.e., μ is conjugate to λ under the dot action of W_ℓ), and hence Φ_λ and Φ_μ are W-conjugate. It is well known that given $\lambda \in X^+$, $\Delta_\zeta(\lambda)$ and $\nabla_\zeta(\lambda)$ both have formal characters (with respect to the action of the "torus" $U^0_\zeta(\mathfrak{g})$) equal to $\chi(\lambda)$, the Weyl character associated to the dominant weight λ. Thus, taking $V = \Delta_\zeta(\lambda)$ or $\nabla_\zeta(\lambda)$, one has $\dim_t V = D_\lambda(t)/D_0(t)$ by Lemma 2.3.

Recall that W_ℓ is generated as a group by the fundamental system $S_\ell \subset W_\ell$. Given $I \subseteq S_\ell$, set $W_{\ell,I} = \langle I\rangle \leqslant W_\ell$, and set $W_\ell^I = \{w \in W_\ell\colon\ l(w) \leqslant l(ws) \text{ for all } s \in W_{\ell,I}\}$. Let $\leqslant$ denote the Chevalley–Bruhat partial ordering on W_ℓ. Given $y \leqslant w$ in W_ℓ, $P_{y,w}(q)$ is the Kazhdan–Lusztig polynomial associated to the pair (y,w). In [6], Deodhar introduced two generalizations of the $P_{y,w}$'s, called parabolic Kazhdan–Lusztig polynomials, which depend on a choice of subset $I \subseteq S_\ell$, and a choice of a root u of the equation $u^2 = q + (q-1)u$, i.e., $u = -1$ or $u = q$. Given $I \subseteq S_\ell$, and given $(y,w) \in W_\ell^I \times W_\ell^I$ with $y \leqslant w$, the parabolic Kazhdan–Lusztig polynomial $P^{I,-1}_{y,w}$ associated to the root $u = q$ is related to the usual Kazhdan–Lusztig polynomials by the following equation [6, Remark 3.8]:

$$P^{I,-1}_{y,w} = \sum_{x\in W_I,\, yx\leqslant w} (-1)^{l(x)} P_{yx,w}. \tag{3.0.3}$$

We are following the notational convention of [19], so the superscript in $P^{I,a}_{y,w}$ indicates the opposite root of the equation $u^2 = q + (q-1)u$; see [19, Remark 2.1]. If $y \nleqslant w$, then $P^{I,-1}_{y,w} = 0$.

By [19, Corollary 4.1], the coefficients of the $P^{I,-1}_{y,w}$ are non-negative integers. In fact, the coefficients are interpreted there as multiplicities of composition factors in Hodge modules associated to Schubert varieties; an alternate approach using affine Hecke algebras is provided in [12].

Fix $\lambda^- \in \overline{C}^-_{\mathbb{Z}}$. The stabilizer in W_ℓ of λ^- is defined by $W_{\ell,\lambda^-} = \{w \in W_\ell \mid w \cdot \lambda^- = \lambda^-\}$; it is generated as a group by the set $I := W_{\ell,\lambda^-} \cap S_\ell$ [16, II.6.3]. Then $W_{\ell,\lambda^-} = W_{\ell,I} := \langle I \rangle \leqslant W_\ell$ is a parabolic subgroup of W_ℓ. If $w \in W_\ell$ is minimal dominant for λ^-, then $w \in W^I_\ell$.

Proposition 3.1. *Let $w \in W_\ell$ be minimal dominant for λ^-, and write $\lambda = w \cdot \lambda^-$. Let $I \subseteq S_\ell$ be such that $W_{\ell,\lambda^-} = W_{\ell,I}$. Then*

$$\operatorname{ch} L_\zeta(\lambda) = \sum_{y \in W^I_\ell} (-1)^{l(w)-l(y)} P^{I,-1}_{y,w}(1) \operatorname{ch} \Delta_\zeta\bigl(y \cdot \lambda^-\bigr). \tag{3.1.1}$$

Proof. As mentioned in the introduction (see also [27, §§6–7]), the Lusztig character formula

$$\operatorname{ch} L_\zeta(\lambda) = \sum_{y \in W_\ell, y \leqslant w, y\cdot\lambda^- \in X^+} (-1)^{l(w)-l(y)} P_{y,w}(1) \operatorname{ch} \Delta_\zeta\bigl(y \cdot \lambda^-\bigr) \tag{3.1.2}$$

holds. If $y \in W_\ell$ is not dominant for λ^-, then $\operatorname{ch} \Delta_\zeta(y \cdot \lambda^-) = 0$. Also, if $y \nleqslant w$, then $P_{y,w} = 0$. Then (3.1.2) can be rewritten as

$$\begin{aligned}\operatorname{ch} L_\zeta(\lambda) &= \sum_{y \in W^I_\ell} (-1)^{l(w)-l(y)} \Bigl(\sum_{x \in W_{\ell,I}} (-1)^{l(x)} P_{yx,w}(1)\Bigr) \operatorname{ch} \Delta_\zeta\bigl(y \cdot \lambda^-\bigr) \\ &= \sum_{y \in W^I_\ell} (-1)^{l(w)-l(y)} P^{I,-1}_{y,w}(1) \operatorname{ch} \Delta_\zeta\bigl(y \cdot \lambda^-\bigr)\end{aligned}$$

by (3.0.3). □

Now choose $J \subseteq \Pi$ such that Φ_{λ^-} is W-conjugate to Φ_J. Let

$$\mathfrak{u}_J = \sum_{\alpha \in \Phi^+ \setminus \Phi^+_J} \mathfrak{g}_{-\alpha} \subset \mathfrak{g} \tag{3.1.3}$$

be the nilpotent radical of the (negative) standard parabolic subalgebra $\mathfrak{p}_J \supseteq \mathfrak{b}$ determined by J. If $J = \emptyset$, denote $\mathfrak{u}_\emptyset$ simply by $\mathfrak{u}$, the nilpotent radical of $\mathfrak{b}$. Let G be the simple complex algebraic group with Lie algebra $\mathfrak{g}$. Recall that $G \cdot \mathfrak{u}_J$ is a closed, irreducible subvariety of the nullcone $\mathcal{N}(\mathfrak{g})$ of $\mathfrak{g}$.

The following theorem was first stated in [22, Theorem 6.1]. We also refer the reader to [3], which considers in addition the situation when $\ell \leqslant h$ and when $(l, r) \neq 1$ for r a bad prime of Φ.

Theorem 3.2. *Let $\lambda \in X^+$, and choose $J \subseteq \Pi$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda)) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda)) = G \cdot \mathfrak{u}_J$.*

We remark that the subset $J \subseteq \Pi$ and the element $w \in W$ in Theorem 3.2 may not be unique. However, the Johnston–Richardson theorem [17] guarantees that if $J, K \subseteq \Pi$ are such that Φ_J is conjugate to Φ_K under W, then $G \cdot \mathfrak{u}_J = G \cdot \mathfrak{u}_K$, so the variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda))$ is well-defined. Also, the equality $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda)) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda))$ may be seen as follows. First, $\Phi_\lambda = w_0(\Phi_{-w_0\lambda})$, so $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda)) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(-w_0\lambda))$. Next, $\Delta_\zeta(\lambda) = \nabla_\zeta(-w_0\lambda)^*$, so the equality $\mathcal{V}_{u_\zeta(\mathfrak{g})}(\Delta_\zeta(\lambda)) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(-w_0\lambda))$ follows as in [23, Remark 5.3] from the fact that the small quantum group $u_\zeta(\mathfrak{g})$ is a quasitriangular Hopf algebra [20, Example 8.16].

Now let M be a finite-dimensional type-1 $U_\zeta(\mathfrak{g})$-module. Before proving the main theorem, we collect some information concerning the support varieties $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ and $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$. First, by [11, Lemma 2.6], there exists a rational B-algebra isomorphism $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{b}), \mathbb{C}) \cong S^\bullet(\mathfrak{u}^*)$, and by [11, Theorem 3], there exists a rational G-algebra isomorphism $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) \cong \mathbb{C}[\mathcal{N}]$. Under these identifications, the restriction map $\mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) \to \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C})$ induced by the inclusion $u_\zeta(\mathfrak{b}) \subset u_\zeta(\mathfrak{g})$ is simply the restriction of functions from $\mathcal{N}$ to $\mathfrak{u}$. In particular, the restriction map is surjective.

Now, from the inclusion of algebras $u_\zeta(\mathfrak{b}) \subset u_\zeta(\mathfrak{g})$ we get the commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), \mathbb{C}) & \longrightarrow & \mathrm{H}^\bullet(u_\zeta(\mathfrak{g}), M \otimes M^*) & = & \mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{g})}(M, M) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow & & \downarrow \\ \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), \mathbb{C}) & \longrightarrow & \mathrm{H}^\bullet(u_\zeta(\mathfrak{b}), M \otimes M^*) & = & \mathrm{Ext}^\bullet_{u_\zeta(\mathfrak{b})}(M, M), \end{array}$$

where the vertical maps are the obvious restriction maps, and the horizontal maps are induced by the $u_\zeta(\mathfrak{g})$-module homomorphism $\mathbb{C} \to M \otimes M^* \cong \mathrm{End}_k(M)$, $1 \mapsto \mathrm{Id}_M$. From the commutativity of the diagram and the surjectivity of the leftmost restriction homomorphism, we conclude that there exists a closed embedding $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M) \subseteq \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \cap \mathcal{N}(\mathfrak{b}) = \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u}$.

The support variety $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ is naturally an algebraic G-variety, hence is a union of G-orbits. The dimension of $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ as an algebraic variety is the maximum of the dimensions of the G-orbits in $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$. Similarly, $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$ is naturally an algebraic B-variety, and its dimension is the maximum of the dimensions of the B-orbits in $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M)$. Since $\mathcal{V}_{u_\zeta(\mathfrak{b})}(M) \subseteq \mathcal{V}_{u_\zeta(\mathfrak{g})}(M) \cap \mathfrak{u}$, it follows by a result of Spaltenstein [13, Proposition 6.7] that $\dim \mathcal{V}_{u_\zeta(\mathfrak{b})}(M) \leqslant \frac{1}{2} \dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$.

We are now ready to prove the main theorem.

Theorem 3.3. *Let $\lambda \in X^+$, and choose $J \subseteq \Pi$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then*

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}\big(L_\zeta(\lambda)\big) = G \cdot \mathfrak{u}_J.$$

Proof. We first claim that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) \subseteq G \cdot \mathfrak{u}_J$. This is proved in [22, §5], but it is easily deduced from the previous theorem: If μ is linked to λ and is minimal among all dominant weights $\leqslant \lambda$, then $L_\zeta(\mu) = \nabla_\zeta(\mu)$, and Φ_μ is W-conjugate to Φ_λ. Then Theorem 3.2 and the Johnston–Richardson theorem [17] imply that $\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\mu)) = G \cdot \mathfrak{u}_J = \mathcal{V}_{u_\zeta(\mathfrak{g})}(\nabla_\zeta(\lambda))$. More generally, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of finite-dimensional $u_\zeta(\mathfrak{g})$-modules, then $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M_{\sigma(1)}) \subseteq \mathcal{V}_{u_\zeta(\mathfrak{g})}(M_{\sigma(2)}) \cup \mathcal{V}_{u_\zeta(\mathfrak{g})}(M_{\sigma(3)})$ for any permutation σ of $\{1, 2, 3\}$ [23, Lemma 5.2]. Thus, the full claim follows from an evident induction argument, again using Theorem 3.2 together with the remarks at the start of this section.

We next estimate the dimension of $\mathcal{V}_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda))$. We have $\dim \mathcal{V}_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda)) = c_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda))$, the complexity of $L_\zeta(\lambda)$ as a $u_\zeta(\mathfrak{b})$-module. By [21, Theorem 3.4.1(a)],[4] the complexity $c_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda))$ satisfies the inequality $c_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda)) \geqslant |\Phi^+| - d + 1$, where d is any positive integer such that $\Psi_\ell(t)^d$ does not divide the generic dimension $\dim_t L_\zeta(\lambda) \in \mathbb{Z}[t, t^{-1}]$. According to the character formula (3.1.1) in Proposition 3.1, Lemma 2.3, and the remarks at the end of the second paragraph of this section, we have

$$\dim_t L_\zeta(\lambda) = \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) D_{y\cdot\lambda^-}(t)/D_0(t),$$

where $I \subseteq S_\ell$ is such that $W_{\ell,\lambda^-} = W_{\ell,I}$. Since $D_0(t)$ is relatively prime to $\Psi_\ell(t)$, to determine a lower bound for $c_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda))$, it suffices to determine the multiplicity with which $\Psi_\ell(t)$ occurs as a factor in $D_0(t) \cdot \dim_t L_\zeta(\lambda)$. Equivalently, it suffices to determine the multiplicity with which the primitive ℓ-th root of unity $\zeta \in \mathbb{C}$ occurs as a root of the Laurent polynomial $D_0(t) \cdot \dim_t L_\zeta(\lambda)$.

Set $f(t) = D_0(t) \cdot \dim_t L_\zeta(\lambda)$. If $f^{(i)}(\zeta) = 0$ for all $0 \leqslant i < n$, but $f^{(n)}(\zeta) \neq 0$, then ζ occurs as a root of f with multiplicity exactly equal to n. Set $s = |\Phi^+_{\lambda^-}|$. Then $s = |\Phi^+_{y\cdot\lambda^-}|$ for any $y \in W_\ell$ by [21, (3.4.2)]. We want to show that $n = s$. Certainly $n \geqslant s$, because $\Psi_\ell(t)$ occurs as a factor of $D_{y\cdot\lambda^-}(t)$ precisely s times by the discussion following Lemma 2.3. Then, to prove $n = s$, we must show that $f^{(s)}(\zeta) \neq 0$.

Applying Theorem 2.4, we get

$$\begin{aligned} f^{(s)}(\zeta) &= \sum_{y \in W_\ell^I} (-1)^{l(w)-l(y)} P_{y,w}^{I,-1}(1) D^{(s)}_{y\cdot\lambda^-}(\zeta) \\ &= \sum_{y \in W_\ell^I} (-1)^{l(w)-(a_{\lambda^-})} P_{y,w}^{I,-1}(1)(s!)\Bigl(\prod_{\alpha \in \Phi^+_{y\cdot\lambda^-}} 2d_\alpha\langle y\cdot\lambda^- + \rho, \alpha^\vee\rangle\Bigr)\zeta^{-s} E_{\lambda^-}(\zeta) \\ &= \bigl((-1)^{l(w)-(a_{\lambda^-})}(s!)\zeta^{-s} E_{\lambda^-}(\zeta)\bigr) \cdot \Bigl(\sum_{y \in W_\ell^I} P_{y,w}^{I,-1}(1)\Bigl(\prod_{\alpha \in \Phi^+_{y\cdot\lambda^-}} 2d_\alpha\langle y\cdot\lambda^- + \rho, \alpha^\vee\rangle\Bigr)\Bigr). \end{aligned}$$

The first term in the product of the last line is non-zero. The second term in the product is a sum of non-negative integers (by the positivity property for the parabolic Kazhdan–Lusztig polynomials). Since $P_{w,w}^{I,-1}(1) = 1$, we conclude that the second term in the product is a strictly positive integer, hence that $f^{(s)}(\zeta) \neq 0$.

[4] Although cast in the situation of algebraic groups, the results of [21, §3] apply here. In fact, [21, Theorem 3.4.1(a)] depends formally on the arguments given in [21, §§3.2–3.3]. In our case, we replace the r-th infinitesimal subgroup B_r by $u_\zeta(\mathfrak{b})$ and TB_r by $U_\zeta^0(\mathfrak{g})u_\zeta(\mathfrak{g})$. Taking $P_\bullet \to L_\zeta(\lambda) \to 0$ to be a minimal projective resolution in the category of type-1 integrable $U_\zeta^0(\mathfrak{g})u_\zeta(\mathfrak{g})$-modules, the critical description [21, (3.2.2)] holds in our case, and the remaining arguments follow as for algebraic groups. As pointed out in [21, Footnote 2], this argument is in some sense given in [22]. In the proof of [21, Theorem 3.3.1], one should first tensor M by an appropriate one-dimensional character so that $\dim_t M$ is a polynomial in t; this will not affect the complexity of M, and will ensure that the only poles of $q(t)$ occur at roots of unity. Also, in the second paragraph of the proof, the phrase "the orders of these poles" should be replaced by "the multiplicative orders of these roots of unity," in the second line the two occurrences of d should be replaced by b, and the phrase "so that a_{ij}" should be replaced by "so that for all sufficiently large j, a_{ij}".

Now $\mathcal{V}_{u_\zeta(\mathfrak{b})}(L_\zeta(\lambda))$ has dimension at least $|\Phi^+| - s = |\Phi^+| - |\Phi^+_{\lambda^-}|$. By the discussion preceding the theorem, we have $\dim \mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) \geqslant |\Phi| - |\Phi_{\lambda^-}| = \dim G \cdot \mathfrak{u}_J$. □

4. Results in positive characteristic

In this section, G is a simple, simply-connected algebraic group defined over an algebraically closed field k of positive characteristic p. We leave to the reader the routine task of extending these results to reductive groups. Fix a maximal torus $T \subset G$, and let Φ be the root system of T acting on the Lie algebra $\mathfrak{g}$. Most of the previous notation, with ℓ set equal to p, carries over to G with only small changes (e.g., $B \supset T$ is a Borel subgroup whose opposite B^+ defines the set Φ^+ of positive roots, etc.). The Lie algebra $\mathfrak{g}$ carries a restricted structure; let $u(\mathfrak{g})$ denote its restricted enveloping algebra. We assume that $p > h$, so that, using [9] and [1], the cohomology algebra $\mathrm{H}^\bullet(u(\mathfrak{g}), k)$ is isomorphic to $k[\mathcal{N}]$, the coordinate ring of the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$. The result below concerns the support varieties $\mathcal{V}_{u(\mathfrak{g})}(L(\lambda))$ of the irreducible G-modules $L(\lambda)$, $\lambda \in X^+$. If $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_p^+$ (the restricted weights), then $\mathcal{V}_{u(\mathfrak{g})}(L(\lambda)) = \mathcal{V}_{u(\mathfrak{g})}(L(\lambda_0))$. Therefore, in computing support varieties for irreducible modules, it suffices to consider only those having restricted highest weights.

Let $\lambda = w \cdot \lambda^- \in X^+$, $\lambda^- \in \overline{C}^-_{\mathbb{Z}}$, $w \in W_p$. Assume that w is minimal dominant for λ^-. The Lusztig character formula asserts

$$\mathrm{ch}\ L(\lambda) = \sum_{y \in W_p, y \leqslant w, y \cdot \lambda^- \in X^+} (-1)^{l(w)-l(y)} P_{y,w}(1)\, \mathrm{ch}\ \Delta\big(y \cdot \lambda^-\big), \tag{4.0.1}$$

where $\Delta(y \cdot \lambda^-)$ is the Weyl module for G of highest weight $y \cdot \lambda^-$. As mentioned in the introduction, (4.0.1) holds for all restricted dominant weights λ, provided that the prime p is sufficiently large (the lower bound on p depending on the root system).[5]

Let $J \subseteq \Pi$ such that Φ_λ is W-conjugate to Φ_J. By [21, Proposition 7.4.1], $\mathcal{V}_{u(\mathfrak{g})}(L(\lambda)) \subseteq G \cdot \mathfrak{u}_J$, where $\mathfrak{u}_J$ is defined as in (3.1.3). With this fact, the proof of the following result is exactly analogous to that of Theorem 3.3 (replacing $\Psi_\ell(t)$ by $\Psi_p(t)$, etc.).

Theorem 4.1. *Assume that G is a simple, simply-connected algebraic group over an algebraically closed field k of characteristic $p > h$. Assume that the Lusztig character formula* (4.0.1) *holds for all restricted dominant weights. Then, for $\lambda \in X^+$ and $J \subseteq \Pi$ with $w(\Phi_\lambda) = \Phi_J$,*

$$\mathcal{V}_{u(\mathfrak{g})}\big(L(\lambda)\big) = G \cdot \mathfrak{u}_J.$$

Remark 4.2. (a) Suppose $p = h$. It may no longer hold that $A := \mathrm{H}^{2\bullet}(u(\mathfrak{g}), k) \cong k[\mathcal{N}]$. Even so, it has been proved that the algebraic variety defined by the affine algebra A is *homeomorphic* to $\mathcal{N}$ [25,26]. In this case, we identify $\mathcal{V}_{u(\mathfrak{g})}(L(\lambda))$ with its image in $\mathcal{N}$, and Theorem 4.1 holds with the condition "$p > h$" replaced by the condition "$p \geqslant h$".

[5] The Lusztig character formula is also known to hold for restricted weights in the following low rank cases (assuming $p \geqslant h$): (1) type A_1, $p \geqslant 2 = h$; (2) type A_2, $p \geqslant 3 = h$; (3) type B_2, $p > 4 = h$; (4) type G_2, $p > 9 = 2h - 3$; (5) type A_3, $p > 4 = h$; (6) type A_4, $p = 5$ or $p = 7$. Case (6) for $p = 5$ is due independently to L. Scott (working with undergraduates) and to A. Buch and N. Lauritzen. The case $p = 7$ is due to L. Scott (again working with undergraduates). Both of these cases required extensive computer application. For more details and references, see [24].

(b) For a restricted Lie algebra $\mathfrak{g}$ with restriction map $x \mapsto x^{[p]}$ and restricted enveloping algebra $u(\mathfrak{g})$, and for a finite-dimensional $u(\mathfrak{g})$-module M, the support variety $\mathcal{V}_{u(\mathfrak{g})}(M)$ has an alternate, more concrete description as the set of all $x \in \mathfrak{g}$ such that $x^{[p]} = 0$ and the induced operator x_M on M has an $r \times r$ Jordan block of size $r < p$. In other words, for $0 \neq x \in \mathfrak{g}$ satisfying $x^{[p]} = 0$, $x \notin \mathcal{V}_{u(\mathfrak{g})}(M)$ if and only if M is projective over the subalgebra of $u(\mathfrak{g})$ generated by x (cf. [10]). At present there is no known concrete realization in $\mathcal{N}$ for the support varieties of modules over the small quantum group.

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