Conics in the Grothendieck ring

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Abstract

The subring of the Grothendieck ring of $k$-varieties generated by smooth conics is described, giving many zero divisors. The proof uses only elementary projective geometry.

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The aim of this note is to describe the subring of the Grothendieck ring generated by smooth conics. As a ring this is quite complicated, with many zero divisors, but the description of the defining relations is entirely elementary.

Definition 1. Let $k$ be a field. The Grothendieck ring of $k$-varieties, denoted by $K_0[\text{Var}_k]$ is defined as follows.

 Its additive group is the Abelian group whose generators are the isomorphism classes of reduced, quasiprojective $k$-schemes and the relations are

$$[X] = [Y] + [X \setminus Y],$$

whenever $Y$ is a closed subscheme of $X$.

Multiplication is defined by $[X] \cdot [Y] = [X \times_k Y].$
The Grothendieck ring of $k$-varieties is still very poorly understood. In characteristic zero, the quotient of $K_0[\text{Var}_k]$ by the ideal generated by $[\mathbb{A}^1]$ is naturally isomorphic to the ring $\mathbb{Z}[SB_k]$, where $\mathbb{Z}[SB_k]$ is the free abelian group generated by the stable birational equivalence classes of smooth, projective, irreducible $k$-varieties and multiplication is given by the product of varieties [Lar-Lun]. (The cited paper proves this over algebraically closed fields only, but the proof works over any field of characteristic zero using the birational factorization theorem as given in [AKMW , Remark 2 after Theorem 0.3.1]. Note also that the product of two irreducible $k$-varieties is not necessarily irreducible, so $\mathbb{Z}[SB_k]$ is not a monoid ring if $k$ is not algebraically closed.)

Zero divisors in the Grothendieck ring of $\mathbb{C}$-varieties were found by [Poonen].

Here we give further examples of nontrivial behavior of these rings by studying products of conics. This gives interesting examples only when the field $k$ is not algebraically closed.

**Theorem 2.** Let $k$ be a number field or the function field of an algebraic surface over $\mathbb{C}$. Let $C_i : i \in I$ and $C'_j : j \in J$ be two collections of smooth conics defined over $k$ (repetitions allowed). The following are equivalent.

1. $\prod_{i \in I} C_i = \prod_{j \in J} C'_j$ in the Grothendieck ring.
2. $\prod_{i \in I} C_i$ and $\prod_{j \in J} C'_j$ are birational.
3. $\prod_{i \in I} C_i$ and $\prod_{j \in J} C'_j$ are stably birational.
4. $|I| = |J|$ and the two subgroups of the Brauer group (cf. (7)) generated by the conics $\langle C_i : i \in I \rangle \subset \text{Br}(k)_2$ and $\langle C'_j : j \in J \rangle \subset \text{Br}(k)_2$ are the same.

**Remark 3.** (1) The precise conditions on $k$ for the proof to work are given in (8). These are satisfied for many other fields, but fail for function fields of more than 2 variables. It is not clear to me, however, if any condition is needed on $k$ or not.

(2) Any isomorphism of two products $\prod_{i \in I} C_i$ and $\prod_{j \in J} C'_j$ is given in the obvious way: by a one-to-one map $g : I \to J$ and isomorphisms $C_i \cong C'_{g(i)}$.

This can be proved many ways. Here is one using extremal rays.

If $X$ is any projective variety, the cone of curves of $X \times \mathbb{P}^1$ is generated by the cone of curves of $X \cong X \times \{0\}$ and by $\{x\} \times \mathbb{P}^1$. Using this repeatedly, we obtain that the cone of curves of $\mathbb{P}^1)^m$ is generated by the fibers of the $m$ coordinate projections $(\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-1}$. Thus the $|I|$ coordinate projections

$$\pi_j : \prod_{i \in I} C_i \to \prod_{i \in I, i \neq j} C_i$$

are in one-to-one correspondence with the extremal rays of $\prod_{i \in I} C_i$. Hence the product structure can be recovered from the intrinsic geometry of $\prod_{i \in I} C_i$.

**Corollary 4.** Let $k$ be a number field or the function field of an algebraic surface over $\mathbb{C}$. 
(1) Let \( G \subset \text{Br}(k) \) be a finite subgroup with basis \( B_1, \ldots, B_s \). Then \([B_1 \times \cdots \times B_s]\) depends only on \( G \). Let us denote it by \( C(G) \).

(2) The Grothendieck ring of conics is the free abelian group generated by the elements \( C(G) \cdot [\mathbb{P}^1]^m \) with multiplication

\[
C(G_1) \cdot C(G_2) = C(\langle G_1, G_2 \rangle) \cdot [\mathbb{P}^1]^\dim G_1 + \dim G_2 - \dim \langle G_1, G_2 \rangle.
\]

where \( \dim G \) denotes dimension as an \( \mathbb{F}_2 \) vector space.

Remark 5. The last description shows that the Grothendieck ring of conics does not have nilpotents. Indeed, given an element \( g = \sum \gamma_{G_0} C(G) \cdot [\mathbb{P}^1]^m \) let \( G_0 \) be minimal such that \( \gamma_{G_0} \neq 0 \) for some \( m \), chosen also minimal. Then the coefficient of \( C(G_0) \cdot [\mathbb{P}^1]^{s m + (s-1) \dim G_0} \) in \( g^s \) is \( \gamma_{G_0}^s \neq 0 \).

The simplest example of nontrivial birational maps between products of conics is the following. The whole description of the Grothendieck ring of conics is only a more elaborate version it.

Example 6. Let \( C \) be a smooth plane conic. Then \( C \times \mathbb{P}^1 \) is birational to \( C \times C \), and they have the same class in the Grothendieck ring of \( k \)-varieties.

Thus \([C] \cdot ([\mathbb{P}^1] - [C]) = 0\) and \([C]\) is a zero divisor in the Grothendieck ring of \( k \)-varieties if \( C \) has no \( k \)-points.

Proof. \( C \subset \mathbb{P}^2 \) is a conic and we think of \( \mathbb{P}^1 \) as a line in the same \( \mathbb{P}^2 \).

Given \( p, q \in C \), the line connecting them intersects \( \mathbb{P}^1 \) in a point \( \phi(p, q) \cdot (p, \phi(p, q)) \) gives a rational map \( C \times C \dashrightarrow C \times \mathbb{P}^1 \). Conversely, given \( p \in C \) and \( r \in \mathbb{P}^1 \) the line connecting them intersects \( C \) in a further point \( \phi^{-1}(p, r) \).

Let \( s, s' \in C(\overline{k}) \) be the two intersection points of \( C \) and \( \mathbb{P}^1 \). \( \phi \) is not defined at the pairs \( (s, s') \) and \( (s', s) \). \( \phi^{-1} \) is not defined at the pairs \( (s, s) \) and \( (s', s') \). Easy computation shows that \( \phi \) becomes an isomorphism after we blow up the indeterminacy loci. The blown-up surface is denoted by \( B(C \times C) \). As \( k \)-schemes, \( (s, s') \cup (s', s) \) and \( (s, s) \cup (s', s') \) are both isomorphic to \( \text{Spec}_k k(s) \).

Thus \([C \times C]\) and \([C \times \mathbb{P}^1]\) can both be written as

\[
[B(C \times C)] - [\mathbb{P}^1] \cdot [\text{Spec}_k k(s)] + [\text{Spec}_k k(s)].
\]

In order to see that \([C]\) is a zero divisor in the Grothendieck ring of \( k \)-varieties, we need to prove that \([\mathbb{P}^1] - [C]\) is not zero. By [Lar-Lun], it is sufficient to prove that \( \mathbb{P}^1 \) and \( C \) are not stably birational. This is however easy, since having \( k \)-points is a stably birational invariant. \( \square \)

7 (Products of conics and the Brauer group).

Below we give an elementary geometric description of a partially defined operation which we call the Brauer product of conics. For number fields and for \( C_2 \) fields the
Brauer product of conics is always defined, and the resulting group is the same as the 2-torsion subgroup $Br(k)_2$ of the Brauer group $Br(k)$. (See [Serre, X.4–7] for a good introduction and basic properties.)

Let $k$ be a field and $C_1, C_2$ two smooth conics defined over $k$. The Brauer product of the two conics is defined as follows. (I warn the reader in advance that this definition only works because on a conic the Hilbert scheme of points is isomorphic to the Hilbert scheme of degree 1 divisors.)

Start with $C_1 \times C_2$. As a first approximation, we construct a 3-dimensional variety, denoted by $P(C_1, C_2)$. We would like to say that $P(C_1, C_2)$ is the 3-dimensional “linear system” of divisors of bidegree $(1, 1)$ on $C_1 \times C_2$. The problem is that in general no such divisor is defined over $k$. Thus we look at the linear system $\lvert -K \rvert$ where $K = K_{C_1 \times C_2}$ is the canonical class. This corresponds to divisors of bidegree $(2, 2)$. Then $P(C_1, C_2) \subset \lvert -K \rvert$ is the subscheme consisting of those divisors which are everywhere double. Over $k$ we recognize this as the (doubled) elements of the linear system $\mathcal{O}(1, 1)$.

Alternatively, the Hilbert scheme $\text{Hilb}(C_1 \times C_2)$ has an irreducible component parametrizing divisors of bidegree $(1, 1)$. This is again $P(C_1, C_2)$.

Thus $P(C_1, C_2)$ is isomorphic to $\mathbb{P}^3$ over $\bar{k}$.

There is a natural embedding $C_1 \times C_2 \hookrightarrow P(C_1, C_2)$ where we map a point $(p, q) \in C_1 \times C_2$ to the divisor $2((p) \times C_2 + C_1 \times (q))$.

In general this is all one can do. There are, however, important cases when such a product $P(C_1, C_2)$ contains a degree 1 smooth curve (a line over $\bar{k}$) defined over $k$. In this case I call this degree 1 curve the Brauer product of $C_1$ and $C_2$ and denote it by $C_1 \ast C_2$. (The terminology “Brauer product” does not seem to be standard.)

It turns out that this is well defined up to isomorphism.

To see this, let $P$ be a 3-dimensional $k$-variety which is isomorphic to $\mathbb{P}^3$ over $\bar{k}$. Let $L_1, L_2 \subset P$ be degree 1 smooth curves defined over $k$ and let $L' \subset P$ be another such curve disjoint from both. (Over an infinite field we can obtain $L'$ as the image of $L_1$ by a general automorphism of $P$.) Then $L_1$ and $L_2$ are both isomorphic to the Hilbert scheme of degree 1 surfaces containing $L'$.

$C \ast C$ is always defined and it is isomorphic to $\mathbb{P}^1$. Indeed, the diagonal $\Delta \subset C \times C$ is defined over $k$ thus $P(C, C)$ is $k$-isomorphic to $\mathbb{P}^3_k$. Hence the Brauer group of conics is a 2-group.

**Lemma 8.** For a field $k$ the following two conditions are equivalent.

1. The Brauer product of 2 smooth conics is always defined.
2. For any two smooth conics $C_1, C_2$ defined over $k$ there is a degree 2 extension $k'/k$ such that both $C_1$ and $C_2$ have $k'$-points.

**Proof.** Let $L \subset P(C_1, C_2)$ be a degree 1 curve defined over $k$. Then $L \cap (C_1 \times C_2)$ is a degree 2 subscheme defined over $k$ with residue field $k'$. By projection to the factors, $C_1, C_2$ both have points in $k'$.

Conversely, if $C_1, C_2$ both have points in $k'$ then so does their product. The unique line in $P(C_1, C_2)$ passing through a $k'$ point is defined over $k$. \(\square\)
The following result is well known in various forms, see for instance [Artin, p. 209] or [Sarkisov, Thm. 5.7].

**Proposition 9.** The conditions in (8) hold in the following two cases:

1. \( k \) is a number field.
2. \( k \) is the function field of an algebraic surface over an algebraically closed field.

More generally, for \( C_2 \)-fields.

**Proof.** Here is a geometric version of some of the classical proofs.

Let \( G(1, P(C_1, C_2)) \) denote the Grassmannian of lines in \( P(C_1, C_2) \). We need to prove that it has a \( k \)-point.

More generally, let \( P \) be a \( k \)-variety which is isomorphic to \( P^n \) over \( \bar{k} \) and assume that there is a quadric hypersurface \( Q \subset P \) defined over \( k \). As explained in [Artin, 4.5] the Grassmannian of lines \( G(1, P) \) is embedded into \( P^{n+1} - 1 \) the usual way.

For \( n = 3 \) the Grassmannian of lines \( G(1, P) \) is thus a quadric in \( P^5 \), and so it has a point over any \( C_2 \) field, proving the second part.

If \( k = \mathbb{R} \) then a \( \mathbb{C} \)-point of \( P \) and its conjugate determine a real line, so \( G(1, P)(\mathbb{R}) \neq \emptyset \). Thus \( G(1, P) \) is a quadric in 6 variables which has a point in all real completions of \( k \). Therefore \( G(1, P) \) has a \( k \)-point by the Hasse–Minkowski theorem. \( \Box \)

**10 (Proof of (2)).** The key step is the following generalization of (6):

**Lemma 11.** Let \( C_1, C_2 \) be smooth conics such that their Brauer product \( C_1 * C_2 \) is defined. Then

1. \( C_1 \times C_2 \) is birational to \( C_1 \times (C_1 * C_2) \).
2. \([C_1 \times C_2] = [C_1 \times (C_1 * C_2)]\) in the Grothendieck ring.

**Proof.** First we write down a rational map \( \phi : C_1 \times C_2 \dasharrow C_1 * C_2 \). Then we check that \( \phi \) and the first projection \( \pi_1 : C_1 \times C_2 \rightarrow C_1 \) give a birational map

\[
(\pi_1, \phi) : C_1 \times C_2 \dasharrow C_1 \times (C_1 * C_2).
\]

Finally we see that this gives an identity in the Grothendieck ring.

**Geometric description:** By assumption there is a degree 2 point \( Q \in C_1 \times C_2 \subset P(C_1, C_2) \). Let \( L' \) be the unique degree 1 curve through \( Q \) and let \( L \in P(C_1, C_2) \) be any degree 1 curve disjoint from \( L' \). Projection from \( L' \) to \( L \) gives \( \phi \).

**Algebraic description:** Assume for simplicity that the characteristic is different from 2. If the common point is in the field \( k(\sqrt{a}) \), we can assume that the conics are given by equations

\[
C_1 = (x_1^2 - ax_2^2 - bx_3^2 = 0) \quad \text{and} \quad C_2 = (y_1^2 - ay_2^2 - cy_3^2 = 0).
\]

Then their Brauer product can be given as

\[
C_1 \ast C_2 = (z_1^2 - az_2^2 - bcz_3^2 = 0).
\]
and \( \phi \) is given by
\[
(z_1 : z_2 : z_3) = (x_1y_1 + ax_2y_2 : x_1y_2 + x_2y_1 : x_3y_3).
\]

\( \phi^{-1} \) is obtained as follows. Pick a point \( p \in C_1 \) and \( r \in L \cong C_1 \ast C_2 \). \( \{p\} \times C_2 \) embeds as a line into \( P(C_1, C_2) \) and \( \phi^{-1}(p, r) \) is the intersection point of this line with the plane \( \langle L', r \rangle \) spanned by \( L' \) and \( r \). This is not defined only if \( \{p\} \times C_2 \subset \langle L', r \rangle \).

Thus we see that \( C_1 \times C_2 \) becomes isomorphic to \( C_1 \times (C_1 \ast C_2) \) after we blow up subschemes isomorphic to \( Q \) in both of them. As in (6) this shows that \( [C_1 \times C_2] = [C_1 \times (C_1 \ast C_2)] \).

Assume that the subgroup \( G_I \subset \text{Br}(k)_2 \) generated by the \( C_i \)-s is the same as the subgroup \( G_J \subset \text{Br}(k)_2 \) generated by the \( C'_j \)-s. Fix a minimal generating set \( \{B_s : s \in S\} \) of \( G \). By a simple group theoretic lemma (13) and a repeated application of (11), \( \prod_{i \in I} C_i \) is birational to \( (\mathbb{P}_k^{1})^{I - |S|} \times \prod_{s \in S} B_s \) and they have the same class in the Grothendieck ring. The same holds for \( \prod_{j \in J} C'_j \).

Conversely, assume that \( G_I \neq G_J \). We may assume that \( G_J \not\subset G_I \) and so there is an index \( j_0 \) such that the class of \( C'_j \) is not in \( G_I \). We claim that in this case there is no rational map from \( \prod_{i \in I} C_i \) to \( C'_j \), hence no rational map from \( \prod_{i \in I} C_i \) to \( \prod_{j \in J} C'_j \). Thus they are not birational and not even stably birational, hence they represent different elements of the Grothendieck ring by [Lar-Lun].

The proof is by induction on \( |I| \), the case \( |I| = 0 \) being clear. Pick \( i_0 \in I \) and set \( I' := I \setminus \{i_0\} \) and \( K = k(C_{i_0}) \). By (14), the kernel of \( G_I \to \text{Br}(k)_2 \) is generated by \( C_{i_0} \) and so the class of \( C'_j \) in \( \text{Br}(k)_2 \) is not in the subgroup \( G_{I'} \subset \text{Br}(K)_2 \) generated by the \( C_i \)-s for \( i \in I' \). By induction, there is no \( k(C_{i_0}) \)-map from \( \prod_{i \in I} C_i \) to \( C'_j \), and so no \( k \)-map from \( \prod_{i \in I} C_i \) to \( C'_j \). This completes the proof of (2).

It is clear that all the possible products \( \prod_{i \in I} C_i \) generate the Grothendieck ring as an additive abelian group, and we have established that each such product is identical to a unique element of the form
\[
[\mathbb{P}_k^{1})^{I - |S|} \cdot \prod_{s \in S} [B_s] = [\mathbb{P}_k^{1})^{I - |S|} \cdot C(G).
\]

This gives the description in (4).
In the above proof we used only the relations given by (11), and this gives the following description of the Grothendieck ring.

**Corollary 12.** Let $k$ be a number field or the function field of an algebraic surface over $\mathbb{C}$. The Grothendieck ring of conics is isomorphic to the polynomial ring generated by the isomorphism classes of smooth conics modulo the ideal generated by the relations $[C_1] \cdot [C_2] - [C_1] \cdot [C_1 * C_2]$.

**Lemma 13.** Let $G$ be a finite abelian 2-group with a minimal generating set $b_1, \ldots, b_m$. Let $e_1, \ldots, e_s$ be any generating collection of elements of $G$, repetitions allowed. Then $e_1, \ldots, e_s$ can be transformed into the collection $b_1, \ldots, b_m, 0, \ldots, 0$ by repeated application of the following operation:

Pick $e_i, e_j$ and replace $e_j$ by $e_i + e_j$.

**Lemma 14.** Let $C, C'$ be smooth conics and $g : C \to C'$ a rational map. Then either $C' \cong \mathbb{P}^1$ or $C \cong C'$.

Therefore, if $C' \not\cong C$, $\mathbb{P}^1$ then $C'$ does not have a $k(C)$-point.

**Proof.** Let $G \subset C \times C'$ be the graph of $g$. It is a divisor of bidegree $(1, \deg g)$. A class of bidegree $(0, 2)$ is defined over $k$, so we obtain that either the linear system $|O(1, 0)|$ or the linear system $|O(1, 1)|$ has a member over $k$. In the first case $C'$ has a $k$-point and $C' \cong \mathbb{P}^1$ and in the second case we get a graph of an isomorphism. □

It is possible that (2) holds for higher dimensional Severi–Brauer varieties as well. In (15) we state and prove the stable birational part, that is, the equivalence of (2.3) and (2.4), for arbitrary Severi–Brauer varieties. (For the basic definitions, see [Serre, X.4].)

The higher dimensional analog of (2.2) seems much harder, as it would be a generalization of the Amitsur conjecture asserting that two Severi–Brauer varieties are isomorphic iff they generate the same subgroup of $\text{Br}(k)$.

Equality in the Grothendieck ring may be even harder to show, and I do not know the answer, not even for 2-dimensional Severi–Brauer varieties.

**Theorem 15.** Two products $\prod_{i \in I} P_i$ and $\prod_{j \in J} P'_j$ of Severi–Brauer varieties are stably birational iff the subgroup $\langle P_i : i \in I \rangle \subset \text{Br}(k)$ is the same as the subgroup $\langle P'_j : j \in J \rangle \subset \text{Br}(k)$.

I only indicate the two main steps of the argument. The general version of (14) is a result of [Amitsur, Thm. 9.3]. See also [Serre, X.4.Exercise 2].

**Lemma 16.** Let $P_1, P_2$ be Severi–Brauer varieties. There is a rational map $\phi : P_1 \dashrightarrow P_2$ iff $P_2$ is in the subgroup of the Brauer group generated by $P_1$. 
The following higher dimensional version of (11) gives only a stable birational equivalence. I do not know how to get rid of the extra projective space factors.

**Lemma 17.** Let $P_1, P_2, P_3$ be Severi–Brauer varieties. Assume that $\langle P_1, P_2 \rangle = \langle P_1, P_3 \rangle$ in $\operatorname{Br}(k)$. Then there is a birational map

$$\phi : P_1 \times P_2 \times \mathbb{P}^{\dim P_3} \sim P_1 \times P_3 \times \mathbb{P}^{\dim P_2}.$$ 

**Proof.** Consider $P_1 \times P_2 \times P_3$. Over the function field $k(P_1 \times P_2)$ both $P_1$ and $P_2$ are trivial, and so is $P_3$ since $P_3 \in \langle P_1, P_2 \rangle$. Thus the projection $P_1 \times P_2 \times P_3 \to P_1 \times P_2$ has a section and so there is a birational map

$$P_1 \times P_2 \times P_3 \sim P_1 \times P_2 \times \mathbb{P}^{\dim P_3}.$$ 

Interchanging the roles of $P_2$ and $P_3$ we get another birational map

$$P_1 \times P_2 \times P_3 \sim P_1 \times P_2 \times \mathbb{P}^{\dim P_2}. $$

Putting them together gives $\phi$. □

The above geometric approach also leads to a simple proof of the following.

**Theorem 18 ([Albert]).** Let $P$ be a 3-dimensional Severi–Brauer variety. Then $P$ is the product of 2 conics, $P = P(C_1, C_2)$, iff $P$ contains a quadric.

**Proof.** If $P = P(C_1, C_2)$ then the embedding $C_1 \times C_2 \hookrightarrow P$ gives a smooth quadric.

Conversely, if $Q \subset P$ is a smooth quadric which is isomorphic to the product of 2 conics $C_1, C_2$ (over the base field), then $P = P(C_1, C_2)$. Thus we need to find a decomposable quadric in $P$.

As noted in [Artin, 4.5], the existence of a quadric implies that the Grassmannian of lines $G(1, P)$ is embedded as a quadric in $\mathbb{P}^5$.

If the base field is finite, then $P \cong \mathbb{P}^3$ and we are done.

For an infinite base field $k$, a general $k$-line in $\mathbb{P}^5$ intersects $G(1, P)$ in 2 points which correspond to skew lines $L \cup L' \subset P_k$. Thus $P$ contains a conjugate pair of skew lines.

The linear system of quadrics in $P$ containing $L \cup L'$ has dimension 3, hence there is a smooth $k$-quadric $Q \subset P$ which contains $L \cup L'$.

The two families of lines on $Q$ are now both defined over $k$; one family contains both $L, L'$ the other contains neither. □

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