Enhanced 2-categories and limits for lax morphisms

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Abstract

We study limits in 2-categories whose objects are categories with extra structure and whose morphisms are functors preserving the structure only up to a coherent comparison map, which may or may not be required to be invertible. This is done using the framework of 2-monads. In order to characterize the limits which exist in this context, we need to consider also the functors which do strictly preserve the extra structure. We show how such a 2-category of weak morphisms which is "enhanced", by specifying which of these weak morphisms are actually strict, can be thought of as category enriched over a particular base cartesian closed category $\mathscr{F}$. We give a complete characterization, in terms of $\mathscr{F}$-enriched category theory, of the limits which exist in such 2-categories of categories with extra structure.
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1. Introduction

Just as sets with algebraic structure are often conveniently described as the algebras for a monad, *categories* with algebraic structure are often conveniently described as the algebras for a *2-monad* (see [2]). By a 2-monad we mean a *strict* 2-monad, i.e. a **Cat**-enriched monad, and likewise its algebras satisfy their laws strictly. Experience shows that even when the "algebraic structure" borne by a category or family of categories satisfies laws only up to specified

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isomorphisms, it is always possible, and usually more convenient, to describe it using strict algebras for a strict 2-monad.

Thus, for example, there are 2-monads on the 2-category **Cat** whose algebras are monoidal categories, strict monoidal categories, symmetric monoidal categories, categories with finite products, categories with finite products and finite coproducts connected by a distributive law, categories with finite limits, categories with countable limits, and so on. A structure such as that of cartesian closed category is more delicate, since the internal hom is contravariant in the first variable; to deal with it, one can work not over **Cat** itself, but over the 2-category **Cat**$_g$ of categories, functors, and natural isomorphisms. There is a 2-monad on **Cat**$_g$ whose algebras are cartesian closed categories; similarly there are 2-monads on **Cat**$_g$ for monoidal closed categories, symmetric monoidal closed categories, and elementary toposes.

Moreover, weak algebras can often be reduced directly to strict algebras. In good situations, such as when the base 2-category $\mathscr{K}$ is locally presentable and the 2-monad T has a rank, there is another 2-monad T' whose strict algebras are the weak T-algebras. (This follows from the general theory of "weak morphism classifiers", which we will recall in Section 2.4, using an auxiliary 2-monad whose algebras are 2-monads.)

However, even though we can usually consider only strict algebras for strict 2-monads, no such simplification is possible for *morphisms* between such algebras; the strict morphisms are generally too strict and we must consider weaker notions. Thus, for any 2-monad T on a 2-category $\mathscr{K}$, in addition to the 2-category T-Alg$_s$ of T-algebras and strict T-morphisms (this is the **Cat**-enriched Eilenberg–Moore category), we have the 2-categories T-Alg, T-Alg$_l$, and T-Alg$_c$, whose objects are (strict) T-algebras and whose morphisms are *pseudo*, *lax*, and *colax* T-morphisms, respectively. Pseudo T-morphisms are defined to preserve T-algebra structure up to a (suitably coherent) isomorphism, while lax and oplax ones preserve it only up to a transformation in one direction or the other. Lax monoidal functors, for instance, are ubiquitous in mathematics, pseudo ones are also common, and strict ones are quite rare. The properties of the 2-categories T-Alg, T-Alg$_l$, and T-Alg$_c$ are therefore of interest; our present concern is with the limits that they admit, in the 2-categorical sense (cf. [11]). (Of course, T-Alg$_s$ admits all limits that $\mathscr{K}$ does, by general enriched category theory.)

In the case of T-Alg, this question was answered in [2]. For any 2-monad T on a complete 2-category $\mathscr{K}$, the 2-category T-Alg admits PIE-limits; that is, all limits constructible from products, inserters, and equifiers (see [19]). In particular, this includes all lax limits and pseudo limits, and therefore all bilimits; thus from the "fully weak" point of view of bicategories, T-Alg has all the limits one might ask for. Moreover, the PIE-limits in T-Alg also satisfy an additional strictness property: for each of products, inserters, and equifiers, there is a specified set of limit projections each of which is a strict T-morphism, and which jointly "detect strictness" of T-morphisms.

For T-Alg$_l$ and T-Alg$_c$ the question is more difficult, and the existing answers less complete. It was shown in [17] that T-Alg$_l$ admits the following limits whenever $\mathscr{K}$ does:

- All *oplax* limits.
- All limits of diagrams consisting of strict T-morphisms (that is, the inclusion functor T-Alg$_s \to T$-Alg$_l$ preserves limits).
- Equifiers of pairs of 2-cells $\alpha, \beta : g \Rightarrow f$ where g is a strict T-morphism.
- Inserters of pairs of morphisms $g, f : A \Rightarrow B$ where g is a strict T-morphism.
- Comma objects (g/f) where g is a strict T-morphism.

Once again, each of these limits has the property that some or all of the limit projections are strict T-morphisms, and that they jointly "detect strictness".

The main result of this paper is a complete characterization of those limits which lift to $T\text{-Alg}_l$ for any 2-monad T. As is evident from the examples above, such a characterization must involve, not only $T\text{-Alg}_l$ itself, but its relationship to $T\text{-Alg}_s$. The obvious relationship is the existence of the inclusion functor $T\text{-Alg}_s \to T\text{-Alg}_l$, which is the identity on objects, faithful (on 1-morphisms), and locally fully faithful. A fundamental observation is that the following notions are essentially equivalent:

 (i) A 2-functor which is the identity on objects, faithful, and locally fully faithful.
(ii) A category enriched over the cartesian closed category $\mathscr{F}$ whose objects are functors that are fully faithful and injective on objects. We sometimes call such functors *full embeddings*.

Therefore, rather than viewing $T\text{-Alg}_s$ and $T\text{-Alg}_l$ as two 2-categories related by a functor, we can combine them together into a *single* structure $T\text{-}\mathbb{A}\text{lg}_l$, which happens to be a category enriched over $\mathscr{F}$, i.e. an $\mathscr{F}$-**category**. Intuitively, an $\mathscr{F}$-category has objects, two types of morphism of which one is a special "stricter" case of the other, and 2-cells between these morphisms.

In working with $\mathscr{F}$-categories, we of course need words for the two types of morphism. In the $\mathscr{F}$-category $T\text{-}\mathbb{A}\text{lg}_l$ they are called "strict" and "lax", but there are also other important $\mathscr{F}$-categories one might consider, such as $T\text{-Alg}_s \to T\text{-Alg}$ (where they are "strict" and "pseudo") and $T\text{-Alg} \to T\text{-Alg}_l$ (where they are "pseudo" and "lax"). Thus, in order to avoid favoring one of these cases in our terminology, we introduce new words for the two types of morphism in a general $\mathscr{F}$-category: we call them **tight** and **loose**.

Remark 1.1. Since a full embedding is injective on objects, a loose morphism can "be tight" in at most one way. More generally, we could consider $\mathscr{F}'$-categories, where $\mathscr{F}'$ is the cartesian closed category whose objects are fully faithful functors. $\mathscr{F}'$-categories correspond to 2-functors which are merely the identity on objects and locally fully faithful. In an $\mathscr{F}'$-category, it may be possible to make a given loose morphism into a tight morphism in more than one way (although all such "tightenings" will be isomorphic).

Since "tightenings are unique" in the fundamental examples such as $T\text{-Alg}_s \to T\text{-Alg}_l$, and since it is slightly easier to say that such-and-such a morphism "is tight" than to say that it "can be made into a tight morphism" or "is equipped with a tight morphism structure", we have chosen to work in the slightly more restrictive setting of $\mathscr{F}$-categories. However, it is not hard to generalize all of our results to $\mathscr{F}'$-categories.

The introduction of $\mathscr{F}$-categories allows us to make use of the familiar and powerful language of enriched category theory when discussing the limits that lift to $T\text{-Alg}_l$. In this language, these limits will be characterized by certain *weights* $\Phi : \mathbb{D} \to \mathbb{F}$, where $\mathbb{D}$ is a small $\mathscr{F}$-category and $\mathbb{F}$ denotes $\mathscr{F}$ regarded as an $\mathscr{F}$-category. All the limits mentioned above that exist in $T\text{-Alg}_l$, together with strictness and strictness-detection of some of their projections, can now be described precisely as certain $\mathscr{F}$-weighted limits in the $\mathscr{F}$-category $T\text{-}\mathbb{A}\text{lg}_l$.

Of course we then have to actually give the characterization of the $\mathscr{F}$-weighted limits that lift. This turns out to be a refinement of the notion of *flexible* limit from [3], which we now describe. Recall that for any **Cat**-weight $\Phi : \mathscr{D} \to \textbf{Cat}$, there is another **Cat**-weight $Q\Phi : \mathscr{D} \to \textbf{Cat}$ such that *pseudo* natural transformations $\Phi \rightsquigarrow \Psi$ are in bijection with *strict* natural transformations $Q\Phi \to \Psi$. A weight is said to be *flexible* if the projection $q : Q\Phi \to \Phi$ has a strictly natural

section (it always has a pseudonatural section). Every PIE-weight is flexible, but the converse is false: the splitting of idempotents is a flexible weight that is not a PIE-weight (and in a certain sense it is the "only" such, since together with PIE-limits it generates all flexible limits).

Now Q defines a comonad on the 2-category of **Cat**-weights, which is in fact pseudo-idempotent in the sense of [12]—so that in particular, a weight admits at most one Q-coalgebra structure, up to unique isomorphism. In fact we shall show that *the PIE-weights are precisely the Q-coalgebras*. (This has independently and separately been observed by John Bourke and by Richard Garner.) Note that being flexible is "half" of being a Q-coalgebra. What is missing is an associativity axiom for the section $s : \Phi \to Q\Phi$, and it turns out to be exactly this additional axiom which guarantees that Φ-weighted limits lift from $\mathcal{K}$ to T-Alg for any T.

To generalize this statement to T-Alg$_l$, we first have to replace 2-categories by $\mathscr{F}$-categories and then pseudo morphisms by lax ones. Roughly, a *pseudo $\mathscr{F}$-transformation* consists of a pseudo transformation on the loose morphisms which becomes strictly natural when restricted to the tight ones, and likewise for lax and oplax $\mathscr{F}$-transformations (see Section 4.1). As is the case for 2-categories, such transformations are classified by $\mathscr{F}$-comonads $\mathcal{Q}_p$, $\mathcal{Q}_l$, and $\mathcal{Q}_c$, respectively.

It turns out that the $\mathscr{F}$-limits which lift to T-$\mathbb{A}$lg$_w$, where w is one of p, l, c, are always $\mathcal{Q}_{\bar{w}}$-coalgebras, where $\bar{w}$ denotes w with sense reversed: $\bar{p} = p$, $\bar{l} = c$, and $\bar{c} = l$. But this $\mathcal{Q}_{\bar{w}}$-coalgebra structure is not quite enough for the $\mathscr{F}$-limits to lift. There is also an additional "tightness" condition; this is what ensures that the projections detect tightness, as is necessary for an $\mathscr{F}$-limit. We call these $(w$-)**rigged** weights; they provide our promised characterization of the limits which lift to T-$\mathbb{A}$lg$_w$.

By the phrase *enhanced 2-category theory* in the title, we mean to indicate the study of structures akin to $\mathscr{F}$-categories, which combine a 2-category with additional data, to be studied as a unit. Two existing notions that can also be viewed as enhanced 2-categories are *proarrow equipments* and *double categories*, both of which are in fact quite closely related to $\mathscr{F}$-categories. For instance, from an $\mathscr{F}$-category we can construct a double category whose horizontal arrows are its tight arrows, whose vertical arrows are its loose arrows, and whose squares are 2-cells

This double category comes with a *connection* in the sense of [4] (although there the focus was on edge-symmetric double categories), and in fact $\mathscr{F}$-categories are essentially equivalent to double categories with connections. The perspective of $\mathscr{F}$-categories, however, has the advantage that we can deploy all the tools of enriched category theory. In fact, the framework of double categories allows a clear explanation of which limits should lift [7], as well as their universal property with respect to strict maps, but does not seem easily to capture the universal property with respect to weak maps.

Then again, the *proarrow equipments* of [25] can be identified (modulo questions of 2-categorical strictness) with $\mathscr{F}$-categories in which every tight morphism has a loose right adjoint. This condition is motivated by the example where tight arrows are functors and loose arrows are profunctors (also called "modules" or "distributors"), but it is not satisfied in the examples we are interested in such as T-$\mathbb{A}$lg$_l$. (Some authors, such as [24], have also used the term *equipment*

without this extra condition.) In terms of double categories with a connection, the condition that tight maps have loose right adjoints corresponds to also having an "op-connection"; see [20].

The title of the paper conveys our belief that the introduction of $\mathscr{F}$-categories is a contribution of equal importance to the actual characterization of the limits that lift to $\mathscr{F}$-categories of weak morphisms. In fact, there is also a version of our main result using only 2-categories (see Section 6.4), which in the case $w = p$ recovers the result that PIE-limits lift to T-Alg. However, while PIE-limits, that is to say Q_p-coalgebras, are plentiful and useful in the 2-categorical context, there seem to be fairly few Q_l- or Q_c-coalgebras until we pass to the $\mathscr{F}$-categorical context. Thus, in the lax case, the passage to $\mathscr{F}$-categories significantly enlarges the class of limits possessed by T-Alg$_l$ that we can describe.

There are many possible variations on the themes considered here. For example, one could consider $\mathscr{F}$-categories with lax morphisms as the loose maps and pseudo morphisms as the tight ones. This gives rise to a different notion of rigged weight. Then again, one could extend the very notion of $\mathscr{F}$-category to allow strict, pseudo, and lax morphisms to be encoded into the structure; or, more radically, to combine both lax and colax morphisms. As a final example, one could consider $\mathscr{F}$-bicategories, in which composition is only associative and unital up to isomorphism. The obvious example is $\mathbb{P}$rof, in which the tight morphisms are functors and the loose ones are profunctors. This is known to admit many bicolimits (see [22]). We hope to address some of these issues in a future paper.

2. 2-Categorical preliminaries

We begin with some background material from 2-category theory. Most of this is standard, but Lemma 2.5 and the terminological conventions of Section 2.3 do not appear to be in the literature.

2.1. Monads in 2-categories

A **monad** in a 2-category $\mathscr{K}$ on an object A is a monoid in the monoidal category $\mathscr{K}(A, A)$; thus it consists of a morphism $t : A \to A$ and 2-cells $\mu : tt \to t$ and $\eta : 1 \to t$ satisfying the usual identities. We write $\mathbb{\triangle}_+$ for the algebraic simplex category (a skeleton of the category of finite totally ordered sets). Since $\mathbb{\triangle}_+$ is the free strict monoidal category containing a monoid, a monad in $\mathscr{K}$ is equivalently a strict monoidal functor $\mathbb{\triangle}_+ \to \mathscr{K}(A, A)$, or a strict 2-functor $\mathcal{B}\mathbb{\triangle}_+ \to \mathscr{K}$, where $\mathcal{B}$ indicates that we regard a strict monoidal category as a 2-category with one object.

An **object of algebras** or **Eilenberg–Moore object** for a monad in $\mathscr{K}$ is an object A^t with a forgetful morphism $u : A^t \to A$ such that for every object X, composing with u exhibits an isomorphism

$$\mathscr{K}\left(X, A^t\right) \cong \mathscr{K}(X, A)^{\mathscr{K}(X,t)}$$

between $\mathscr{K}(X, A^t)$ and the usual Eilenberg–Moore category of the ordinary monad $\mathscr{K}(X, t)$ on the ordinary category $\mathscr{K}(X, A)$ induced by whiskering with t. Of course, an object A^t with this property may or may not exist for given $\mathscr{K}$ and t. Making this universal property explicit, it says that u is a "t-algebra" in the sense that we have a 2-cell $\alpha : tu \to u$ such that the usual diagrams for an algebra commute:

$$
\begin{array}{ccc}
ttu & \xrightarrow{\ t\alpha\ } & tu \\
\mu u \downarrow & & \downarrow \alpha \\
tu & \xrightarrow{\ \alpha\ } & u
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
u & \xrightarrow{\ \eta u\ } & tu \\
& {}_{1_u}\searrow & \downarrow \alpha \\
& & u
\end{array}
$$

and u is the universal t-algebra, in a suitable 2-dimensional sense. It was shown in [9, I,7.12.4] that A^t can be described as a *lax limit* of the diagram $\mathcal{B}\triangle_+ \to \mathcal{K}$. The lax *colimit* of this diagram turns out to be the **Kleisli object** A_t, while if we consider $\triangle_+^{\mathrm{op}}$ instead we obtain Eilenberg–Moore and Kleisli objects for *comonads*.

If (A, t) and (B, s) are monads in $\mathcal{K}$, a **lax monad morphism** is a morphism $f : A \to B$ together with a 2-cell $\overline{f} : sf \to ft$ satisfying suitable axioms. (In [21] these were called *monad functors*.) These are the lax morphisms of algebras for a suitable 2-monad or 2-comonad, and also the lax natural transformations between 2-functors $\mathcal{B}\triangle_+ \to \mathcal{K}$. A **monad 2-cell** $\alpha : (f, \overline{f}) \to (g, \overline{g})$ is a 2-cell $\alpha : f \to g$ in $\mathcal{K}$ such that $s\alpha.\overline{f} = \overline{g}.\alpha t$.

We write $\mathrm{Mnd}_l(\mathcal{K})$ for the 2-category of monads, lax monad morphisms, and monad 2-cells. There is a functor $\mathcal{K} \to \mathrm{Mnd}_l(\mathcal{K})$ assigning to each object its identity monad, and $\mathcal{K}$ has Eilenberg–Moore objects if and only if this functor has a right adjoint. In particular, any lax monad morphism $(A, t) \to (B, s)$ induces a morphism $A^t \to B^s$ in a functorial way.

Dually, **colax monad morphisms** (also known as *monad opfunctors*) come with a 2-cell $ft \to sf$ and induce morphisms between Kleisli objects. There is a 2-category $\mathrm{Mnd}_c(\mathcal{K})$ of monads and colax monad morphisms, and $\mathcal{K}$ has Kleisli objects if and only if the inclusion $\mathcal{K} \to \mathrm{Mnd}_c(\mathcal{K})$ has a left adjoint.

A **distributive law** between monads t and s on the same object A consists of a 2-cell $st \to ts$ satisfying suitable axioms. This is equivalent to giving a compatible monad structure on the composite ts, and to giving a lifting of t to the Eilenberg–Moore object A^s, and also to giving an extension of s to the Kleisli object A_t. It is also equivalent to giving a monad in $\mathrm{Mnd}_l(\mathcal{K})$ on the object (A, s), and to giving a monad in $\mathrm{Mnd}_c(\mathcal{K})$ on the object (A, t). When $\mathcal{K}$ has Eilenberg–Moore objects, the EM-object-assigning functor $\mathrm{Mnd}_l(\mathcal{K}) \to \mathcal{K}$ takes each distributive law to the above-mentioned lifting, and similarly for $\mathrm{Mnd}_c(\mathcal{K})$ and the extensions to Kleisli objects.

It follows that there are four different 2-categories whose objects are distributive laws in $\mathcal{K}$. We will need the following description of one of them, which is easily verified by writing out the axioms.

Lemma 2.1. *The* 2-*category* $\mathrm{Mnd}_l(\mathrm{Mnd}_c(\mathcal{K}))$ *can be described as follows.*

- *Its objects are distributive laws* $k : SR \to RS$ *on an object* $\mathscr{A}$ *in* $\mathcal{K}$.
- *Its morphisms from* k_1 *to* k_2 *are morphisms* $F : \mathscr{A}_1 \to \mathscr{A}_2$ *in* $\mathcal{K}$ *equipped with 2-cells* $\psi : S_2 F \to F S_1$ *making it a lax morphism of monads from* S_1 *to* S_2 *and* $\chi : F R_1 \to R_2 F$ *making it a colax morphism of monads from* R_1 *to* R_2, *and such that the following diagram commutes:*

$$
\begin{array}{ccccc}
S_2 F R_1 & \xrightarrow{\ \psi R_1\ } & F S_1 R_1 & \xrightarrow{\ F k_1\ } & F R_1 S_1 \\
{}_{S_2\chi}\downarrow & & & & \downarrow {}_{\chi S_1} \\
S_2 R_2 F & \xrightarrow[\ k_2 F\ ]{} & R_2 S_2 F & \xrightarrow[\ R_2\psi\ ]{} & R_2 F S_1
\end{array}
\tag{1}
$$

- *Its 2-cells from F to G are 2-cells $\alpha : F \to G$ in $\mathcal{K}$ which are both colax monad 2-cells and lax monad 2-cells.*

Corollary 2.2. *If $\mathcal{K}$ has Kleisli objects and (F, ψ, χ) is a morphism in $\mathrm{Mnd}_l(\mathrm{Mnd}_c(\mathcal{K}))$ as in Lemma 2.1, then its extension $\overline{F} : (\mathscr{A}_1)_{R_1} \to (\mathscr{A}_2)_{R_2}$ to Kleisli objects is naturally a lax monad morphism from $\overline{S_1}$ to $\overline{S_2}$.*

Proof. The Kleisli-object-assigning 2-functor $\mathrm{Mnd}_c(\mathcal{K}) \to \mathcal{K}$ induces a 2-functor $\mathrm{Mnd}_l(\mathrm{Mnd}_c(\mathcal{K})) \to \mathrm{Mnd}_l(\mathcal{K})$. □

2.2. 2-Monads

A **2-monad** is a monad in the 2-category **2Cat** of 2-categories, 2-functors, and 2-natural transformations. For a 2-monad T on a 2-category $\mathcal{K}$, we write $T\text{-Alg}_s = \mathcal{K}^T$ for its 2-category of algebras. Explicitly, an object of $T\text{-Alg}_s$ is a (strict) T-algebra, consisting of an object $A \in \mathcal{K}$ and a morphism $a : TA \to A$ such that

$$
\begin{array}{ccc}
T^2A & \xrightarrow{Ta} & TA \\
{\scriptstyle \mu A}\downarrow & & \downarrow{\scriptstyle a} \\
TA & \xrightarrow{a} & A
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{\eta A} & TA \\
& {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\
& & A
\end{array}
$$

commute (strictly). A morphism in $T\text{-Alg}_s$ from (A, a) to (B, b) is called a **strict T-morphism**; it consists of a morphism $f : A \to B$ in $\mathcal{K}$ such that

$$
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
A & \xrightarrow{f} & B
\end{array}
$$

commutes (strictly). Finally, a 2-cell in $T\text{-Alg}_s$ from f to g is called a T-**transformation**, and consists of a 2-cell $\alpha : f \to g$ in $\mathcal{K}$ such that

$$
TA \underset{Tg}{\overset{Tf}{\Rightarrow_{T\alpha}}} TB \xrightarrow{b} B \qquad = \qquad TA \xrightarrow{a} A \underset{g}{\overset{f}{\Rightarrow_{\alpha}}} B
$$

However, we also have various weaker notions of morphism between T-algebras. A **lax T-morphism** $(f, \overline{f}) : (A, a) \to (B, b)$ consists of $f : A \to B$ and a 2-cell

$$
\begin{array}{ccc}
TA & \xrightarrow{\;Tf\;} & TB \\
{\scriptstyle a}\big\downarrow & \overset{\;\;}{\underset{\bar{f}}{\Nearrow}} & \big\downarrow{\scriptstyle b} \\
A & \xrightarrow{\;f\;} & B
\end{array}
$$

such that certain diagrams of 2-cells commute [2]. It is a **colax T-morphism** if $\bar{f}$ goes in the other direction, and a **pseudo T-morphism** if $\bar{f}$ is an isomorphism. We write $T\text{-Alg}_l$, $T\text{-Alg}_c$, and $T\text{-Alg} = T\text{-Alg}_p$ for the 2-categories of (strict) T-algebras and lax, colax, and pseudo T-morphisms, respectively (each with an appropriate notion of T-transformation).

Example 2.3. When $\mathscr{K} = [\mathrm{ob}\,\mathscr{D}, \mathbf{Cat}]$ for a small 2-category $\mathscr{D}$ and T is the 2-monad whose algebras are 2-functors $\mathscr{D} \to \mathbf{Cat}$, then lax, oplax, and pseudo T-morphisms coincide with lax, oplax, and pseudo natural transformations.

Remark 2.4. We will use the generic word *weak* to refer to pseudo, lax, or colax without prejudice. We use the letter w as a decoration or subscript to stand for one of p (pseudo), l (lax), or c (colax). Thus, for instance, for any w and any 2-monad T, we have a 2-category $T\text{-Alg}_w$. We write $\bar{w}$ to denote w with sense reversed, i.e. $\bar{p} = p$, $\bar{l} = c$, and $\bar{c} = l$.

If T and S are 2-monads on 2-categories $\mathscr{A}$ and $\mathscr{B}$, and $(F, \psi) : (\mathscr{A}, T) \to (\mathscr{B}, S)$ is a lax morphism of monads in **2Cat**, then as well as a 2-functor $T\text{-Alg}_s \to S\text{-Alg}_s$, it also induces a 2-functor $T\text{-Alg}_w \to S\text{-Alg}_w$ in a straightforward way.

Moreover, each 2-category $T\text{-Alg}_w$, like $T\text{-Alg}_s$, comes equipped with a forgetful 2-functor $U_w : T\text{-Alg}_w \to \mathscr{K}$ and a transformation $TU_w \to U_w$ which again makes U_w into a strict T-algebra. The difference is that now the transformation $TU_w \to U_w$ is only pseudo, oplax, or lax natural, respectively as $w = p$, l, or c (note the inversion of lax and oplax).

It is shown in [15] that composing with U_w induces an isomorphism

$$
\mathrm{Nat}_{\bar{w}}(\mathscr{X}, T\text{-Alg}_w) \cong \mathrm{Nat}_{\bar{w}}(\mathscr{X}, T)\text{-Alg}_w \tag{2}
$$

where $\mathrm{Nat}_{\bar{w}}(\mathscr{X}, \mathscr{Y})$ denotes the 2-category of 2-functors and $\bar{w}$-natural transformations between 2-categories $\mathscr{X}$ and $\mathscr{Y}$, and $\mathrm{Nat}_{\bar{w}}(\mathscr{X}, T)$ is the 2-monad induced on $\mathrm{Nat}_{\bar{w}}(\mathscr{X}, \mathscr{K})$ by composition with T. This should be compared with the universal property of $T\text{-Alg}_s$, which asserts that

$$
\mathrm{Nat}_s(\mathscr{X}, T\text{-Alg}_s) \cong \mathrm{Nat}_s(\mathscr{X}, T)\text{-Alg}_s
$$

where $\mathrm{Nat}_s(\mathscr{X}, \mathscr{Y}) = [\mathscr{X}, \mathscr{Y}]$ denotes the 2-category of 2-functors and strict 2-natural transformations.

Since an $\mathscr{F}$-categorical version of (2) will be central to our characterization theorem, we recall briefly the idea behind it. Suppose for simplicity that $\mathscr{X} = 2$ and $w = l$. Then an object of $\mathrm{Oplax}(2, T\text{-Alg}_l)$ is simply a lax T-morphism $(f, \bar{f}) : (A, a) \to (B, b)$. On the other hand, an $\mathrm{Oplax}(2, T)$-algebra consists of a morphism $f : A \to B$ in $\mathscr{K}$ (that is, an object of $\mathrm{Oplax}(2, \mathscr{K})$) together with an oplax natural transformation from Tf to f; this consists of morphisms $a : TA \to A$ and $b : TB \to B$ and a 2-cell $\bar{f} : b.Tf \to f.a$. The algebra axioms then

assert precisely that (A, a) and (B, b) are T-algebras and $(f, \overline{f})$ is a lax T-morphism. This shows the bijection on objects.

Now a morphism in $\mathrm{Oplax}(2, T\text{-}\mathrm{Alg}_l)$ from $(f, \overline{f})$ to $(g, \overline{g}) : (C, c) \to (D, d)$ consists of lax T-morphisms $(h, \overline{h}) : (A, a) \to (C, c)$ and $(k, \overline{k}) : (B, b) \to (D, d)$, together with a T-transformation $\alpha : (k, \overline{k})(f, \overline{f}) \to (g, \overline{g})(h, \overline{h})$. On the other hand, a lax morphism of $\mathrm{Oplax}(2, T)$-algebras consists of an oplax transformation from $f : A \to B$ to $g : C \to D$, hence morphisms $h : A \to C$ and $k : B \to D$ and a 2-cell $\alpha : kf \to gh$, together with a modification consisting of 2-cells $\overline{h} : c.Th \to h.a$ and $\overline{k} : d.Tk \to k.b$. The requisite axioms then assert precisely that $(h, \overline{h})$ and $(k, \overline{k})$ are lax T-morphisms and α is a T-transformation.

Finally, of particular importance are those 2-monads T such that for T-algebras (A, a) and (B, b), every morphism $f : A \to B$ in $\mathscr{K}$ supports a unique structure of w-T-morphism. Following [12], we call such a 2-monad w-**idempotent** (also in use is **(co-)KZ**, since they were first isolated by Kock and later Zöberlein, cf. [14,26,13]). The following conditions are known to be equivalent to lax-idempotence of T:

- For every T-algebra (A, a), we have $a \dashv \eta_A$ with identity counit.
- For every $A \in \mathscr{K}$, we have $\mu_A \dashv \eta_{TA}$ with identity counit.
- For every $A \in \mathscr{K}$, we have $T\eta_A \dashv \mu_A$ with identity unit.

For colax-idempotence, the adjunctions go the other way, and for pseudo-idempotence, they are adjoint equivalences. Moreover, if T is w-idempotent for some w, then:

- any two T-algebra structures on $A \in \mathscr{K}$ are isomorphic via a unique isomorphism of the form $(1_A, \alpha)$, and
- for any two w-T-morphisms $(f, \overline{f}), (g, \overline{g}) : (A, a) \rightrightarrows (B, b)$, any 2-cell $\alpha : f \to g$ in $\mathscr{K}$ is a T-transformation.

In particular, if T is w-idempotent, then the forgetful functor $T\text{-}\mathrm{Alg}_w \to \mathscr{K}$ is 2-fully-faithful (an isomorphism on hom-categories).

2.3. 2-Comonads

In this section we briefly treat the dual case of 2-comonads. A 2-comonad on a 2-category $\mathscr{K}$ is of course the same as a 2-monad on $\mathscr{K}^{\mathrm{op}}$, so formally there is not much to say. (As usual, $\mathscr{K}^{\mathrm{op}}$ denotes reversal of 1-cells, but not 2-cells.) However, since there has been little discussion of comonads in the 2-dimensional context, we describe the conventions we adopt, and some of their ramifications.

Just as for 2-monads, we only consider the strict notion of 2-comonad, consisting of a 2-functor W equipped with 2-natural transformations $d : W \to W^2$ and $e : W \to 1$ satisfying the usual laws. Once again, we consider only strict coalgebras, consisting of an object C equipped with a morphism $c : C \to WC$, once again satisfying the usual laws. We need say nothing here about the notion of strict morphism and pseudo morphism of coalgebras; what is worth pointing out is the meaning of lax and colax.

Our starting point is the fact that if T is an endo-2-functor on a 2-category $\mathscr{K}$, with a right adjoint T^*, then to give a 2-monad structure on T is equivalent to giving a 2-comonad structure on T^*, and furthermore the Eilenberg–Moore 2-categories $T\text{-}\mathrm{Alg}_s$ and $T^*\text{-}\mathrm{Coalg}_s$ agree. We shall define lax and colax morphisms of coalgebras in such a way that the isomorphism

T-Alg$_s \cong T^*$-Coalg$_s$ extends to an isomorphism T-Alg$_w \cong T^*$-Coalg$_w$. This way, in concrete cases where the algebras/coalgebras are understood, we may speak of w-morphisms without specifying whether these are defined using T or T^*.

A lax morphism of T-algebras $(A, a) \to (B, b)$ involves a morphism $f : A \to B$ in $\mathscr{K}$ equipped with a suitable 2-cell

$$
\begin{array}{ccc}
TA & \xrightarrow{\ Tf\ } & TB \\
a \downarrow & \ \ \Swarrow_{\bar{f}} & \downarrow b \\
A & \xrightarrow[\ f\ ]{} & B
\end{array}
$$

This corresponds, under the adjunction $T \dashv T^*$, to a 2-cell

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\tilde{a} \downarrow & \ \ \Swarrow_{\tilde{f}} & \downarrow \tilde{b} \\
T^*A & \xrightarrow[\ T^*f\ ]{} & T^*B
\end{array}
$$

Accordingly, for a 2-comonad W and W-coalgebras (C, c) and (D, d), we define a *lax morphism* of W-coalgebras to be a morphism $f : C \to D$ equipped with a 2-cell

$$
\begin{array}{ccc}
C & \xrightarrow{\ f\ } & D \\
c \downarrow & \ \ \Swarrow_{\tilde{f}} & \downarrow d \\
WC & \xrightarrow[\ Wf\ ]{} & WD
\end{array}
$$

satisfying the usual coherence conditions (in dual form). We write W-Coalg$_l$ for the 2-category of W-coalgebras, lax W-morphisms, and W-transformations, and W-Coalg$_c$ and W-Coalg for the evident variants.

Now a 2-monad T on $\mathscr{K}$ also induces a 2-comonad T^{op} on $\mathscr{K}^{\mathrm{op}}$, and the diagram for a lax morphism of T-algebras, when drawn in $\mathscr{K}^{\mathrm{op}}$, becomes the diagram below (drawn with two different orientations to make the comparison easier).

$$
\begin{array}{ccccccc}
TA & \xleftarrow{\ Tf\ } & TB & \qquad & B & \xrightarrow{\ f\ } & A \\
a \uparrow & \ \Swarrow_{\bar{f}} & \uparrow b & \qquad & b \downarrow & \ \tilde{f}\nearrow & \downarrow a \\
A & \xleftarrow[\ f\ ]{} & B & \qquad & TB & \xrightarrow[\ Tf\ ]{} & TA
\end{array}
$$

Thus we have a *colax* morphism of coalgebras, so T^{op}-Coalg$_c = (T$-Alg$_l)^{\mathrm{op}}$, and more generally T^{op}-Coalg$_w = (T$-Alg$_{\bar{w}})^{\mathrm{op}}$ for any w.

Finally, just as a 2-monad T is called w-*idempotent* when the forgetful 2-functor T-$\mathrm{Alg}_w \to \mathcal{K}$ is fully faithful, we say that a 2-comonad W is w-**idempotent** when the 2-functor W-$\mathrm{Coalg}_w \to \mathcal{K}$ is fully faithful; i.e. when every morphism between W-coalgebras admits a unique structure of w-W-morphism. Since W-$\mathrm{Coalg}_w \cong W^{\mathrm{op}}$-$\mathrm{Alg}_{\bar{w}}$, this is equivalent to the 2-monad W^{op} being $\bar{w}$-idempotent. On the other hand, if $W = T^*$ for a 2-monad T, then W is w-idempotent if and only if T is w-idempotent.

As a case of particular interest, if $\mathcal{D}$ and $\mathcal{K}$ are 2-categories with $\mathcal{K}$ complete and cocomplete (such as **Cat**), then the forgetful 2-functor $[\mathcal{D}, \mathcal{K}] \to [\mathrm{ob}\,\mathcal{D}, \mathcal{K}]$ has both adjoints, and is monadic and comonadic. If T and T^* are the corresponding monad and comonad, then lax T-morphisms, which as we have just seen are the same as lax T^*-morphisms, can be identified with lax natural transformations, and similarly in the pseudo and colax cases.

2.4. Weak morphism classifiers

If T is a 2-monad on a 2-category $\mathcal{K}$ and the 2-category T-Alg_s admits a certain kind of 2-colimit called a w-**codescent object**, then the (non-full) inclusion T-$\mathrm{Alg}_s \to T$-Alg_w has a left adjoint Q_w, whose value at a T-algebra (A, a) is the w-codescent object of the diagram

$$T^3 A \underset{\underset{T^2 a}{\longrightarrow}}{\overset{\overset{mTA}{\longrightarrow}}{\overset{TmA}{\longrightarrow}}} T^2 A \underset{\underset{Ta}{\longrightarrow}}{\overset{\overset{mA}{\longrightarrow}}{\underset{\longleftarrow}{TiA}}} T A$$

For instance, in the case $w = l$, this means that we have a universal map $z : TA \to Q_l(A, a)$ equipped with a 2-cell $\zeta : z.mA \to z.Ta$ satisfying two compatibility conditions.

The fact that Q_w is left adjoint to the inclusion of T-$\mathrm{Alg}_s \to T$-Alg_w means that w-morphisms $A \rightsquigarrow B$ are in bijection with strict morphisms $Q_w A \to B$, and likewise for 2-cells between them. The functor Q_w is called the w-**morphism classifier**; see [2]. Note that Q_p is traditionally denoted $(-)'$.

Several conditions on T ensuring that T-Alg_s has w-codescent objects are considered in [16], including:

- $\mathcal{K}$ is cocomplete, and T has a rank (that is, its 2-functor part preserves α-filtered colimits for some α).
- $\mathcal{K}$ has, and T preserves, w-codescent objects.
- $\mathcal{K}$ has, and T preserves coinserters and coequifiers.

Dually, if W is a 2-comonad such that W-Alg_s has w-descent objects, then the inclusion W-$\mathrm{Alg}_s \to W$-Alg_w has a right adjoint R_w called the w-**morphism coclassifier**. Thus w-W-morphisms $A \rightsquigarrow B$ are in bijection with strict morphisms $A \to R_w B$, and likewise for 2-cells.

Finally, if a 2-monad T has a right adjoint T^*, which becomes a 2-comonad with the same algebras and morphisms as in Section 2.3, then T-$\mathrm{Alg}_s = T^*$-Alg_s has all limits and colimits that $\mathcal{K}$ does. Thus, if $\mathcal{K}$ has w-descent and w-codescent objects, then T-$\mathrm{Alg}_s \to T$-Alg_w has both left and right adjoints, giving natural bijections between weak morphisms $A \rightsquigarrow B$, strict morphisms $Q_w A \to B$, and strict morphisms $A \to R_w B$, and likewise for 2-cells.

We write Q_w equally for the left adjoint T-$\mathrm{Alg}_w \to T$-Alg_s and for the composite T-$\mathrm{Alg}_s \hookrightarrow T$-$\mathrm{Alg}_w \xrightarrow{Q_w} T$-$\mathrm{Alg}_s$. In this latter incarnation, Q_w is a 2-comonad on T-Alg_s, since it is a right adjoint followed by its left adjoint. Moreover, since T-$\mathrm{Alg}_s \hookrightarrow T$-$\mathrm{Alg}_w$ is the identity on

objects, the adjunction $T\text{-Alg}_w \rightleftarrows T\text{-Alg}_s$ can be identified with the co-Kleisli adjunction for this comonad. Dually, when R_w exists for a comonad W, it is a 2-monad on $W\text{-Alg}_s$ for which $W\text{-Alg}_w$ is the Kleisli category.

The components of the counit of the adjunction defining Q_w are strict T-morphisms $q_A : Q_w A \to A$, and the components of the unit are w-T-morphisms $p_A : A \rightsquigarrow Q_w A$. The triangle identities for the adjunction say that $q \circ p = 1_A$ and $q \circ Q_w(p) = 1_{Q_w A}$. Of course, q is also the counit of the comonad Q_w, and $Q_w(p)$ is its comultiplication.

Lemma 2.5. *Let T be a 2-monad on a 2-category $\mathcal{K}$. Suppose that $T\text{-Alg}_s$ admits w-codescent objects, so that the 2-comonad Q_w on $T\text{-Alg}_s$ which is the classifier for weak morphisms exists. If moreover $\mathcal{K}$ admits $\bar{w}$-limits of arrows, then Q_w is w-idempotent.*

Proof. We write this out in the case $w = l$, and we write Q for Q_l. The proof is based on an argument in [2] in the pseudo setting.

Let $W = FU$ be the comonad on $T\text{-Alg}_s$ generated by the adjunction. Write $w : W \to 1$ for the counit and $d : W \to W^2$ for the comultiplication. Then Q is given by the l-codescent object of

$$
W^3 \; \overset{\substack{wW^2 \\ \overrightarrow{} \\ WwW \\ \overrightarrow{} \\ W^2 w \\ \overrightarrow{}}}{} \; W^2 \; \overset{\substack{wW \\ \overrightarrow{} \\ d \\ \overleftarrow{} \\ Ww \\ \overrightarrow{}}}{} \; W
$$

via a map $z : W \to Q$ and 2-cell $\zeta : z.wW \to z.Ww$. Let $p : U \to UQ$ be the composite

$$
U \xrightarrow{\;iU\;} TU = UW \xrightarrow{\;Uz\;} UQ.
$$

Then $p : A \to QA$ becomes a lax T-morphism $(p, \bar{p}) : (A, a) \to Q(A, a)$ where $\bar{p} = \zeta.iTA$.

By assumption, $\mathcal{K}$ has oplax limits of arrows. By [17, Theorem 3.2], therefore, $T\text{-Alg}_l$ also has oplax limits of arrows, and the projections are strict and jointly detect strictness. (This will also be a special case of our main theorem; see Section 3.5.2.) Let
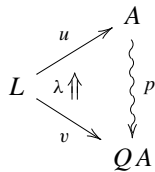
be the oplax limit of p in $T\text{-Alg}_l$, so that u and v are strict and jointly detect strictness. There is a unique lax morphism $(d, \bar{d}) : A \to L$ with $u(d, \bar{d}) = 1$, $v(d, \bar{d}) = (p, \bar{p})$, and λd the identity.

This map $(d, \bar{d}) : A \to L$ factorizes through $(p, \bar{p}) : A \to QA$ via a unique strict map $c : QA \to L$. Now uc is strict and $uc(p, \bar{p}) = u(d, \bar{d}) = 1$, so $uc = q : QA \to A$. Similarly vc is strict and $vc(p, \bar{p}) = v(d, \bar{d}) = (p, \bar{p})$ and so $vc = 1$. It follows that $\lambda c : vc \to puc$ is a 2-cell $\eta : 1 \to pq$ in $T\text{-Alg}_l$. We shall show that it is the unit of an adjunction $q \dashv p$ with identity counit; in other words, that $q\eta$ and ηp are both identities.

Now $\eta p = \lambda c p = \lambda d$, which is an identity by definition of d. On the other hand $\eta : 1 \to pq$ and so $q\eta : q \to qpq = q$. Since q is strict, $q\eta$ will be an identity if and only if $q\eta p$ is; but this follows immediately from the fact that ηp is an identity.

Thus $q \dashv p$ with identity counit, and so $Qq \dashv Qp$ with identity counit. But $Qp : Q \to Q^2$ is the comultiplication of the comonad and q is its counit, so Q is lax-idempotent. $\square$

In particular, for the case of pseudo morphisms, we have $pq \cong 1_{Q_p A}$, so that p and q are inverse adjoint equivalences in $T\text{-Alg}_p$ [2]. In this case, $q : Q_p A \to A$ is a *cofibrant replacement* in a suitable Quillen model structure on $T\text{-Alg}_s$ whose homotopy 2-category is $T\text{-Alg}_p$; see [18]. The cofibrant objects, traditionally called **flexible** algebras, are those for which there exists a *strict T-morphism* $s : A \to Q_p A$ with $qs = 1_A$. In this case we also have $sq \cong 1_{Q_p A}$, so that q is an equivalence in $T\text{-Alg}_s$ as well. Of course, any coalgebra for the comonad Q_p is flexible, but not every flexible object is a Q_p-coalgebra. Note, though, that the flexible objects are precisely the retracts of Q_p-coalgebras; see [8,6] for a general theory of such "algebraic cofibrancy".

3. Enriched category theory over $\mathscr{F}$

3.1. The cartesian closed category $\mathscr{F}$

Let $\mathbf{Cat}^2$ be the category of arrows in $\mathbf{Cat}$; we denote by $\mathscr{F}$ its full subcategory determined by the functors which are injective on objects and fully faithful. We sometimes call such functors *full embeddings*. Thus an object of $\mathscr{F}$ is a full embedding

$$A_\tau \overset{j}{\hookrightarrow} A_\lambda$$

and a morphism in $\mathscr{F}$ is a commutative square

$$
\begin{array}{ccc}
A_\tau & \overset{j_A}{\hookrightarrow} & A_\lambda \\
{\scriptstyle f_\tau}\downarrow & & \downarrow{\scriptstyle f_\lambda} \\
B_\tau & \underset{j_B}{\hookrightarrow} & B_\lambda
\end{array}
$$

Since j_B is monic in $\mathbf{Cat}$, in such a commutative square f_τ is determined uniquely, if it exists, by f_λ. We speak of A_τ as the **tight** part of A and A_λ as the **loose** part, and similarly for f_τ and f_λ.

Note that $\mathscr{F}$ is naturally a 2-category: its 2-cells $\alpha : f \to g$, as inherited from $\mathbf{Cat}^2$, are commuting diagrams of 2-cells of the form

$$
\begin{array}{ccc}
A_\tau & \overset{j_A}{\hookrightarrow} & A_\lambda \\
{\scriptstyle g_\tau}\left(\underset{\alpha_\tau}{\overset{\Leftarrow}{}}\right){\scriptstyle f_\tau} & & {\scriptstyle g_\lambda}\left(\underset{\alpha_\lambda}{\overset{\Leftarrow}{}}\right){\scriptstyle f_\lambda} \\
B_\tau & \underset{j_B}{\hookrightarrow} & B_\lambda
\end{array}
$$

Since j_B is fully faithful, such a 2-cell $\alpha : f \to g$ is determined by $\alpha_\lambda : f_\lambda \to g_\lambda$.

Now, since full embeddings are the right class of a factorization system on **Cat** (the left class consists of functors that are surjective on objects), $\mathscr{F}$ is reflective in $\mathbf{Cat}^2$ and therefore complete and cocomplete, with limits formed pointwise. Colimits in $\mathscr{F}$ are formed by taking the colimit in $\mathbf{Cat}^2$, then applying the reflection, which amounts to taking the full embedding part of the (surjective on objects, full embedding) factorization of a functor.

Moreover, $\mathscr{F}$ is cartesian closed. This can be seen as an instance of the Day reflection theorem [5], or can be checked directly. To see explicitly what the internal hom $[B, C]$ in $\mathscr{F}$ must be, it is convenient to introduce two special objects of $\mathscr{F}$. We denote by 1_τ the terminal object $1 \to 1$ of $\mathscr{F}$, and we denote by 1_λ the object $0 \to 1$, where 0 is the empty category and 1 the terminal category. Note that 1_τ and 1_λ together generate $\mathscr{F}$ as a 2-category; in fact, they are the representables in $\mathbf{Cat}^2$, seen as objects of $\mathscr{F}$. Moreover, for any $A \in \mathscr{F}$ we have

$$A_\tau \cong \mathscr{F}(1_\tau, A) \quad \text{and}$$

$$A_\lambda \cong \mathscr{F}(1_\lambda, A)$$

where $\mathscr{F}(-, -)$ denotes the **Cat**-valued hom of the 2-category $\mathscr{F}$. In particular, this tells us that we must have

$$[B, C]_\tau \cong \mathscr{F}(1_\tau, [B, C]) \cong \mathscr{F}(1_\tau \times B, C) \cong \mathscr{F}(B, C)$$

and

$$[B, C]_\lambda \cong \mathscr{F}(1_\lambda, [B, C]) \cong \mathscr{F}(1_\lambda \times B, C) \cong \mathbf{Cat}(B_\lambda, C_\lambda).$$

That is, a tight object of $[B, C]$ is simply a morphism $B \to C$ in $\mathscr{F}$, while a loose object of $[B, C]$ is a functor $B_\lambda \to C_\lambda$, and the morphisms in either case are natural transformations

$$B_\lambda \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} C_\lambda$$

(As with the 2-cells in $\mathscr{F}$, in the tight case this uniquely determines a compatible transformation between tight parts.) Comparing $[B, C]_\tau$ with $[B, C]_\lambda$, we can say informally that a morphism in $[B, C]$ is tight just when it "preserves tightness", in the sense that it takes tight objects of B to tight objects of C.

We can equivalently construct the full embedding $[B, C]_\tau \hookrightarrow [B, C]_\lambda$ using the following pullback:

$$
\begin{array}{ccc}
[B, C]_\tau & \overset{j_{[B,C]}}{\hookrightarrow} & [B, C]_\lambda \\
\downarrow & & \| \\
& & [B_\lambda, C_\lambda] \\
& & \downarrow {\scriptstyle [j_B, C_\lambda]} \\
[B_\tau, C_\tau] & \underset{[B_\tau, j_\lambda]}{\hookrightarrow} & [B_\tau, C_\lambda]
\end{array}
$$

In practice, many full subcategories are *replete*, in the sense that any object isomorphic to one in the subcategory is itself in the subcategory. A non-replete full subcategory is equivalent to its repletion as a category, but *not* as an object of $\mathscr{F}$.

There is also a larger sub-2-category $\mathscr{F}'$ of $\mathbf{Cat}^2$ containing all the fully faithful functors, not necessarily injective on objects. As mentioned in the introduction, all our results have straight-forward extensions to $\mathscr{F}'$-categories, but we shall not mention them.

3.2. $\mathscr{F}$-categories

Since $\mathscr{F}$ is cartesian closed, we can now consider the notion of $\mathscr{F}$-**category**, or category enriched in $\mathscr{F}$. Of course, an $\mathscr{F}$-category $\mathbb{A}$ has a collection of objects, together with hom-objects $\mathbb{A}(x, y)$ in $\mathscr{F}$ and composition and identity maps also in $\mathscr{F}$. Each hom-object $\mathbb{A}(x, y)$ thus consists of two categories $\mathbb{A}(x, y)_\tau$ and $\mathbb{A}(x, y)_\lambda$ related by a full embedding. It is easy to see that the categories $\mathbb{A}(x, y)_\tau$ must form the hom-categories of a 2-category $\mathscr{A}_\tau$. Likewise, the categories $\mathbb{A}(x, y)_\lambda$ must form a 2-category $\mathscr{A}_\lambda$ with the same objects, and the full embeddings relating them must fit together into a 2-functor $J_\mathbb{A} : \mathscr{A}_\tau \to \mathscr{A}_\lambda$ which is the *identity on objects*, *faithful*, and *locally fully faithful*. Furthermore, any such 2-functor determines a unique $\mathscr{F}$-category, so we will generally identify $\mathscr{F}$-categories with such 2-functors.

We refer to morphisms in $\mathscr{A}_\tau$ as **tight morphisms** and those in $\mathscr{A}_\lambda$ as **loose morphisms**. We generally write tight morphisms with straight arrows $A \to B$ and loose ones with wavy arrows $A \rightsquigarrow B$. We will also omit the subscript on J when it is evident from context, and since it is the identity on objects, we will not notate its application to objects. However, we will usually notate J when applied to morphisms, or when composed with other 2-functors.

Remark 3.1. In terms of the generating objects 1_τ and 1_λ introduced in Section 3.1, we have two monoidal functors $\mathscr{F}(1_\tau, -), \mathscr{F}(1_\lambda, -) : \mathscr{F} \to \mathbf{Cat}$ related by a monoidal transformation arising from the inclusion $1_\lambda \hookrightarrow 1_\tau$. The 2-categories $\mathscr{A}_\tau$ and $\mathscr{A}_\lambda$ and the 2-functor $J_\mathbb{A}$ are then the "change of base" of the $\mathscr{F}$-category $\mathbb{A}$ along these functors and transformation.

When we write our $\mathscr{F}$-categories as 2-functors in this way, an $\mathscr{F}$-**functor** $F : \mathbb{A} \to \mathbb{B}$ consists of 2-functors $F_\tau : \mathscr{A}_\tau \to \mathscr{B}_\tau$ and $F_\lambda : \mathscr{A}_\lambda \to \mathscr{B}_\lambda$ making the evident square commute. Since $J_\mathbb{B}$ is monic in **2Cat**, F_τ is determined uniquely, if it exists, by F_λ; thus we can say informally that an $\mathscr{F}$-functor $\mathbb{A} \to \mathbb{B}$ is a 2-functor $\mathscr{A}_\lambda \to \mathscr{B}_\lambda$ which "preserves tightness".

Likewise, an $\mathscr{F}$-**natural transformation** $m : F \to G$ reduces to a pair of 2-natural transformations $m_\tau : F_\tau \to G_\tau$ and $m_\lambda : F_\lambda \to G_\lambda$ subject to the evident condition. Since $J_\mathbb{A}$ is the identity on objects, the components of m_λ are determined by those of m_τ, so its existence is a mere additional property imposed on m_τ ("naturality with respect to loose maps, in addition to tight ones"). On the other hand, since $J_\mathbb{B}$ is faithful, we can equally regard the existence of m_τ as a property of m_λ, namely that all of its components are tight.

Example 3.2. Any 2-category $\mathscr{K}$ may be regarded as an $\mathscr{F}$-category $\mathbb{K}$ in which $\mathscr{K}_\tau = \mathscr{K}_\lambda = \mathscr{K}$, so that "all morphisms are tight". We call such an $\mathscr{F}$-category **chordate**. On the other hand, we may instead take $\mathscr{K}_\lambda = \mathscr{K}$ but let $\mathscr{K}_\tau$ be the locally full sub-2-category of $\mathscr{K}$ containing all the objects but only the identity morphisms ("only identities are tight"). We call such an $\mathscr{F}$-category **inchordate**. Note that in the inchordate case $\mathscr{K}_\tau$ is *not* generally a discrete 2-category: it contains only the identity 1-morphisms of $\mathscr{K}$, but all the endo-2-cells of these.

For an abstract point of view, recall that $\mathscr{F}(1_\lambda, -) : \mathscr{F} \to \mathbf{Cat}$ induces a 2-functor $\mathscr{F}\mathbf{Cat} \to$ **2Cat** which sends an $\mathscr{F}$-category to its loose part. This 2-functor has a left adjoint which sends a 2-category to the corresponding inchordate $\mathscr{F}$-category, and a right adjoint which sends a 2-category to the corresponding chordate $\mathscr{F}$-category. The latter 2-functor can also be induced directly by the finite-product-preserving functor $\mathbf{Cat} \to \mathscr{F}$ which sends a category C to its identity functor.

Example 3.3. In our motivating examples of $\mathscr{F}$-categories, the objects are some sort of category with structure, the tight morphisms are the functors which preserve the structure strictly, the loose morphisms are the functors which preserve the structure in some weaker sense, and the 2-cells are suitably compatible with the extra structure. For example, for any 2-monad T on a 2-category $\mathscr{K}$ and any $w = p, c, l$, the inclusion $J : T\text{-Alg}_s \to T\text{-Alg}_w$ is a prototypical $\mathscr{F}$-category, which we denote $T\text{-}\mathbb{Alg}_w$. It comes with a forgetful $\mathscr{F}$-functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$, where $\mathbb{K}$ is the chordate $\mathscr{F}$-category associated to $\mathscr{K}$.

Example 3.4. Another important class of $\mathscr{F}$-categories, less relevant in this paper, is where $\mathscr{K}_\lambda$ is some 2-category of interest, and $\mathscr{K}_\tau$ the sub-2-category of left adjoints.

Example 3.5. Our last, and very important, example of an $\mathscr{F}$-category comes from the general fact that any monoidal closed category is enriched over itself. We shall write $\mathbb{F}$ for the $\mathscr{F}$-category which arises from $\mathscr{F}$ in this way. The hom-objects of $\mathbb{F}$ are, of course, given by the cartesian closed internal hom of $\mathscr{F}$ as described in Section 3.1. Thus, the objects of $\mathbb{F}$ are the objects of $\mathscr{F}$, its tight morphisms are the morphisms of $\mathscr{F}$, its loose morphisms are functors between loose parts, and its 2-cells are transformations between the latter. In particular, the 2-category $\mathbb{F}_\tau$ is just $\mathscr{F}$ with its 2-category structure as mentioned previously, while $\mathbb{F}_\lambda$ can be obtained as the fully-faithful reflection of the composite $\text{cod} \circ N$, as in the following diagram:

$$
\begin{array}{ccc}
\mathbb{F}_\tau & \xrightarrow{\ N\ } & \mathbf{Cat}^2 \\
{\scriptstyle J_{\mathbb{F}}}\big\downarrow & & \big\downarrow{\scriptstyle \text{cod}} \\
\mathbb{F}_\lambda & \xhookrightarrow[\ M\ ]{} & \mathbf{Cat}
\end{array}
\qquad (3)
$$

Here N is the inclusion, $J_{\mathbb{F}}$ is the identity on objects, and M is 2-fully-faithful (i.e. an isomorphism on hom-categories). We shall sometimes, as in this diagram, display fully faithful maps using a hooked arrow, and bijective-on-objects ones using a bar at the tip of the arrow. Since $\text{cod} \circ N$ is locally fully faithful, so is $J_{\mathbb{F}}$. (M is actually an equivalence of 2-categories, but it is important to maintain the distinction between $\mathbb{F}_\lambda$ and $\mathbf{Cat}$, since $J_{\mathbb{F}}$ is the identity on objects but $M \circ J_{\mathbb{F}}$ is not.)

Since $\mathscr{F}$ is a complete and cocomplete symmetric monoidal closed category, we have all of the basic machinery of enriched category theory at our disposal. In Sections 3.3–3.5 we will discuss enriched functor categories, limits, and colimits in the particular case of enrichment over $\mathscr{F}$.

3.3. Functor $\mathscr{F}$-categories

Given $\mathscr{F}$-categories $\mathbb{D}$ and $\mathbb{K}$ with $\mathbb{D}$ small, we can form the functor $\mathscr{F}$-category $[\mathbb{D}, \mathbb{K}]$ whose objects are $\mathscr{F}$-functors from $\mathbb{D}$ to $\mathbb{K}$. A morphism in $[\mathbb{D}, \mathbb{K}]_\tau$ is just an $\mathscr{F}$-natural transformation, while a morphism in $[\mathbb{D}, \mathbb{K}]_\lambda$ between $F, G : \mathbb{D} \to \mathbb{K}$ consists of a 2-natural transformation $F_\lambda \to G_\lambda$. We also have 2-cells in $[\mathbb{D}, \mathbb{K}]_\lambda$, which are modifications between the 2-natural transformations $F_\lambda \to G_\lambda$ just considered. In Section 4.1, we shall consider weakenings of these notions, where the morphisms are not required to be strictly natural.

We now turn to the special case where $\mathbb{K} = \mathbb{F}$. An $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{F}$ is often called a *weight*; it amounts to a commutative square of 2-functors as in the left half of the following diagram (the right half simply reproduces (3)):

$$
\begin{array}{ccccc}
\mathscr{D}_\tau & \xrightarrow{\ G_\tau\ } & \mathscr{F}_\tau & \xrightarrow{\ N\ } & \mathbf{Cat}^2 \\
{\scriptstyle J_\mathbb{D}}\Big\downarrow & & {\scriptstyle J_\mathbb{F}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{cod}} \\
\mathscr{D}_\lambda & \xrightarrow[\ G_\lambda\ ]{} & \mathscr{F}_\lambda & \xrightarrow[\ M\ ]{} & \mathbf{Cat}
\end{array}
$$

Since M is 2-fully-faithful and $J_\mathbb{D}$ is the identity on objects, G_λ is uniquely determined by G_τ and the composite MG_λ. On the other hand, G_τ is uniquely determined by the composites $\mathrm{dom}\, NG_\tau$, $\mathrm{cod}\, NG_\lambda$, and a 2-natural transformation $\mathrm{dom}\, NG_\tau \to \mathrm{cod}\, NG_\tau$ whose components are full embeddings; we then also need $\mathrm{cod}\, NG_\tau = MG_\lambda J_\mathbb{D}$. Thus, altogether, to give a weight is to give 2-functors $\Phi_\tau : \mathscr{D}_\tau \to \mathbf{Cat}$ and $\Phi_\lambda : \mathscr{D}_\lambda \to \mathbf{Cat}$, and a 2-natural transformation $\varphi : \Phi_\tau \to \Phi_\lambda J_\mathbb{D}$ whose components are full embeddings; we write Φ for such a weight $(\Phi_\tau, \Phi_\lambda, \varphi)$.

Suppose now that $\Psi = (\Psi_\tau, \Psi_\lambda, \psi)$ is another such weight. We compute the $\mathscr{F}$-valued hom $[\mathbb{D}, \mathbb{F}](\Phi, \Psi)$, using our description of functor $\mathscr{F}$-categories. An object of $[\mathbb{D}, \mathbb{F}](\Phi, \Psi)_\lambda$ (that is, a loose morphism in $[\mathbb{D}, \mathbb{F}]$) is just a 2-natural transformation between the corresponding 2-functors $\mathscr{D}_\lambda \to \mathscr{F}_\lambda$, but since M is 2-fully-faithful, that is the same as a 2-natural transformation $\Phi_\lambda \to \Psi_\lambda$. A morphism in $[\mathbb{D}, \mathbb{F}](\Phi, \Psi)_\lambda$ (that is, a 2-cell in $[\mathbb{D}, \mathbb{F}]$) is a modification between 2-naturals $\Phi_\lambda \to \Psi_\lambda$. In other words,

$$[\mathbb{D}, \mathbb{F}](\Phi, \Psi)_\lambda = [\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, \Psi_\lambda).$$

A morphism $\Phi \to \Psi$ is tight when, seen as a 2-natural $\Phi_\lambda \to \Psi_\lambda$, it restricts to a 2-natural transformation $\Phi_\tau \to \Psi_\tau$; in other words, we have a pullback

$$
\begin{array}{ccc}
[\mathbb{D}, \mathbb{F}](\Phi, \Psi)_\tau & \longrightarrow \quad [\mathbb{D}, \mathbb{F}](\Phi, \Psi)_\lambda = & [\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, \Psi_\lambda) \\
\Big\downarrow & & \Big\downarrow {\scriptstyle [J_\mathbb{D}, \mathbf{Cat}]} \\
& & [\mathscr{D}_\tau, \mathbf{Cat}](\Phi_\lambda J_\mathbb{D}, \Psi_\lambda J_\mathbb{D}) \\
& & \Big\downarrow {\scriptstyle [\mathscr{D}_\tau, \mathbf{Cat}](\varphi, \Psi_\lambda J_\mathbb{D})} \\
[\mathscr{D}_\tau, \mathbf{Cat}](\Phi_\tau, \Psi_\tau) & \xrightarrow[\ \ [\mathscr{D}_\tau, \mathbf{Cat}](\Phi_\tau, \psi)\ \ ]{} & [\mathscr{D}_\tau, \mathbf{Cat}](\Phi_\tau, \Psi_\lambda J_\mathbb{D})
\end{array}
$$

A particular important class of weights are the representables: for any $\mathscr{F}$-category $\mathbb{K}$ and any object X of $\mathbb{K}$, we have a representable $\mathscr{F}$-functor $R = \mathbb{K}(X, -) : \mathbb{K} \to \mathbb{F}$. In terms of the previous paragraph, we have $\mathbb{K}(X, -)_{\tau} = \mathscr{K}_{\tau}(X, -)$ and $\mathbb{K}(X, -)_{\lambda} = \mathscr{K}_{\lambda}(X, -)$, while $r : R_{\tau} \to R_{\lambda} J_{\mathbb{K}}$ is the map

$$J_{\mathbb{K}} : \mathscr{K}_{\tau}(X, -) \to \mathscr{K}_{\lambda}(J_{\mathbb{K}} X, J_{\mathbb{K}} -) = \mathscr{K}_{\lambda}(X, J_{\mathbb{K}} -)$$

given by the action of $J_{\mathbb{K}}$ on hom-categories.

3.4. $\mathscr{F}$-weighted limits

Suppose now $S : \mathbb{D} \to \mathbb{A}$ is an $\mathscr{F}$-functor, of which we shall shortly consider the Φ-weighted limit, and let A be an object of $\mathbb{A}$. These induce an $\mathscr{F}$-functor $\mathbb{A}(A, S) : \mathbb{D} \to \mathbb{F}$. Writing this functor in the form $\Psi = (\Psi_{\tau}, \Psi_{\lambda}, \psi)$ of Section 3.3, we have $\Psi_{\tau} = \mathscr{A}_{\tau}(A, S_{\tau})$, $\Psi_{\lambda} = \mathscr{A}_{\lambda}(A, S_{\lambda})$, and ψ given by

$$
\begin{array}{ccc}
\mathscr{D}_{\tau} & \xrightarrow{S_{\tau}} & \mathscr{A}_{\tau} \\
& & \quad\searrow^{\mathscr{A}_{\tau}(A,-)} \\
J_{\mathbb{D}} \downarrow & J_{\mathbb{A}} \downarrow \quad J_{\mathbb{A}} \Downarrow & \textbf{Cat} \\
& & \quad\nearrow \\
\mathscr{D}_{\lambda} & \xrightarrow{S_{\lambda}} & \mathscr{A}_{\lambda} \quad \mathscr{A}_{\lambda}(A,-)
\end{array}
$$

We are now ready to consider the weighted limit $\{\Phi, S\}$, for some weight $\Phi : \mathbb{A} \to \mathbb{F}$. This limit, if it exists, is characterized by an isomorphism

$$\mathbb{A}\big(A, \{\Phi, S\}\big) \cong [\mathbb{D}, \mathbb{F}]\big(\Phi, \mathbb{A}(A, S)\big)$$

in $\mathscr{F}$, natural in A. This involves an isomorphism

$$\mathscr{A}_{\lambda}\big(A, \{\Phi, S\}\big) \cong [\mathscr{D}_{\lambda}, \textbf{Cat}]\big(\Phi_{\lambda}, \mathscr{A}_{\lambda}(A, S_{\lambda})\big)$$

and a left leg for the square

$$
\begin{array}{ccc}
\mathscr{A}_{\tau}(A, \{\Phi, S\}) & \xrightarrow{\hspace{3cm}} & \mathscr{A}_{\lambda}(A, \{\Phi, S\}) \\
\vdots & & \downarrow{\scriptstyle \cong} \\
\vdots & & [\mathscr{D}_{\lambda}, \textbf{Cat}](\Phi_{\lambda}, \mathscr{A}_{\lambda}(A, S_{\lambda})) \\
\vdots & & \downarrow{\scriptstyle [J_{\mathbb{D}}, \textbf{Cat}]} \\
\vdots & & [\mathscr{D}_{\tau}, \textbf{Cat}](\Phi_{\lambda} J_{\mathbb{D}}, \mathscr{A}_{\lambda}(A, S_{\lambda} J_{\mathbb{D}})) \\
\vdots & & \downarrow{\scriptstyle [\mathscr{D}_{\lambda}, \textbf{Cat}](\varphi, 1)} \\
[\mathscr{D}_{\tau}, \textbf{Cat}](\Phi_{\tau}, \mathscr{A}_{\tau}(A, S_{\tau})) & \xrightarrow{\hspace{1.5cm}} & [\mathscr{D}_{\tau}, \textbf{Cat}](\Phi_{\tau}, \mathscr{A}_{\lambda}(A, J_{\mathbb{A}} S_{\tau}))
\end{array} \tag{4}
$$

so that it becomes a pullback. The first isomorphism says that $\{\Phi, S\}$ is the 2-categorical limit $\{\Phi_\lambda, S_\lambda\}$ in $\mathscr{A}_\lambda$; we shall write $p_\lambda : \Phi_\lambda \to \mathscr{A}_\lambda(\{\Phi, S\}, S_\lambda)$ for the corresponding unit. The pullback square (4) specifies a further universal property involving the tight maps, which we now analyze.

First of all, to give a dotted map making the square commute is to give a map $p_\tau : \Phi_\tau \to \mathscr{A}_\tau(\{\Phi, S\}, S_\tau)$ making

$$
\begin{array}{ccc}
\Phi_\tau & \xrightarrow{\quad \varphi \quad} & \Phi_\lambda J \\
{\scriptstyle p_\tau} \downarrow & & \downarrow {\scriptstyle p_\lambda J} \\
\mathscr{A}_\tau(\{\Phi, S\}, S_\tau) & \xrightarrow[\quad J_\mathbb{A} \quad]{} & \mathscr{A}_\lambda(\{\Phi, S\}, S_\lambda)
\end{array}
$$

commute. Since the bottom leg of this latter square is injective on objects and fully faithful, such a p_τ is unique if it exists, and will exist if and only if for each $D \in \mathbb{D}$ the composite

$$
\Phi_\tau D \xrightarrow{\ \varphi D\ } \Phi_\lambda D \xrightarrow{\ p_\lambda D\ } \mathscr{A}_\lambda\big(\{\Phi, S\}, SD\big)
$$

takes its values in $\mathscr{A}_\tau(\{\Phi, S\}, SD)$. In other words, for any $D \in \mathbb{D}$ and any $a \in \Phi_\tau D$, the morphism $p_{\lambda, \varphi(a)} : \{\Phi, S\} \to SD$ is tight.

Second, we require that the resulting square (4) be a pullback. Since the horizontals are fully faithful, we need only check the universal property at the level of objects. This says that a loose morphism $h : A \to \{\Phi, S\}$ is tight if the composite

$$
\Phi_\tau D \xrightarrow{\ \varphi D\ } \Phi_\lambda D \xrightarrow{\ p_\lambda D\ } \mathscr{A}_\lambda\big(\{\Phi, S\}, SD\big) \xrightarrow{\ \mathscr{A}_\lambda(h, SD)\ } \mathscr{A}_\lambda(A, SD)
$$

takes its values in $\mathscr{A}_\tau(A, SD)$ for each $D \in \mathbb{D}$. In other words, h is tight if $p_{\lambda, \varphi(a)} \circ h$ is tight for each $a \in \Phi_\tau D$. We express this by saying that the $p_{\lambda, \varphi(a)}$ **jointly detect tightness**.

Combining the two conditions, we have:

Proposition 3.6. *Let* $\Phi : \mathbb{D} \to \mathbb{F}$ *be a weight and* $S : \mathbb{D} \to \mathbb{K}$ *an* $\mathscr{F}$-*functor. An* $\mathscr{F}$-*categorical limit* $\{\Phi, S\}$ *in* $\mathbb{K}$ *is a 2-categorical limit* $\{\Phi_\lambda, S_\lambda\}$ *in* $\mathscr{K}_\lambda$ *with the extra property that for* $D \in \mathbb{D}$ *and* $a \in \Phi_\tau(D)$, *the projections* $p_{\lambda, \varphi(a)} : \{\Phi_\lambda, S_\lambda\} \to SD$ *are tight and jointly detect tightness.*

In particular, if $\mathbb{K}$ is chordate, then an $\mathscr{F}$-limit $\{\Phi, S\}$ in $\mathbb{K}$ is nothing but a 2-categorical limit $\{\Phi_\lambda, S_\lambda\}$ in the 2-category $\mathscr{K}_\tau = \mathscr{K}_\lambda$.

Finally, we consider two slightly different notions of "limits lifting along a functor", one well-known and one less so.

Definition 3.7. Let $\mathscr{V}$ be a closed symmetric monoidal category, $U : \mathscr{A} \to \mathscr{B}$ a $\mathscr{V}$-functor, and $\Phi : \mathscr{D} \to \mathscr{V}$ a $\mathscr{V}$-weight. If for any diagram $G : \mathscr{D} \to \mathscr{A}$ such that the limit $\{\Phi, UG\}$ exists, the limit $\{\Phi, G\}$ also exists and is preserved by U, we say that Φ-weighted limits **lift along** U or **lift to** $\mathscr{A}$, or that U **lifts** Φ-weighted limits. If U furthermore reflects all such Φ-weighted limits, we say that it **creates** them.

The notion of "creation" of limits is standard, although there is some variation in its usage. Some authors use it only when $\mathscr{B}$ has all Φ-weighted limits, which we do not generally require. Others require that $\{\Phi, G\}$ must map exactly onto $\{\Phi, UG\}$ and be literally unique with that property; this will be true for the $\mathscr{F}$-functors $U : \mathscr{A} \to \mathscr{B}$ we consider in this context, but we shall neither use nor verify it.

Note that if Φ-weighted limits lift along U, then any Φ-weighted cone over G in $\mathscr{A}$ which maps to a limiting cone in $\mathscr{B}$ must factor through the limit $\{\Phi, G\}$ in $\mathscr{A}$ by a map which is inverted by U. Thus, if U is conservative, reflection is automatic, and so lifting and creating are equivalent.

The $\mathscr{F}$-functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ is always conservative, since the underlying ordinary category of $T\text{-}\mathbb{Alg}_w$ consists of *strict T-morphisms*. Thus, in this case there is no difference between lifting and creation. By contrast, the 2-functor $U_w : T\text{-}\mathrm{Alg}_w \to \mathscr{K}$ is not in general conservative for $w = l$ or c, and in this case the limits which lift are *not* generally reflected. (Thus the statements in [17] referring to "creation" of limits are only about "lifting" of limits according to our present terminology.) We regard this as another advantage of $\mathscr{F}$-categories over 2-categories.

3.5. Examples of weights

In this section we consider a few specific examples of weights, and describe the corresponding notions of limit. For now, we focus on examples which lift to $T\text{-}\mathbb{Alg}_w$ for some w, and are thus of interest in our primary examples. We will mention some more "pathological" examples in Section 6.3. We shall describe in each case what is known about lifting the limit to $T\text{-}\mathbb{Alg}_w$ for a 2-monad T, as in Example 3.3; we shall see in Proposition 5.6 that this implies a lifting result for any $\mathscr{F}$-monad T.

3.5.1. Tight limits

Let $\mathscr{D}$ be a 2-category, and $\mathbb{D}$ the corresponding chordate $\mathscr{F}$-category, with $\mathscr{D}_\tau = \mathscr{D}_\lambda = \mathscr{D}$. A 2-functor $M : \mathscr{D}_\tau \to \mathbf{Cat}$ gives rise to an $\mathscr{F}$-weight $\Phi : \mathbb{D} \to \mathbb{F}$ with $\Phi_\tau = \Phi_\lambda = M$, and φ the identity. Then for any weight $\Psi : \mathbb{D} \to \mathbb{F}$, the $\mathbb{F}$-valued hom $[\mathbb{D}, \mathbb{F}](\Phi, \Psi)$ is given by the full embedding

$$[\mathscr{D}, \mathbf{Cat}](M, \Psi_\tau) \xrightarrow{[\mathscr{D}, \mathbf{Cat}](M, \psi)} [\mathscr{D}, \mathbf{Cat}](M, \Psi_\lambda).$$

A diagram $S : \mathbb{D} \to \mathbb{A}$ of shape $\mathbb{D}$ is just a 2-functor $S : \mathscr{D} \to \mathscr{A}_\tau$; then the corresponding loose part $\mathscr{D}_\lambda = \mathscr{D} \to \mathscr{A}_\lambda$ is the composite $J_{\mathbb{A}} S : \mathscr{D}_\tau \to \mathscr{A}_\lambda$. Thus the tight part of the universal property of the limit $\{\Phi, S\}$ says that $\mathscr{A}_\tau(A, \{\Phi, S\}) \cong [\mathscr{D}, \mathbf{Cat}](M, \mathscr{A}_\tau(A, S))$, and so $\{\Phi, S\}$ is the 2-categorical limit $\{M, S\}$ in $\mathscr{A}_\tau$; while the loose part says that this limit is preserved by $J_{\mathbb{A}} : \mathscr{A}_\tau \to \mathscr{A}_\lambda$. We call a limit of this type **tight**.

In the case where $\mathbb{A}$ is $T\text{-}\mathbb{Alg}_w$ for a 2-monad T, such tight limits amount to limits in $T\text{-}\mathrm{Alg}_s$ preserved by the inclusion $T\text{-}\mathrm{Alg}_s \to T\text{-}\mathrm{Alg}_w$. These tight limits do lift to $T\text{-}\mathbb{Alg}_w$ for any 2-monad T and any w; see [2] and [17, Proposition 4.1].

3.5.2. The oplax limit of a loose morphism

Let $\mathscr{D}_\lambda$ be the arrow category $2 = \{d \to c\}$, seen as a locally discrete 2-category, let $\mathscr{D}_\tau$ be the discrete 2-category with two objects, and let $J_{\mathbb{D}} : \mathscr{D}_\tau \to \mathscr{D}_\lambda$ be the evident inclusion. Then a diagram $S : \mathbb{D} \to \mathbb{A}$ is precisely a loose morphism in $\mathbb{A}$; we shall write it as $s : Sd \to Sc$.

Let $\Phi_\lambda : \mathscr{D}_\lambda \to \mathbf{Cat}$ be the 2-functor which picks out the functor $c : 1 \to 2$, let $\Phi_\tau : \mathscr{D}_\tau \to \mathbf{Cat}$ be the 2-functor constant at the terminal category 1, and let $\varphi : \Phi_\tau \to \Phi_\lambda J_{\mathbb{D}}$ have components $1 : 1 \to 1$ and $c : 1 \to 2$.

As always, a limit $\{\Phi, S\}$ must in particular be a limit $\{\Phi_\lambda, S_\lambda\}$, which means an oplax limit in $\mathscr{A}_\lambda$ of the arrow $s : Sd \to Sc$. This consists of an object L with loose morphisms $u : L \rightsquigarrow Sd$ and $v : L \rightsquigarrow Sc$, and a 2-cell $\sigma : v \to su$, these data being universal in $\mathscr{A}_\lambda$. The tight aspect of the universal property now says that we have a pullback

$$
\begin{array}{ccc}
\mathscr{A}_\tau(A, L) & \longrightarrow & \mathscr{A}_\lambda(A, L) \\
\vdots & & \downarrow \\
\mathscr{A}_\tau(A, Sd) \times \mathscr{A}_\tau(A, Sc) & \longrightarrow & \mathscr{A}_\lambda(A, Sd) \times \mathscr{A}_\lambda(A, Sc)
\end{array}
$$

in $\mathbf{Cat}$. In other words, u and v are tight and jointly detect tightness.

In particular, this means that we have a bijection between loose morphisms $x : A \rightsquigarrow L$ and pairs of loose morphisms $u : A \rightsquigarrow Sd$, $v : A \rightsquigarrow Sc$ equipped with a 2-cell $\xi : v \to su$; and similarly a bijection between tight morphisms $x : A \to L$ and pairs of tight morphisms $u : A \to Sd$, $v : A \to Sc$ equipped with a 2-cell $\xi : J(v) \to s \circ J(u)$.

By [17, Theorem 3.2], oplax limits of loose morphisms lift to $T\text{-}\mathbb{Alg}_l$ for any 2-monad T. Dually, lax limits of loose morphisms (where the 2-cell σ is reversed) lift to $T\text{-}\mathbb{Alg}_c$. By [2, Remark 2.7], lax, oplax, and also pseudo limits (the case where σ is invertible) of loose morphisms all lift to $T\text{-}\mathbb{Alg}_p$ for any 2-monad T.

3.5.3. Inserters

Let $\mathbb{D}$ be the inchordate $\mathscr{F}$-category on a parallel pair of arrows $(a \rightrightarrows b)$; thus a functor $\mathbb{D} \to \mathbb{K}$ is a parallel pair of loose morphisms in $\mathbb{K}$. Let $\Phi_\lambda : \mathscr{D}_\lambda \to \mathbf{Cat}$ pick out the two distinct functors $1 \rightrightarrows 2$, and let $\Phi_\tau(a) = 1$ and $\Phi_\tau(b) = 0$.

A Φ-weighted limit of $f, g : A \rightsquigarrow B$ is in particular an inserter, i.e. a morphism $i : I \to A$ with a 2-cell $fi \rightsquigarrow gi$ which is universal among such. Moreover, the morphism i (but not the composites fi and gi) must be tight and must detect tightness. We call such a limit a p-**rigged inserter**; by [2, Proposition 2.2], such inserters lift to $T\text{-}\mathbb{Alg}_p$ for any 2-monad T.

Now instead suppose we take $\mathscr{D}_\tau$ to be the arrow category 2, equipped with one of its inclusions into the parallel pair $\mathscr{D}_\lambda$. We let Φ_λ be as before, with the tight morphism in $\mathbb{D}$ going to the functor $d : 1 \to 2$, and we let Φ_τ be constant at 1.

In this case, a $\mathbb{D}$-diagram in $\mathbb{K}$ is a parallel pair $f, g : A \to B$ where f is tight and g is loose, and a Φ-weighted limit of such is an inserter $fi \to gi$ such that i is tight and detects tightness; thus fi is also tight. We call such a limit an l-**rigged inserter**; by [17, Proposition 4.4], such inserters lift to $T\text{-}\mathbb{Alg}_l$ for any 2-monad T.

If instead we require g to be tight, we obtain the notion of c-**rigged inserter**, which lifts to $T\text{-}\mathbb{Alg}_c$.

3.5.4. Equifiers

Let $\mathbb{D}$ be the inchordate $\mathscr{F}$-category on a parallel pair of 2-cells (between a parallel pair of morphisms $a \rightrightarrows b$), so that a $\mathbb{D}$-shaped diagram is a parallel pair of 2-cells between a parallel pair of loose morphisms. Let Φ_λ be the diagram

$$1 \overset{\Downarrow\Downarrow}{\underset{}{\rightrightarrows}} 2$$

in **Cat**, where the two parallel 2-cells are equal, and let $\Phi_\tau(a) = 1$ and $\Phi_\tau(b) = 0$.

Then a Φ-weighted limit of $\alpha, \beta : f \rightrightarrows g$ is an equifier, i.e. a morphism e such that $\alpha.e = \beta.e$ which is universal with this property, such that moreover e is tight and detects tightness. We call such a limit a p-**rigged equifier**; by [2, Proposition 2.3], such equifiers lift to T-$\mathbb{Alg}_p$ for any 2-monad T.

Now suppose that in $\mathbb{D}$ we require the morphism that is the domain of the 2-cells to be tight, and take Φ_τ to be constant at 1. Then a $\mathbb{D}$-diagram is a parallel pair of 2-cells whose common domain is tight, and a Φ-weighted limit is again an equifier which again is tight and detects tightness. We call such a limit an l-**rigged equifier**; by [17, Proposition 4.3], such equifiers lift to T-$\mathbb{Alg}_l$ for any 2-monad T. Of course, dually we have c-**rigged equifiers**, which lift to T-$\mathbb{Alg}_c$.

3.5.5. Descent objects

Write $\mathbb{1} = \{0\}$ for the terminal category, $\mathbb{2} = \{0 \to 1\}$ for the free-living arrow, and $\mathbb{3} = \{0 \to 1 \to 2\}$ for the free-living composable pair. Consider the functors

$$\mathbb{1} \overset{\delta_0}{\underset{\delta_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\leftarrow\sigma-}}}} \mathbb{2} \overset{\delta_0}{\underset{\delta_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{-\delta_1\twoheadrightarrow}}}} \mathbb{3}$$

where δ_i is the inclusion which omits i. Let $\mathscr{D}_\lambda$ be the locally discrete sub-2-category of **Cat** generated by the functors in the diagram, and let $\Phi_\lambda : \mathscr{D}_\lambda \to$ **Cat** be the inclusion. A diagram $G : \mathscr{D}_\lambda \to \mathscr{A}$ in a 2-category has the form

$$A_0 \overset{\delta_0}{\underset{\delta_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\leftarrow\sigma-}}}} A_1 \overset{\delta_0}{\underset{\delta_2}{\overset{\longrightarrow}{\underset{\longrightarrow}{-\delta_1\twoheadrightarrow}}}} A_2$$

and the Φ_λ-weighted limit of this diagram we call an l-**descent object**. This is an object A universally equipped with a morphism $p : A \to A_0$ and a 2-cell $\pi : \delta_1 a \to \delta_0 a$ such that $\delta_0\pi.\delta_2\pi = \delta_1\pi$ and $\sigma\pi = 1$.

Let $\mathscr{D}_\tau$ be the sub-2-category of $\mathscr{D}_\lambda$ generated by the functors

$$\mathbb{1} \overset{\delta_0}{\underset{\leftarrow\sigma-}{\longrightarrow}} \mathbb{2} \overset{\delta_0}{\underset{-\delta_1\twoheadrightarrow}{\longrightarrow}} \mathbb{3}$$

and let $\Phi_\tau : \mathscr{D}_\tau \to$ **Cat** be the 2-functor constant at $\mathbb{1}$. Define $\varphi : \Phi_\tau \to \Phi_\lambda J_{\mathbb{D}}$ to have components at $\mathbb{1}$, $\mathbb{2}$, and $\mathbb{3}$ given by the identity, $\delta_0 : \mathbb{1} \to \mathbb{2}$, and $\delta_0\delta_0 : \mathbb{1} \to \mathbb{3}$, respectively. This now gives a weight $\Phi : \mathbb{D} \to \mathbb{F}$. The Φ-weighted limit of a diagram

$$A_0 \xrightarrow[]{\delta_0} \overset{\delta_0}{\xleftarrow{\;\;\sigma\;\;}} A_1 \xrightarrow[]{\delta_1} \xrightarrow{\delta_1} A_2$$

in $\mathbb{A}$ is an l-descent object ($p : A \to A_0$, $\pi : \delta_1 p \to \delta_0 p$) in $\mathscr{A}_\lambda$ for which p is tight and detects tightness. We call such a limit an l-**rigged** l-**descent object**.

The lifting of such descent objects to T-$\mathbb{A}\mathrm{lg}_l$ for a 2-monad T was not considered explicitly in [17], but it could be treated by similar techniques to those used there: either by giving a direct construction, or by constructing the descent object using inserters and equifiers.

Dually, there are c-descent objects, in which the direction of the 2-cell π is reversed, and the corresponding c-**rigged** c-**descent objects** can be shown to lift to T-$\mathbb{A}\mathrm{lg}_c$. There are also p-descent objects, in which π is required to be invertible, and these lift to T-Alg.

Note that $\mathscr{D}_\lambda$ admits an automorphism which swaps the **Cat**-weights for l-descent objects and c-descent objects, but this is no longer true for $\mathbb{D}$ and the rigged weights.

3.5.6. Eilenberg–Moore objects

Recall from Section 2.1 that monads in a 2-category $\mathscr{K}$ are in bijection with 2-functors from $\mathscr{B}\mathbb{\Delta}_+$ to $\mathscr{K}$. We may regard $\mathscr{B}\mathbb{\Delta}_+$ as an inchordate $\mathscr{F}$-category $\mathbb{D}$, so that $\mathscr{D}_\lambda = \mathscr{B}\mathbb{\Delta}_+$ and $\mathscr{D}_\tau$ is the terminal 2-category.

If s is a monad on an object $A \in \mathscr{K}$, and $S : \mathscr{B}\mathbb{\Delta}_+ \to \mathscr{K}$ is the corresponding 2-functor, an Eilenberg–Moore object A^s for s is a limit of S weighted by a particular weight $\mathscr{B}\mathbb{\Delta}_+ \to$ **Cat** which we shall call Φ_λ. We obtain a weight $\Phi : \mathbb{D} \to \mathbb{F}$ by setting $\Phi_\tau = \mathbb{1}$ and $\varphi : \Phi_\tau \to \Phi_\lambda J$ the map which picks out the "generating projection".

An $\mathbb{F}$-functor $S : \mathbb{D} \to \mathbb{K}$ is equivalently a monad s on some object A of $\mathscr{K}_\lambda$, and a Φ-weighted limit of S is now an Eilenberg–Moore object $u : A^s \to A$ for s in $\mathscr{K}_\lambda$ for which u is tight and detects tightness.

By [17, Proposition 4.5], limits of this sort lift to T-$\mathbb{A}\mathrm{lg}_c$ for any 2-monad T. Dually, Eilenberg–Moore objects of comonads lift to T-$\mathbb{A}\mathrm{lg}_l$. They also lift to T-$\mathbb{A}\mathrm{lg}_p$, by the results of [2], since they can be constructed using inserters and equifiers.

3.5.7. Powers (cotensors)

Let $\mathbb{D}$ be the terminal $\mathscr{F}$-category, with $\mathscr{D}_\tau = \mathscr{D}_\lambda = \mathbb{1}$. Then a weight $\Phi : \mathbb{D} \to \mathbb{F}$ consists of an object $X = (x : X_\tau \to X_\lambda)$ of $\mathbb{F}$. A Φ-weighted limit is called a **power** or cotensor.

A diagram $S : \mathbb{D} \to \mathbb{K}$ consists of an object S of $\mathbb{K}$. The power of S by X is written $X \pitchfork S$. The loose part of its universal property says that it is the 2-categorical power $L = X_\lambda \pitchfork S$ in $\mathscr{K}_\lambda$, defined by a natural isomorphism

$$\mathscr{K}_\lambda(A, L) \cong \mathbf{Cat}\big(X_\lambda, \mathscr{K}_\lambda(A, S)\big).$$

The tight part says that a morphism $f : A \to L$ is tight if and only if the corresponding $\tilde{f} : X_\lambda \to \mathscr{K}_\lambda(A, S)$ restricts to give the dotted part of the following commutative square.

$$
\begin{array}{ccc}
X_\tau & \dashrightarrow & \mathscr{K}_\tau(A, S) \\
\scriptstyle x \downarrow & & \downarrow \scriptstyle J_{\mathbb{K}} \\
X_\lambda & \xrightarrow{\;\tilde{f}\;} & \mathscr{K}_\lambda(A, S)
\end{array}
$$

In other words, the projections $X_\lambda \pitchfork S \to S$ which correspond to objects of X_τ are tight and jointly detect tightness.

Notice, in particular, that if X and $\mathbb{K}$ are both chordate, then the tight part of the universal property is automatic. Then X-powers lift to T-$\mathbb{Alg}_w$, for any w and any 2-monad T [2].

On the other hand, if X_τ is empty, then all maps $A \to L$ must be tight. In particular, if X_τ is empty and X_λ is terminal, then all maps $A \to L$ are tight and they are in bijection with loose maps $A \to S$, and finally this bijection extends to 2-cells. Thus L is a (slightly odd) kind of "loose morphism coclassifier". Such a limit does not generally lift to T-$\mathbb{Alg}_w$.

4. Weak aspects of $\mathscr{F}$-category theory

In the previous section we developed some of the standard enriched-categorical notions in the case of enrichment over $\mathscr{F}$. In this section we turn to those notions where some weakness is involved; this is of course absent from general enriched category theory.

4.1. Weak $\mathscr{F}$-natural transformations

We begin by generalizing the notions of lax, oplax, and pseudo natural transformations from the 2-categorical setting to the $\mathscr{F}$-categorical setting. Given $\mathscr{F}$-categories $\mathbb{D}$ and $\mathbb{K}$, we can define an $\mathscr{F}$-category $\mathrm{Nat}_w(\mathbb{D}, \mathbb{K})$ for each flavour of weakness ($w = p, c, l$), where in each case the objects are the $\mathscr{F}$-functors from $\mathbb{D}$ to $\mathbb{K}$.

Given $\mathscr{F}$-functors $M, N : \mathbb{D} \to \mathbb{K}$, we define a **loose w-natural transformation** $f : M \to N$ to be a w-natural transformation (of 2-functors) $f_\lambda : M_\lambda \to N_\lambda$, such that $f_\lambda J_\mathbb{D} : M_\lambda J_\mathbb{D} \to N_\lambda J_\mathbb{D}$ is strictly 2-natural. Such a loose w-natural transformation is **tight** when its components $f_\lambda D : M_\lambda D \to N_\lambda D$ are all tight; this amounts to giving a 2-natural transformation $f_\tau : M_\tau \to N_\tau$ such that $f_\lambda J_\mathbb{D} = J_\mathbb{K} f_\tau$. Note that even a tight w-natural transformation is more general than an $\mathscr{F}$-natural transformation, in which f_λ would also have to be 2-natural.

Finally, a **modification** between w-natural transformations $f, g : M \to N$ is a modification $f_\lambda \to g_\lambda$. When f and g are tight, such a modification induces a unique modification $f_\tau \to g_\tau$, since $J_\mathbb{K}$ is locally fully faithful.

We define the $\mathscr{F}$-category $\mathrm{Nat}_w(\mathbb{D}, \mathbb{K})$ in the obvious way: its objects are $\mathscr{F}$-functors $\mathbb{D} \to \mathbb{F}$, its tight and loose morphisms are tight and loose w-natural transformations, respectively, and its 2-cells are modifications. For $w = p, l, c$ we may write $\mathbb{Ps}(\mathbb{D}, \mathbb{K})$, $\mathbb{Lax}(\mathbb{D}, \mathbb{K})$, and $\mathbb{Oplax}(\mathbb{D}, \mathbb{K})$ respectively.

Note in particular that being *weak* is independent of being *loose*; thus a strict transformation can be either tight or loose (these are the tight and loose morphisms in $[\mathbb{D}, \mathbb{K}]$) and likewise a w-transformation can be either tight or loose (these are the tight and loose morphisms in $\mathrm{Nat}_w(\mathbb{D}, \mathbb{K})$).

4.2. Weak $\mathscr{F}$-transformation classifiers

We now use the 2-categorical weak morphism classifiers from Section 2.4 to build corresponding classifiers for weak $\mathscr{F}$-transformations. Specifically, given a small $\mathscr{F}$-category $\mathbb{D}$ and a cocomplete $\mathscr{F}$-category $\mathbb{K}$, we shall construct a left $\mathscr{F}$-adjoint to the inclusion $[\mathbb{D}, \mathbb{K}] \to \mathrm{Nat}_w(\mathbb{D}, \mathbb{K})$.

First of all, let $\mathrm{ob}\,\mathbb{D}$ be the discrete $\mathscr{F}$-category with the same objects as $\mathbb{D}$, and $H : \mathrm{ob}\,\mathbb{D} \to \mathbb{D}$ the inclusion. Restriction along H and left Kan extension induces a comonad $W = W_{\mathbb{D}}$ on $[\mathbb{D}, \mathbb{K}]$. For each $F : \mathbb{D} \to \mathbb{K}$, we form the tight w-codescent object $Q_{\mathbb{D}}F$ of the diagram

$$W^3 F \begin{array}{c} \xrightarrow{wW^2F} \\ \xrightarrow{WwWF} \\ \xrightarrow{W^2wF} \end{array} W^2 F \begin{array}{c} \xrightarrow{wWF} \\ \xleftarrow{dF} \\ \xrightarrow{WwF} \end{array} W F$$

whose universal property means that loose maps $Q_{\mathbb{D}}F \rightsquigarrow G$ in $[\mathbb{D}, \mathbb{K}]$ correspond to w-natural transformations $F_\lambda \to G_\lambda$. They are tight if their components are tight.

Next, let $\mathbb{C}$ be $\mathscr{D}_\tau$, regarded as a chordate $\mathscr{F}$-category, with $J : \mathbb{C} \to \mathbb{D}$ the evident inclusion. For an $\mathscr{F}$-functor $F : \mathbb{D} \to \mathbb{K}$, we may restrict along J and then left Kan extend to obtain an $\mathscr{F}$-functor $\mathrm{Lan}_J(FJ)$, whose universal property means that loose maps $\mathrm{Lan}_J(FJ) \rightsquigarrow G$ correspond to (strict) natural transformations $J_{\mathbb{K}} F_\tau \to J_{\mathbb{K}} G_\tau$. They are tight if their components are tight, or in other words, if they are natural transformations $F_\tau \to G_\tau$.

On the other hand, there is a comonad $W_{\mathbb{C}}$ on $[\mathbb{C}, \mathbb{K}]$ analogous to $W_{\mathbb{D}}$, and if we form the corresponding $Q_{\mathbb{C}}$, then the $\mathscr{F}$-functor $\mathrm{Lan}_J(Q_{\mathbb{C}}(FJ))$ has the universal property that loose maps $\mathrm{Lan}_J(Q_{\mathbb{C}}(FJ)) \rightsquigarrow G$ correspond to w-natural transformations $J_{\mathbb{K}} F_\tau \to J_{\mathbb{K}} G_\tau$. They are tight if their components are tight, or in other words, if they are w-natural transformations $F_\tau \to G_\tau$.

The weak morphism classifier is now given by the pushout $\mathcal{Q}F$ as in

$$\begin{array}{ccc} \mathrm{Lan}_J(Q_{\mathbb{C}}(FJ)) & \longrightarrow & \mathrm{Lan}_J(FJ) \\ \downarrow & & \downarrow \\ Q_{\mathbb{D}}F & \longrightarrow & \mathcal{Q}F \end{array}$$

whose universal property says that a loose morphism $\mathcal{Q}F \rightsquigarrow G$ is a w-natural $F_\lambda \to G_\lambda$ for which the induced $F_\lambda J \to G_\lambda J$ is strict; it is tight if its components are tight. This is exactly what is needed for the weak morphism classifier.

Dually, if $\mathbb{D}$ is small and $\mathbb{K}$ is complete, we have weak morphism coclassifiers. If $\mathbb{K}$ is both complete and cocomplete, then we have both classifiers and coclassifiers for weak morphisms, giving left and right adjoints $\mathcal{Q}_w$ and $\mathcal{R}_w$ to the inclusion $[\mathbb{D}, \mathbb{K}] \to \mathrm{Nat}_w(\mathbb{D}, \mathbb{K})$.

If $\mathbb{K} = \mathbb{F}$, so that $\Phi : \mathbb{D} \to \mathbb{F}$ can be expressed as $\varphi : \Phi_\tau \to \Phi_\lambda J$, we can alternatively construct the weak morphism classifier as follows. For any $\mathscr{F}$-category $\mathbb{D}$, let $Q_w^{\mathbb{D}}$ denote the 2-categorical weak morphism classifier for the 2-monad on $[\mathscr{D}_\tau, \mathbf{Cat}]$ whose category of algebras is $[\mathscr{D}_\lambda, \mathbf{Cat}]$. We call this the **relative w-transformation classifier** for $\mathbb{D}$; its universal property says that strict natural transformations $Q_w^{\mathbb{D}} F \to G$ correspond to w-natural transformations $F \to G$ which become strict when restricted to $\mathscr{D}_\tau$. Then we can define

$$(\mathcal{Q}_w \Phi)_\tau = \Phi_\tau,$$
$$(\mathcal{Q}_w \Phi)_\lambda = Q_w^{\mathbb{D}}(\Phi_\lambda).$$

The structure map $(\mathcal{Q}_w \Phi)_\tau \to (\mathcal{Q}_w \Phi)_\lambda J$ for $\mathcal{Q}_w \Phi$ is given by the composite

$$\Phi_\tau \xrightarrow{\varphi} \Phi_\lambda J \xrightarrow{p} Q_w^{\mathbb{D}}(\Phi_\lambda) J.$$

To see that this is a pointwise full embedding, consider the case $w = l$. Observe that φ is a pointwise full embedding since Φ is a weight, while p is a pointwise full embedding since it has a left adjoint $qJ : Q_w^{\mathbb{D}}(\Phi_\lambda)J \to \Phi_\lambda J$ with identity counit (essentially by the argument given in Lemma 2.5 or see [2]). Thus the composite $p\varphi$ is also a pointwise full embedding.

Unlike in the 2-categorical case, the weak $\mathscr{F}$-transformation classifier $\mathcal{Q}_w$ is seemingly not a special case of any construction that applies to more general $\mathscr{F}$-monads; see Section 4.3.

As in the 2-categorical case, however, composing $\mathcal{Q}_w$ with the inclusion gives a comonad on $[\mathbb{D}, \mathbb{F}]$, which we also call $\mathcal{Q}_w$, and whose co-Kleisli $\mathscr{F}$-category is $\mathrm{Nat}_w(\mathbb{D}, \mathbb{F})$. Similarly, the composite of $\mathcal{R}_w$ with the inclusion gives a monad, also called $\mathcal{R}_w$, whose Kleisli $\mathscr{F}$-category is $\mathrm{Nat}_w(\mathbb{D}, \mathbb{F})$.

We summarize all the weak transformation classifiers we will need in this paper, and their notations, as follows.

- For any small 2-category $\mathscr{D}$ and any cocomplete 2-category $\mathscr{K}$, we have a 2-comonad $Q_w^{\mathscr{D}}$ on $[\mathscr{D}, \mathscr{K}]$, which classifies 2-categorical w-natural transformations.
- For any small $\mathscr{F}$-category $\mathbb{D}$, we have a 2-comonad $Q_w^{\mathbb{D}}$ on $[\mathscr{D}_\lambda, \mathbf{Cat}]$, which classifies 2-categorical w-transformations that become strict when restricted to $\mathscr{D}_\tau$ (the *relative w-transformation classifier*). Comparing universal properties, we see that if $\mathbb{D}$ is inchordate, then $Q_w^{\mathbb{D}} = Q_w^{\mathscr{D}_\lambda}$.
- For any small $\mathscr{F}$-category $\mathbb{D}$ and any cocomplete $\mathscr{F}$-category $\mathbb{K}$, we have an $\mathscr{F}$-comonad $\mathcal{Q}_w^{\mathbb{D}}$ on $[\mathbb{D}, \mathbb{K}]$, which classifies both tight and loose weak $\mathscr{F}$-transformations as defined in Section 4.1. In the case $\mathbb{K} = \mathbb{F}$, we have $(\mathcal{Q}_w^{\mathbb{D}} \Phi)_\tau = \Phi_\tau$ and $(\mathcal{Q}_w^{\mathbb{D}} \Phi)_\lambda = Q_w^{\mathbb{D}}(\Phi_\lambda)$.

We will frequently omit the superscripts and/or subscripts on these classifiers when there is no danger of confusion.

Note that $\mathcal{Q}$ is left adjoint to $\mathcal{R}$, since we have

$$[\mathbb{D}, \mathbb{K}](\mathcal{Q}_w F, G) \cong \mathrm{Nat}_w(\mathbb{D}, \mathbb{K})(F, G) \cong [\mathbb{D}, \mathbb{K}](F, \mathcal{R}_w G).$$

Moreover, we also have the following (standard) "adjointness" with respect to weighted limits.

Lemma 4.1. *For any complete $\mathscr{F}$-category $\mathbb{K}$, any weight $\Phi : \mathbb{D} \to \mathbb{F}$, and any diagram $G : \mathbb{D} \to \mathbb{K}$, we have $\{\Phi, \mathcal{R}_w G\} \cong \{\mathcal{Q}_w \Phi, G\}$.*

Proof. For any $A \in \mathbb{K}$, we have

$$\mathbb{K}\big(A, \{\Phi, \mathcal{R}_w G\}\big) \cong [\mathbb{D}, \mathbb{F}]\big(\Phi, \mathbb{K}(A, \mathcal{R}_w G)\big)$$
$$\cong [\mathbb{D}, \mathbb{F}]\big(\Phi, \mathcal{R}_w \mathbb{K}(A, G)\big)$$
$$\cong [\mathbb{D}, \mathbb{F}]\big(\mathcal{Q}_w \Phi, \mathbb{K}(A, G)\big)$$
$$\cong \mathbb{K}\big(A, \{\mathcal{Q}_w \Phi, G\}\big)$$

where we have used the fact that since $\mathcal{R}$ is a limit construction, it is preserved by the representable $\mathbb{K}(A, -)$. $\quad\square$

In particular, for any $D \in \mathbb{D}$ we have

$$(\mathcal{R}_w G)(D) \cong \big\{ \mathbb{D}(D, -), \mathcal{R}_w G \big\} \cong \big\{ \mathcal{Q}_w \mathbb{D}(D, -), G \big\}$$

so that $\mathcal{R}_w$ is itself a weighted limit construction. (To those who are familiar with the behavior of weighted limits, this is also evident from our construction of $\mathcal{R}$ out of other weighted limits.)

4.3. $\mathscr{F}$-monads

By an $\mathscr{F}$-**monad** we mean, of course, a monad $T : \mathbb{K} \to \mathbb{K}$ in the 2-category $\mathscr{F}\mathbf{Cat}$ of $\mathscr{F}$-categories, $\mathscr{F}$-functors, and $\mathscr{F}$-transformations. In particular, this means that the components of its multiplication and unit are tight, and strictly natural with respect to both tight and loose morphisms. We denote the Eilenberg–Moore object in $\mathscr{F}\mathbf{Cat}$ of such a T by $T\text{-}\mathbb{Alg}_s$. The objects of $T\text{-}\mathbb{Alg}_s$ are the (strict) T_τ-algebras, and the tight morphisms are the strict T_τ-morphisms, which we call **strict T-morphisms**. The loose morphisms in $T\text{-}\mathbb{Alg}_s$, on the other hand, are the strict T_λ-morphisms (where we regard T_τ-algebras as T_λ-algebras in the evident way). $T\text{-}\mathbb{Alg}_s$ has the usual universal property with respect to $\mathscr{F}$-functors $G : \mathbb{X} \to \mathbb{K}$ equipped with a T-algebra structure $TG \to G$ which is $\mathscr{F}$-natural.

If we replace $\mathscr{F}$-naturality in this universal property by weak $\mathscr{F}$-naturality of the three kinds considered in Section 4.1, we thereby characterize a trio $(w = p, c, l)$ of $\mathscr{F}$-categories which we denote $T\text{-}\mathbb{Alg}_w$. Explicitly:

- An object of $T\text{-}\mathbb{Alg}_w$ is a (strict) T_τ-algebra (hence also a T_λ-algebra).
- A tight morphism in $T\text{-}\mathbb{Alg}_w$ is a strict T_τ-morphism (hence also a strict T_λ-morphism).
- A loose morphism in $T\text{-}\mathbb{Alg}_w$ is a w-T_λ-morphism; we call these w-T-**morphisms**.
- A transformation is a T_λ-transformation.

For instance, a loose morphism $(f, \overline{f}) : (A, a) \rightsquigarrow (B, b)$ in $T\text{-}\mathbb{Alg}_l$ consists of a loose morphism $f : A \rightsquigarrow B$ in $\mathbb{K}$ together with a 2-cell

$$
\begin{array}{ccc}
TA & \overset{Tf}{\rightsquigarrow} & TB \\
{\scriptstyle a}\big\downarrow & \overset{}{\underset{\overline{f}}{\Swarrow}} & \big\downarrow{\scriptstyle b} \\
A & \underset{f}{\rightsquigarrow} & B
\end{array}
$$

satisfying the usual axioms. Note that if $\mathbb{K}$ is chordate, so that T is just a 2-monad on the 2-category $\mathscr{K}_\tau = \mathscr{K}_\lambda$, then the $\mathscr{F}$-category $T\text{-}\mathbb{Alg}_w$ defined above can be identified with the $\mathscr{F}$-category of the same name defined in Example 3.3. This is essential for applications to 2-category theory.

The universal property of $T\text{-}\mathbb{Alg}_w$ says that

$$\mathbb{Nat}_{\bar{w}}(\mathbb{X}, T\text{-}\mathbb{Alg}_w) \cong \mathbb{Nat}_{\bar{w}}(\mathbb{X}, T)\text{-}\mathbb{Alg}_w. \tag{5}$$

(Note the reversal of sense in the weak natural transformations, as in Section 2.2.) As with the 2-categorical version (2), we find it helpful to make (5) more explicit in a couple of cases.

Suppose that $w = l$ and that $\mathbb{X}$ is the inchordate 2. Then an object of $\mathbb{O}\mathrm{plax}(\mathbb{X}, T\text{-}\mathbb{A}\mathrm{lg}_l)$ is simply a loose and lax T-morphism $(f, \overline{f}) : (A, a) \rightsquigarrow (B, b)$, as above. On the other hand, an $\mathbb{O}\mathrm{plax}(\mathbb{X}, T)$-algebra consists of a loose morphism $f : A \rightsquigarrow B$ in $\mathscr{K}$ (that is, an object of $\mathbb{O}\mathrm{plax}(\mathbb{X}, \mathbb{K})$) together with an oplax natural transformation from Tf to f whose components are tight (since the structure maps of any algebra for an $\mathscr{F}$-monad are tight, and the tight morphisms in $\mathbb{O}\mathrm{plax}(\mathbb{X}, \mathbb{K})$ have tight components). This consists of tight morphisms $a : TA \to A$ and $b : TB \to B$ and a 2-cell $\overline{f} : b.Tf \to f.a$, and as before the algebra axioms assert precisely that (A, a) and (B, b) are T-algebras and $(f, \overline{f})$ is a lax T-morphism.

Now a loose morphism in $\mathbb{O}\mathrm{plax}(\mathbb{X}, T\text{-}\mathbb{A}\mathrm{lg}_l)$ from $(f, \overline{f})$ to $(g, \overline{g}) : (C, c) \rightsquigarrow (D, d)$ is a loose oplax transformation; thus it consists of loose and lax T-morphisms $(h, \overline{h}) : (A, a) \rightsquigarrow (C, c)$ and $(k, \overline{k}) : (B, b) \rightsquigarrow (D, d)$ (these being loose morphisms in $T\text{-}\mathbb{A}\mathrm{lg}_l$), together with a T-transformation $\alpha : (k, \overline{k})(f, \overline{f}) \to (g, \overline{g})(h, \overline{h})$. On the other hand, a loose and lax morphism of $\mathbb{O}\mathrm{plax}(\mathbb{X}, T)$-algebras consists of a loose oplax transformation from $f : A \to B$ to $g : C \to D$, hence loose morphisms $h : A \rightsquigarrow C$ and $k : B \rightsquigarrow D$ and a 2-cell $\alpha : kf \to gh$, together with a modification consisting of 2-cells $\overline{h} : c.Th \to h.a$ and $\overline{k} : d.Tk \to k.b$. As before, the axioms assert precisely that $(h, \overline{h})$ and $(k, \overline{k})$ are lax T-morphisms and α is a T-transformation.

Finally, a tight morphism in $\mathbb{O}\mathrm{plax}(\mathbb{X}, T\text{-}\mathbb{A}\mathrm{lg}_l)$ from $(f, \overline{f})$ to $(g, \overline{g})$ is a tight oplax transformation; thus it consists of tight and strict T-morphisms $h : (A, a) \to (C, c)$ and $k : (B, b) \to (D, d)$ and a T-transformation $\alpha : kf \to gh$. On the other side, a tight and strict morphism of $\mathbb{O}\mathrm{plax}(\mathbb{X}, T)$-algebras consists of a tight oplax transformation from f to g, hence tight morphisms $h : A \to C$ and $k : B \to D$ and a 2-cell $\alpha : kf \to gh$, such that $c.Th = h.a$ and $d.Tk = k.b$. The axioms say exactly that h and k are strict T-morphisms and α is a T-transformation.

If instead we take $\mathbb{X}$ to be the chordate 2, then in $\mathbb{O}\mathrm{plax}(\mathbb{X}, T\text{-}\mathbb{A}\mathrm{lg}_l)$ we require that f and g must be tight and strict T-morphisms. On the other side, we obtain the fact that f and g are tight from the fact that we have functors out of $\mathbb{X}$, while the fact that $\overline{f}$ and $\overline{g}$ are identities follows from the requirement that an oplax $\mathscr{F}$-transformation be strictly natural with respect to tight morphisms.

Note that in order to make (5) true, we need the "notions of tightness" in $T\text{-}\mathbb{A}\mathrm{lg}_w$ and in $\mathbb{N}\mathrm{at}_w(\mathbb{D}, \mathbb{K})$ to be different. Specifically, the tight morphisms in $T\text{-}\mathbb{A}\mathrm{lg}_w$ are strict, whereas the tight morphisms in $\mathbb{N}\mathrm{at}_w(\mathbb{D}, \mathbb{K})$ are strictly natural only with respect to tight maps. Unfortunately, this seems to mean that unlike the situation in 2-category theory, the weak $\mathscr{F}$-transformation category $\mathbb{N}\mathrm{at}_w(\mathbb{D}, \mathbb{K})$ is not of the form $T\text{-}\mathbb{A}\mathrm{lg}_w$ for any $\mathscr{F}$-monad T.

Finally, as in the 2-categorical case, any lax morphism of $\mathscr{F}$-monads induces a functor between $\mathscr{F}$-categories of algebras and weak morphisms in a straightforward way.

5. Rigged limits lift

Let $\Phi : \mathbb{D} \to \mathbb{F}$ be an $\mathscr{F}$-weight, let $\mathbb{K}$ be an $\mathscr{F}$-category, and let T be an $\mathscr{F}$-monad on $\mathbb{K}$. (The reader is encouraged to think of $\mathbb{K}$ as the chordate $\mathscr{F}$-category associated to a 2-category and of T as arising from a 2-monad, since this is the case of most interest.) Our goal in this section is to show that Φ-weighted limits are created by $U_w : T\text{-}\mathbb{A}\mathrm{lg}_w \to \mathbb{K}$ for any $\mathscr{F}$-monad T if and only if Φ is "rigged" (and we will define what this means). Recall that as remarked after Definition 3.7, since the $\mathscr{F}$-categorical U_w is conservative, it creates any limits that it lifts.

Once again, w could be any of p, l, or c; there will be a notion of "w-rigged" for each value of w. We will state the definitions and theorems of this section for general w, but in most of the

proofs we will describe only the case $w = l$ explicitly. The case $w = c$ is a formal dual; while the proofs may easily be adapted to the case $w = p$ by requiring the relevant 2-cells to be invertible.

5.1. Rigged weights

To make a start, we suppose that our weight $\Phi : \mathbb{D} \to \mathbb{F}$ is a $\mathcal{Q}_{\bar{w}}$-coalgebra, with structure map $s : \Phi \to \mathcal{Q}_{\bar{w}}\Phi$. Note the reversal of sense: we are considering liftings to $T\text{-}\mathbb{A}\mathrm{lg}_w$, but we assume Φ to be a $\mathcal{Q}_{\bar{w}}$-coalgebra.

At the moment, this hypothesis may seem somewhat unmotivated, but at least in the case $w = p$ it is a strengthening of *flexibility*, while the notion of *PIE-limit* (the limits known to lift when $w = p$) is also a strengthening of flexibility. In Section 6.4 we will show that in fact, PIE-limits are precisely the 2-categorical Q-coalgebras. Moreover, one of the results of [17] is that *oplax limits* lift to $T\text{-}\mathrm{Alg}_l$, and the weights for oplax limits are exactly the *cofree Q_c-coalgebras*.

In Section 5.3 we will show that under the hypothesis that Φ is a $\mathcal{Q}_{\bar{w}}$-coalgebra, we can construct, for each $G : \mathbb{D} \to T\text{-}\mathbb{A}\mathrm{lg}_w$, a T-algebra L with the universal property of the limit $\{\Phi_\lambda, G_\lambda\}$ in $(T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$, lifting the limit $\{\Phi_\lambda, UG_\lambda\}$ in $\mathcal{K}_\lambda$. This hypothesis is not, however, enough to get the full universal property of the limit $\{\Phi, G\}$. Recall from Section 3.4 that in addition to being a limit in the 2-category of loose morphisms, an $\mathcal{F}$-limit must "detect tightness".

Example 5.1. Consider powers (Section 3.5.7) by an object $0 \to C$ of $\mathcal{F}$, with C a non-empty category. Let T be a 2-monad on a 2-category $\mathcal{K}$, seen as a chordate $\mathcal{F}$-category, and let (B, b) be a T-algebra. A power of $B \in \mathbb{K}$ by $0 \to C$ consists of a power $C \pitchfork B$ in $\mathcal{K}$; the tight part of the universal property is automatic, since all morphisms in $\mathcal{K}$ are tight. This power will lift to a power of (B, b) in $T\text{-}\mathbb{A}\mathrm{lg}_w$ if (i) the power $C \pitchfork B$ lifts to a power $C \pitchfork (B, b)$ in $(T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$, and (ii) all morphisms into $C \pitchfork (B, b)$ are tight (that is, strict). Now (i) will always hold, but (ii) generally will not, even in the case where C is the terminal category $\mathbb{1}$.

To ensure the tight aspect of the universal property, we must impose an additional condition on Φ. Recall that we write $J : \mathscr{D}_\tau \to \mathscr{D}_\lambda$ for the 2-functor that underlies an $\mathcal{F}$-category $\mathbb{D}$, and $\varphi : \Phi_\tau \to \Phi_\lambda J$ for the structure map of an $\mathcal{F}$-weight Φ.

Definition 5.2. We say that an $\mathcal{F}$-weight $\Phi : \mathbb{D} \to \mathbb{F}$ is w-**rigged** if

1. Φ is a $\mathcal{Q}_{\bar{w}}$-coalgebra, and
2. The induced morphism $\bar{\varphi} : \mathrm{Lan}_J \Phi_\tau \to \Phi_\lambda$ is pointwise surjective on objects.

The extra condition may seem somewhat odd; its importance is due to the following alternative characterization.

Lemma 5.3. *The following are equivalent for a 2-natural transformation $f : \Phi \to \Psi$ between 2-functors $\Phi, \Psi : \mathscr{D} \to \mathbf{Cat}$.*

(i) *f is pointwise surjective on objects.*
(ii) *Precomposition with f reflects identities, i.e. for any $g, h : \Psi \to \Upsilon$ and modification $\beta : g \to h$, if βf is an identity then so is β.*
(iii) *As in* (ii)*, but only when β is known to be an isomorphism.*

Proof. Since $(\beta f)_{d,x} = \beta_{d,f(x)}$ for $d \in \mathscr{D}$ and $x \in \Phi(d)$, if f is pointwise surjective on objects and $(\beta f)_{d,x}$ is an identity for all d and x, then $\beta_{d,y}$ is an identity for all $d \in \mathscr{D}$ and $y \in \Psi(d)$. Thus (i) implies (ii), and clearly (ii) implies (iii), so suppose (iii). Pick any $d_0 \in \mathscr{D}$ and $y_0 \in \Psi(d)$, and let $\Upsilon = \mathrm{Ran}_{d_0} C$ be the co-free diagram at $d_0 \in \mathscr{D}$ on the chaotic category C with two objects 0 and 1; thus a 2-natural transformation $\Psi \to \Upsilon$ is determined by a functor $\Psi(d_0) \to C$. Let $g : \Psi \to \Upsilon$ be determined by the functor $\Psi(d_0) \to C$ constant at 0, and let h be determined by the functor $\Psi(d_0) \to C$ sending y_0 to 1 and everything else to 0. Then there is an invertible modification $\beta : g \overset{\cong}{\to} h$ such that $\beta_{d,y}$ is an identity whenever $d \neq d_0$ or $y \neq y_0$, but β_{d_0,y_0} is not an identity. Thus βf cannot be an identity either, and so there must be some $x \in \Phi(d_0)$ with $f(x) = y_0$. Hence f is pointwise surjective on objects, and so (iii) implies (i). $\square$

Remark 5.4. Our $\mathscr{F}$-categories depend heavily on the class of full embeddings (functors which are injective on objects and fully faithful). These are the right class of a factorization system on **Cat** for which the left class consists of the functors which are surjective on objects. But we have also referred in passing to the more general $\mathscr{F}'$-categories, which involve merely fully faithful functors, and we have promised that all our results extend to the setting of $\mathscr{F}'$-categories. Since fully faithful functors form the right class of a factorization system on **Cat** for which the left class consists of the functors which are bijective on objects, one might guess that the notion of rigging for $\mathscr{F}'$-categories would involve $\varphi : \Phi_\tau \to \Phi_\lambda J$ which is pointwise bijective on objects. This is not the case: we still use surjectivity on objects, and Lemma 5.3 explains why.

On the other hand, we would need to modify surjectivity on objects to obtain a notion of rigging appropriate for $\mathscr{F}$-categories of algebras which combine pseudo and lax morphisms, instead of strict and lax ones, as suggested in the introduction.

The relationship between the two conditions defining a rigged weight is further clarified by the following observation.

Recall that $\mathcal{Q}_\lambda = Q^\mathbb{D}$ is the 2-categorical relative $\bar{w}$-morphism classifier on $[\mathscr{D}_\lambda, \mathbf{Cat}]$, and hence is $\bar{w}$-idempotent. In particular, for $\mathcal{Q}$-coalgebras $\Phi, \Psi : \mathbb{D} \to \mathbb{F}$, any loose map $\Phi \rightsquigarrow \Psi$ in $[\mathbb{D}, \mathbb{F}]$, being a 2-natural transformation $\Phi_\lambda \to \Psi_\lambda$, is automatically a $\bar{w}$-$\mathcal{Q}_\lambda$-morphism. Moreover, any morphism between such loose maps is automatically a $\mathcal{Q}_\lambda$-transformation. The next lemma says that if Φ and Ψ are w-rigged, we also have a corresponding property for tight morphisms.

Lemma 5.5. *If Φ and Ψ are $\mathcal{Q}_{\bar{w}}$-coalgebras, and Φ is w-rigged, then any (strict) $\mathscr{F}$-natural transformation $\Phi \to \Psi$ is automatically a strict $\mathcal{Q}_{\bar{w}}$-morphism.*

Proof. As usual, we write in the case $w = l$. Let $\mathcal{Q} = \mathcal{Q}_c$, let Φ, Ψ have structure maps $s_\Phi : \Phi \to \mathcal{Q}\Phi$ and $s_\Psi : \Psi \to \mathcal{Q}\Psi$, and let $f : \Phi \to \Psi$ be $\mathscr{F}$-natural. Suppose that Φ is w-rigged. We must show that $(\mathcal{Q}f)_\tau (s_\Phi)_\tau = (s_\Psi)_\tau f_\tau$ and $(\mathcal{Q}f)_\lambda (s_\Phi)_\lambda = (s_\Psi)_\lambda f_\lambda$. However, $(s_\Phi)_\tau$ and $(s_\Psi)_\tau$ are the identity, since $\mathcal{Q}_\tau$ is the identity, so the first of these is trivial. Moreover, since $\mathcal{Q}_\lambda$ is colax-idempotent, we have a unique colax $\mathcal{Q}_\lambda$-morphism structure map $\overline{f} : (\mathcal{Q}f)_\lambda.(s_\Phi)_\lambda \to (s_\Psi)_\lambda.f_\lambda$, so it suffices to show that $\overline{f}$ is an identity.

Now by definition, $\overline{f}$ is the composite

$$(\mathcal{Q}_\lambda f_\lambda).s_\Phi \xrightarrow{\eta.(\mathcal{Q}_\lambda f_\lambda).s_\Phi} s_\Psi.q_\Psi.(\mathcal{Q}f)_\lambda.s_\Phi = s_\Psi.f_\lambda.q_\Phi.s_\Phi = s_\Psi f_\lambda$$

where η is the unit of the adjunction $q_\Phi \dashv s_\Phi$. Now let us apply $J_\mathbb{D}$ to this composite and pre-compose with $\varphi : \Phi_\tau \to \Phi_\lambda J$. Since $J(\mathcal{Q}_\lambda f_\lambda).J(s_\Phi).\varphi = J(s_\Psi).\psi.f_\tau$ (using the fact that s_Φ is strictly $\mathscr{F}$-natural), we obtain

$$J(s_\Psi).\psi.f_\tau \xrightarrow{J(\eta.s_\Psi).\psi.f_\tau} J(s_\Psi.q_\Psi.s_\Psi).\psi.f_\tau.$$

But the counit of the adjunction $q_\Phi \dashv s_\Phi$ is an identity, hence so also is $\eta.s_\Phi$. Thus, $J(\overline{f}).\varphi$ is an identity, which equivalently means that

$$\mathrm{Lan}_J \, \Phi_\tau \xrightarrow{\ \overline{\varphi}\ } \Phi_\lambda \overset{\Downarrow \overline{f}}{\underset{}{\rightrightarrows}} \mathcal{Q}_\lambda \Psi_\lambda$$

is an identity. Since $\overline{\varphi}$ is pointwise surjective on objects, by Lemma 5.3 this implies that $\overline{f}$ is an identity, as desired. $\square$

Thus, the forgetful functor $\mathcal{Q}_{\bar{w}}\text{-}\mathbb{Coalg}_{\bar{w}} \to [\mathbb{D}, \mathbb{F}]$, when restricted to w-rigged weights, is fully faithful in the $\mathscr{F}$-enriched sense. In particular, if an $\mathscr{F}$-weight Φ admits two $\mathcal{Q}$-coalgebra structures of which one (hence also the other) is rigged, then the identity $\Phi \to \Phi$ is a strict $\mathcal{Q}$-coalgebra map between them, and hence the two structures coincide. Thus, "being w-rigged" (unlike "being a $\mathcal{Q}$-coalgebra") is a mere property of an $\mathscr{F}$-weight, not structure on it. (However, see also Section 6.2.)

5.2. Reduction to special $\mathbb{K}$

We start with the following simplification, which will be useful at various stages in the proofs.

Proposition 5.6. *For a weight $\Phi : \mathbb{D} \to \mathbb{F}$ and a given choice of w, the following conditions are equivalent:*

(i) *Φ-weighted limits are created by $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ for all $\mathscr{F}$-categories $\mathbb{K}$ and all $\mathscr{F}$-monads T on $\mathbb{K}$;*

(ii) *Φ-weighted limits are created by $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ for all complete $\mathscr{F}$-categories $\mathbb{K}$ and all $\mathscr{F}$-monads T on $\mathbb{K}$;*

(iii) *Φ-weighted limits are created by $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ for all presheaf $\mathscr{F}$-categories $\mathbb{K} = [\mathbb{C}, \mathbb{F}]$ and all $\mathscr{F}$-monads T on $\mathbb{K}$;*

(iv) *Φ-weighted limits are created by $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ for all small $\mathscr{F}$-categories $\mathbb{K}$ and all $\mathscr{F}$-monads T on $\mathbb{K}$;*

(v) *Φ-weighted limits are created by $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ for all chordate $\mathscr{F}$-categories $\mathbb{K}$ and all $\mathscr{F}$-monads T on $\mathbb{K}$.*

Proof. Clearly (i) implies all the other conditions, and (ii) implies (iii). We shall show that (iii) implies (iv), that (iv) implies (i), and that (v) implies (ii).

Suppose (iii) and let T be an $\mathscr{F}$-monad on a small $\mathscr{F}$-category $\mathbb{K}$, and let $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_w$ be an $\mathscr{F}$-functor for which the limit $\{\Phi, UG\}$ exists. Since $\mathbb{K}$ is small, we may form the presheaf $\mathscr{F}$-category $\widehat{\mathbb{K}} = [\mathbb{K}^{\mathrm{op}}, \mathbb{F}]$ and the monad $\widehat{T}$ on $\widehat{\mathbb{K}}$ induced by left Kan extension along T. There

is an induced fully faithful $T\text{-}\mathbb{A}\mathrm{lg}_w \to \widehat{T}\text{-}\mathbb{A}\mathrm{lg}_w$ lifting the Yoneda embedding $\mathbb{K} \to \widehat{\mathbb{K}}$. A $\widehat{T}$-algebra is in the image of $T\text{-}\mathbb{A}\mathrm{lg}_w$ if and only if the underlying object in $\widehat{\mathbb{K}}$ is in the image of the Yoneda embedding. Now the limit $\{\Phi, UG\}$ in $\mathbb{K}$ is preserved by Yoneda, and since $\widehat{\mathbb{K}}$ is a presheaf category this limit lifts to $\widehat{T}\text{-}\mathbb{A}\mathrm{lg}_w$; but now this limit also lies in $T\text{-}\mathbb{A}\mathrm{lg}_w$ and so is a limit there. Thus (iii) implies (iv).

Suppose (iv) and let T be an $\mathscr{F}$-monad on an arbitrary $\mathscr{F}$-category $\mathbb{K}$, and let $G : \mathbb{D} \to T\text{-}\mathbb{A}\mathrm{lg}_w$ be an $\mathscr{F}$-functor for which the limit $\{\Phi, UG\}$ exists. First choose a small full subcategory $\mathbb{B}$ of $\mathbb{K}$ which is closed under the action of T and contains $\{\Phi, UG\}$ and the image of UG. Then the limit $\{\Phi, UG\}$ lifts to $S\text{-}\mathbb{A}\mathrm{lg}_w$, where S is the restriction of T to $\mathbb{B}$. Our lifted limit has the correct universal property in $S\text{-}\mathbb{A}\mathrm{lg}_w$, but we still need to check the universal property in the larger $\mathscr{F}$-category $T\text{-}\mathbb{A}\mathrm{lg}_w$. But this can be done one object at a time: for each object $C \in \mathbb{K}$, we may enlarge $\mathbb{B}$ to a small full subcategory $\mathbb{C}$ of $\mathbb{K}$ having the same properties as before, but also containing C. Now our lifted limit also has the correct universal property in $R\text{-}\mathbb{A}\mathrm{lg}_w$, where R is the restriction of T to $\mathbb{C}$, and since we can do this for any object C, it has the correct universal property in all of $T\text{-}\mathbb{A}\mathrm{lg}_w$. Thus (iv) implies (i).

Suppose (v) and let T be an $\mathscr{F}$-monad on a complete $\mathscr{F}$-category $\mathbb{K}$, and let $G : \mathbb{D} \to T\text{-}\mathbb{A}\mathrm{lg}_w$ be an $\mathscr{F}$-functor. The $\mathscr{F}$-monad T on $\mathbb{K}$ induces a 2-monad T_λ on $\mathscr{K}_\lambda$. As usual, we regard $\mathscr{K}_\lambda$ as a chordate $\mathscr{F}$-category. The canonical $\mathscr{F}$-functor $\mathbb{K} \to \mathscr{K}_\lambda$ lifts to an $\mathscr{F}$-functor $P : T\text{-}\mathbb{A}\mathrm{lg}_w \to T_\lambda\text{-}\mathbb{A}\mathrm{lg}_w$ whose loose part P_λ is 2-fully faithful; a T_λ-algebra (A, a) lies in the image of P_λ if and only if $a : TA \to A$ is tight.

By (v), the limit $L = \{\Phi, UPG\}$ in $\mathscr{K}_\lambda$ lifts to a limit $(L, \ell) = \{\Phi, PG\}$ in $T_\lambda\text{-}\mathbb{A}\mathrm{lg}_w$; this will be a T-algebra if and only if ℓ is tight. Furthermore, since (L, ℓ) is in particular a limit $\{\Phi_\lambda, P_\lambda G_\lambda\}$ in $(T_\lambda\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$, and P_λ is 2-fully faithful and so reflects limits, if (L, ℓ) is a T-algebra then it will be a limit $\{\Phi_\lambda, G_\lambda\}$ in $(T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$.

Now the projections $p_{\lambda, \varphi(a)} : L \to UG_\lambda D$, for $D \in \mathbb{D}$ and $a \in \Phi_\tau D$, are tight in $\mathbb{K}$ and jointly detect tightness. They are also strict T_λ-morphisms, and so the square

$$
\begin{array}{ccc}
TL & \xrightarrow{\;Tp_{\lambda,\varphi(a)}\;} & TUG_\lambda D \\
{\scriptstyle \ell}\big\downarrow & & \big\downarrow \\
L & \xrightarrow[\;p_{\lambda,\varphi(a)}\;]{} & UG_\lambda D
\end{array}
$$

in which the right leg is the structure map of $G_\lambda D$, is commutative, and ℓ will be tight if and only if the common composites $TL \to UG_\lambda D$ are all tight. But the right leg is tight since each GD is a T-algebra, and the top leg is tight since $p_{\lambda, \varphi(a)}$ is a tight projection. Thus ℓ is tight, and (L, ℓ) is the limit $\{\Phi_\lambda, G_\lambda\}$ in $(T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$.

Moreover, since the above projections $p_{\lambda, \varphi(a)}$ are strict T_λ-morphisms, they are also strict T-morphisms, and since they are tight, they are tight morphisms in $T\text{-}\mathbb{A}\mathrm{lg}_w$. Thus, it remains to show that they jointly detect tightness in $T\text{-}\mathbb{A}\mathrm{lg}_w$.

Let (A, a) be a T-algebra, and $(f, \overline{f}) : (A, a) \to (L, \ell)$ a loose morphism in $T\text{-}\mathbb{A}\mathrm{lg}_w$. Suppose that the composite

$$
(A, a) \xrightarrow{\;(f, \overline{f})\;} (L, \ell) \xrightarrow{\;p_{\lambda, \varphi(a)}\;} GD
$$

is tight for each $D \in \mathbb{D}$ and each $a \in \Phi_\tau D$. The projections $p_{\lambda,\varphi(a)}$ jointly detect tightness of T_λ-morphisms (that is, they jointly detect strictness) and so $(f, \bar{f})$ is strict. On the other hand, the projections $p_{\lambda,\varphi(a)} : L \to UGD$ jointly detect tightness of morphisms in $\mathbb{K}$, and so f is tight. Thus $(f, \bar{f})$ is tight in $T\text{-}\mathbb{Alg}_w$, and so (L, ℓ) has the full universal property of $\{\Phi, G\}$. $\quad\square$

Our main interest is in the weights which satisfy (i), but it is also useful to have (v), which says that restricting our attention to 2-monads on 2-categories does not affect the resulting class of $\mathscr{F}$-limits. In other words, the introduction of $\mathscr{F}$-categories has not "changed the problem" from the original 2-categorical question.

The other conditions are more technical. In particular, it will be convenient in our analysis of the lifting of limits to suppose that $\mathbb{K}$ is complete, and by this last result there is no loss of generality in doing so. In fact we could have given still more equivalent conditions; for example that $\mathbb{K}$ was complete and chordate, or small and chordate.

5.3. Lifting of limits

We now embark on the actual proof that rigged limits lift. Suppose that $w = l$ and that $\mathbb{K}$ is complete, and consider a diagram $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_l$ with the limit $L = \{\Phi, UG\} \in \mathbb{K}$. By the isomorphism

$$\mathbb{O}\text{plax}(\mathbb{D}, T\text{-}\mathbb{Alg}_l) \cong \mathbb{O}\text{plax}(\mathbb{D}, T)\text{-}\mathbb{Alg}_l \tag{6}$$

we have a tight oplax $\mathscr{F}$-natural transformation $g : TUG \to UG$, which makes UG into an $\mathbb{O}\text{plax}(\mathbb{D}, T)$-algebra in $\mathbb{O}\text{plax}(\mathbb{D}, \mathbb{K})$. Now consider the upper path around the following diagram

$$\begin{array}{ccc}
\Phi & \xrightarrow{\;\;\eta\;\;} \mathbb{K}(L, UG-) \xrightarrow{\;\;T\;\;} & \mathbb{K}(TL, TUG-) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathbb{K}(TL, g-)} \\
\mathcal{Q}\Phi & \xrightarrow{\qquad\qquad\qquad\zeta\qquad\qquad\qquad} & \mathbb{K}(TL, UG-)
\end{array} \tag{7}$$

in which the first two (horizontal) morphisms are tight and strictly $\mathscr{F}$-natural, while the third (vertical) is tight oplax natural. Therefore, by the universal property of $\mathcal{Q} = \mathcal{Q}_c$, there is a unique tight and strict transformation $\zeta : \mathcal{Q}\Phi \to \mathbb{K}(TL, UG-)$ making the diagram commute. Finally, by the universal property of the limit L, there is a unique tight $\ell : TL \to L$ making the following diagram commute:

$$\begin{array}{ccc}
\Phi & \xrightarrow{\;\;\eta\;\;} & \mathbb{K}(L, UG-) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathbb{K}(\ell, UG-)} \\
\mathcal{Q}\Phi & \xrightarrow{\;\;\zeta\;\;} & \mathbb{K}(TL, UG-)
\end{array} \tag{8}$$

One could now prove directly that this map $\ell : TL \to L$ makes L into a T-algebra, but the calculations are lengthy and not particularly enlightening. Thus, we will instead give a more formal approach involving monad morphisms.

The overall strategy is this. Rather than construct the limit $\{\Phi, G\}$ separately for each $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_l$, we do this functorially. More precisely, we construct the $\mathscr{F}$-functor $\mathbb{Oplax}(\mathbb{D}, T\text{-}\mathbb{Alg}_l) \to T\text{-}\mathbb{Alg}_l$ which (in a later theorem) will turn out to send G to the limit $\{\Phi, G\}$. We use $\mathbb{Oplax}(\mathbb{D}, T\text{-}\mathbb{Alg}_l)$ rather than the simpler $[\mathbb{D}, T\text{-}\mathbb{Alg}_l]$ in order to take advantage of the isomorphism (6). This reduces the problem to constructing an $\mathscr{F}$-functor $\mathbb{Oplax}(\mathbb{D}, T)\text{-}\mathbb{Alg}_l \to T\text{-}\mathbb{Alg}_l$, which in turn can be done by constructing a lax monad morphism from $\mathbb{Oplax}(\mathbb{D}, T)$ to T, since monad morphisms induce liftings not just to their Eilenberg–Moore objects but also to variants using weak morphisms.

Remark 5.7. It turns out that if we make this construction sufficiently functorial in the weight Φ as well, then it is possible to deduce the universal properties of these limits $\{\Phi, G\}$ from their functoriality. In an appendix to the paper, we describe the resulting proof, which treats weighted limits using profunctors. In many ways this gives a fuller picture of the situation, but it is also somewhat longer, so we have chosen here an intermediate approach, which constructs algebra structure using the monad-theoretic ideas of the previous paragraph, but then proves the universal property by showing that the relevant hom-objects can be constructed as certain descent objects.

Proposition 5.8. *If $\mathbb{K}$ is complete and Φ is a $Q_{\bar{w}}$-coalgebra, the $\mathscr{F}$-functor $F = \{\Phi, -\} :$ $[\mathbb{D}, \mathbb{K}] \to \mathbb{K}$ extends to an $\mathscr{F}$-functor $F' : \mathbb{Nat}_{\bar{w}}(\mathbb{D}, \mathbb{K}) \to \mathbb{K}$ which lifts to an $\mathscr{F}$-functor $F'' : \mathbb{Nat}_{\bar{w}}(\mathbb{D}, T\text{-}\mathbb{Alg}_w) \to T\text{-}\mathbb{Alg}_w$.*

Proof. The proof involves three monads and various relationships between them. The first two monads are T itself and the induced $[\mathbb{D}, T]$ on $[\mathbb{D}, \mathbb{K}]$. The $\mathscr{F}$-functor $F : [\mathbb{D}, \mathbb{K}] \to \mathbb{K}$ can be given the structure of a monad morphism from $[\mathbb{D}, T]$ to T. Explicitly, the 2-cell $\psi : TF \to F[\mathbb{D}, T]$ which makes F a monad morphism has component at $G : \mathbb{D} \to \mathbb{K}$ given by the canonical comparison map $T\{\Phi, G\} \to \{\Phi, TG\}$. Alternatively, one can construct the lifting as the composite

$$[\mathbb{D}, T]\text{-}\mathbb{Alg}_s \xrightarrow{\;\cong\;} [\mathbb{D}, T\text{-}\mathbb{Alg}_s] \longrightarrow T\text{-}\mathbb{Alg}_s$$

where the first map is the canonical isomorphism, and the second comes from the fact that limits lift to Eilenberg–Moore objects; then as usual this lifting of F determines a monad morphism structure on F.

Next we need to introduce weakness into the situation. This is done via the weak morphism coclassifier $\mathcal{R} = \mathcal{R}_{\bar{w}}$, seen as a monad on $[\mathbb{D}, \mathbb{K}]$. Recall that $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, \mathbb{K})$ is the Kleisli object for $\mathcal{R}$; in particular weak maps $G \to H$ correspond to strict maps $G \to \mathcal{R}H$. Since the monad $[\mathbb{D}, T]$ on $[\mathbb{D}, \mathbb{K}]$ extends to a monad $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, T)$ on the Kleisli object $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, \mathbb{K})$ of $\mathcal{R}$, there is an induced distributive law $k : [\mathbb{D}, T]\mathcal{R} \to \mathcal{R}[\mathbb{D}, T]$.

The last ingredient is the relationship between F and $\mathcal{R}$. This is where we use the assumption that Φ is a $\mathcal{Q}_{\bar{w}}$-coalgebra. The coaction $s : \Phi \to \mathcal{Q}_{\bar{w}}\Phi$ induces an opaction

$$f : F\mathcal{R} = \{\Phi, \mathcal{R}-\} \cong \{\mathcal{Q}\Phi, -\} \to \{\Phi, -\} = F$$

of $\mathcal{R}$ on F, and so an extension $F' : \mathbb{Nat}_{\bar{w}}(\mathbb{D}, \mathbb{K}) \to \mathbb{K}$ of F to the Kleisli object of $\mathcal{R}$.

Next we put these pieces together to make F' into a monad morphism from $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, T)$ to T. Since $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, T)$ is an extension of T to the Kleisli category of $\mathcal{R}$, and T is an extension of itself to the Kleisli category of the identity monad, by Corollary 2.2 it will suffice to show that the diagram

$$
\begin{array}{ccc}
TF\mathcal{R} \xrightarrow{\ \psi\ } F[\mathbb{D}, T]\mathcal{R} \xrightarrow{\ Fk\ } F\mathcal{R}[\mathbb{D}, T] \\
\ \ \downarrow{\scriptstyle Tf} \hspace{6.5cm} \downarrow{\scriptstyle f[\mathbb{D},T]} \\
TF \xrightarrow{\hspace{4.5cm}\psi\hspace{1.5cm}} F[\mathbb{D}, T]
\end{array}
$$

commutes. This means that for each $G : \mathbb{D} \to \mathbb{K}$ the corresponding diagram

$$
\begin{array}{ccc}
T\{\Phi, \mathcal{R}G\} \longrightarrow \{\Phi, T\mathcal{R}G\} \longrightarrow \{\Phi, \mathcal{R}TG\} \\
\downarrow \hspace{6.5cm} \downarrow \\
T\{\Phi, G\} \xrightarrow{\hspace{5.5cm}} \{\Phi, TG\}
\end{array}
$$

commutes; but this reduces, using the isomorphism $\{\Phi, \mathcal{R}-\} \cong \{\mathcal{Q}\Phi, -\}$ to commutativity of the diagram

$$
\begin{array}{ccc}
T\{\mathcal{Q}\Phi, G\} \longrightarrow \{\mathcal{Q}\Phi, TG\} \\
\ \ \downarrow{\scriptstyle T\{s,G\}} \hspace{4cm} \downarrow{\scriptstyle \{s,TG\}} \\
T\{\Phi, G\} \longrightarrow \{\Phi, TG\}
\end{array}
$$

expressing the naturality of the canonical comparisons (appearing as the horizontal maps) with respect to s.

Finally, we can take F'' to be the composite

$$
\mathbb{Nat}_{\bar{w}}(\mathbb{D}, T\text{-}\mathbb{Alg}_w) \xrightarrow{\ \cong\ } \mathbb{Nat}_{\bar{w}}(\mathbb{D}, T)\text{-}\mathbb{Alg}_w \longrightarrow T\text{-}\mathbb{Alg}_w
$$

in which the second map is the lifting of the above monad morphism F' from $\mathbb{Nat}_{\bar{w}}(\mathbb{D}, T)$ to T.

Since F'' is given by lifting F', for any diagram $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_w$ the induced T-algebra will have underlying object L calculated by composing with U to get $UG : \mathbb{D} \to \mathbb{K}$ and taking the limit $\{\Phi, UG\}$. The algebra structure $\ell : TL \to L$ is given by the composite

$$
T\{\Phi, UG\} \longrightarrow \{\Phi, TUG\} \longrightarrow \{\Phi, \mathcal{R}UG\} \xrightarrow{\ \cong\ } \{\mathcal{Q}\Phi, UG\} \longrightarrow \{\Phi, UG\} \tag{9}
$$

where the first map is the canonical comparison, the second comes from the weakly natural actions $TGD \to GD$, the third is the canonical isomorphism, and the last is induced by composition with $s : \Phi \to \mathcal{Q}\Phi$. This agrees with the earlier description of ℓ.   □

The proposition shows that we have an algebra structure on $L = \{\Phi, UG\}$, and a degree of functoriality, but it does not give any sort of universal property. We now turn to this, starting with the universal property with respect to loose maps.

Theorem 5.9. *Let $\Phi : \mathbb{D} \to \mathbb{F}$ be an $\mathscr{F}$-weight which is a $\mathcal{Q}_{\bar{w}}$-coalgebra, let T be an $\mathscr{F}$-monad on a complete $\mathscr{F}$-category $\mathbb{K}$, and let $G : \mathbb{D} \to T\text{-}\mathrm{Alg}_w$ be an $\mathscr{F}$-functor. Then the T-algebra structure on $\{\Phi, UG\}$ constructed in Proposition 5.8 gives it the universal property of the limit $\{\Phi_\lambda, G_\lambda\}$ in the 2-category $(T\text{-}\mathbb{Alg}_w)_\lambda$.*

Proof. Once again, we write the proof only for the case $w = l$, with $\mathcal{Q} = \mathcal{Q}_c$, and we continue writing L for the limit $\{\Phi, UG\}$ and $\ell : TL \to L$ for its induced algebra structure.

Write M for the 2-monad on $[\mathscr{D}_\tau, \mathbf{Cat}]$ whose Eilenberg–Moore 2-category is $[\mathscr{D}_\lambda, \mathbf{Cat}]$ with forgetful 2-functor $[J_\mathbb{D}, \mathbf{Cat}] : [\mathscr{D}_\lambda, \mathbf{Cat}] \to [\mathscr{D}_\tau, \mathbf{Cat}]$. Thus an M-algebra is a 2-functor $\mathscr{D}_\lambda \to \mathbf{Cat}$, and a colax M-morphism is an oplax natural transformation whose restriction along $J_\mathbb{D}$ is strict. (This is the 2-monad M for which $\mathcal{Q}_\lambda = Q^\mathbb{D}$ is the colax morphism classifier.) We have the 2-category $M\text{-}\mathrm{Alg}_c$ of M-algebras and colax morphisms.

Let $\mathbf{A} = (A, a)$ and $\mathbf{B} = (B, b)$ be T_λ-algebras. Then we have a c-descent object

$$T_\lambda\text{-}\mathrm{Alg}_l(\mathbf{A}, \mathbf{B}) \dashrightarrow \mathscr{K}_\lambda(A, B) \underset{\longrightarrow}{\overset{\rightsquigarrow}{\longleftarrow}} \mathscr{K}_\lambda(TA, B) \overset{\rightsquigarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathscr{K}_\lambda(T^2A, B)$$

in $\mathbf{Cat}$, in which the straight arrows are all given by precomposition by some arrow in $\mathscr{K}_\lambda$, while the wriggly arrows are given by applying T and then composing with b. In particular, for each $D \in \mathscr{D}_\lambda$ we have a c-descent object

$$T_\lambda\text{-}\mathrm{Alg}_l(\mathbf{A}, G_\lambda D) \longrightarrow \mathscr{K}_\lambda(A, UG_\lambda D) \underset{\longrightarrow}{\overset{\rightsquigarrow}{\longleftarrow}} \mathscr{K}_\lambda(TA, UG_\lambda D) \overset{\rightsquigarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathscr{K}_\lambda(T^2A, UG_\lambda D).$$

Now the straight arrows are strictly natural in D with respect to all arrows of $\mathscr{D}_\lambda$, while the wriggly ones are strictly natural with respect to tight maps and oplax natural with respect to loose ones (since G takes tight maps to strict T_λ-morphisms and loose maps to lax T_λ-morphisms). In other words, the straight arrows are strict morphisms of M-algebras, while the wriggly arrows are colax morphisms of M-algebras.

Now as remarked in Section 3.5.5, c-rigged c-descent objects are created, and in particular reflected, by the forgetful $\mathscr{F}$-functor $M\text{-}\mathbb{Alg}_c \to [\mathscr{D}_\tau, \mathbf{Cat}]$ (viewing $[\mathscr{D}_\tau, \mathbf{Cat}]$ as a chordate $\mathscr{F}$-category). Thus we have a c-rigged c-descent object

$$T_\lambda\text{-}\mathrm{Alg}_l(\mathbf{A}, G_\lambda -) \dashrightarrow \mathscr{K}_\lambda(A, UG_\lambda -) \underset{\longrightarrow}{\overset{\rightsquigarrow}{\longleftarrow}} \mathscr{K}_\lambda(TA, UG_\lambda -) \overset{\rightsquigarrow}{\underset{\longrightarrow}{\longrightarrow}} \mathscr{K}_\lambda(T^2A, UG_\lambda -)$$

(10)

in $M\text{-}\mathrm{Alg}_c$, which is to say a c-descent object in $M\text{-}\mathrm{Alg}_c$ for which the projection $T_\lambda\text{-}\mathrm{Alg}_l(\mathbf{A}, G_\lambda -) \to \mathscr{K}_\lambda(A, UG_\lambda -)$ is strict and detects strictness.

Step 1: $\mathcal{Q}\Phi$**-limits lift.** Here is a rough sketch of this step. By Lemma 4.1, the limit $\{\mathcal{Q}\Phi, UG\}$ is also the limit $\{\Phi, \mathcal{R}(UG)\}$. But since $\mathcal{R}$ takes colax transformations to strict ones, $\mathcal{R}(UG)$ lifts to a diagram "$\mathcal{R}(G)$" of *strict* T-morphisms, and so the limit $\{\Phi, \mathcal{R}(UG)\}$ lifts to $T\text{-}\mathrm{Alg}_l$.

We then show that this limit also has the universal property of $\{\mathcal{Q}\Phi_\lambda, G_\lambda\}$ by considering Fig. 1. We are to prove that the two objects at the top are isomorphic. The strategy for this is

Fig. 1. Two descent objects.

to show that the vertical columns exhibit the objects at the top as descent objects, and therefore deduce the invertibility of the top horizontal map from the invertibility of the other horizontal maps.

We now turn to the details, including an explanation of the arrows in Fig. 1. Write $L' = \{\mathcal{Q}\Phi, UG\}$, and $\ell' : TL' \to L'$ for the corresponding structure map constructed as in Proposition 5.8. Since $\mathbb{K}$ is complete, Lemma 4.1 implies that L' is also the limit $\{\Phi, \mathcal{R}(UG)\}$ in $\mathbb{K}$, and hence also the 2-categorical limit $\{\Phi_\lambda, \mathcal{R}(UG)_\lambda\}$ in $\mathcal{K}_\lambda$.

Now recall from Proposition 5.8 that we have a distributive law $k : [\mathbb{D}, T]\mathcal{R} \to \mathcal{R}[\mathbb{D}, T]$, according to which T extends to a monad $\mathbb{O}\mathrm{plax}(\mathbb{D}, T)$ on the Kleisli category $\mathbb{O}\mathrm{plax}(\mathbb{D}, \mathbb{K})$ of $\mathcal{R}$. It also follows that the right adjoint $\mathcal{R} : \mathbb{O}\mathrm{plax}(\mathbb{D}, \mathbb{K}) \to [\mathbb{D}, \mathbb{K}]$ is a lax monad morphism from $\mathbb{O}\mathrm{plax}(\mathbb{D}, T)$ to $[\mathbb{D}, T]$. Therefore, since UG is a $\mathbb{O}\mathrm{plax}(\mathbb{D}, T)$-algebra, $\mathcal{R}(UG)$ is a $[\mathbb{D}, T]$-algebra, with structure map

$$[\mathbb{D}, T]\mathcal{R}(UG) \xrightarrow{k} \mathcal{R}(TUG) \xrightarrow{\mathcal{R}(g)} \mathcal{R}(UG)$$

where $g : TUG_\lambda \to UG_\lambda$ has components given by the structure maps of the T-algebras GD. Since $[\mathbb{D}, T]\text{-}\mathbb{A}\mathrm{lg}_s \cong [\mathbb{D}, T\text{-}\mathbb{A}\mathrm{lg}_s]$, we have a functor $\mathcal{R}(G) : \mathbb{D} \to T\text{-}\mathbb{A}\mathrm{lg}_s$.

In particular, $\mathcal{R}(G)_\lambda$ is a functor $\mathscr{D}_\lambda \to (T\text{-}\mathbb{A}\mathrm{lg}_s)_\lambda$. But $(T\text{-}\mathbb{A}\mathrm{lg}_s)_\lambda$ is the full sub-2-category of $T_\lambda\text{-}\mathrm{Alg}_s$ consisting of those T_λ-algebras with tight structure map, and we know that all 2-categorical limits lift to 2-categories of algebras and strict morphisms, and that these limits are preserved by the functor into the 2-category of weak morphisms.

Therefore, the object L' acquires a T_λ-algebra structure which makes it into the limit $\{\Phi_\lambda, \mathcal{R}(G)_\lambda\}$ in $T_\lambda\text{-}\mathrm{Alg}_s$, hence also in $T_\lambda\text{-}\mathrm{Alg}_l$. Tracing through the definition of this T_λ-algebra structure, we find that it is equal to ℓ' as defined above, and in particular is tight. Thus, the T-algebra (L', ℓ') has the universal property of $\{\Phi_\lambda, \mathcal{R}(G)_\lambda\}$ in $T_\lambda\text{-}\mathrm{Alg}_l$, hence also in its full sub-2-category $(T\text{-}\mathbb{A}\mathrm{lg}_l)_\lambda$. It thus remains only to identify this universal property with that of the desired limit $\{\mathcal{Q}\Phi_\lambda, G_\lambda\}$.

Now the representable $M\text{-}\mathrm{Alg}_c(\Phi_\lambda, -)$ preserves any existing limits, and in particular preserves the descent object (10). But for any **Cat**-weight Ψ, we have $M\text{-}\mathrm{Alg}_c(\Phi_\lambda, \Psi) \cong [\mathscr{D}_\lambda, \mathbf{Cat}](\mathcal{Q}\Phi_\lambda, \Psi)$, and so the left-hand column of Fig. 1 is a descent object in **Cat**.

We have continued to use wriggly arrows in this column, although this has no formal meaning in the 2-category **Cat**, in order to draw attention to the fact that the definition of these arrows

is less straightforward than the straight ones, which are all induced by composition with some arrow in $\mathscr{K}_\lambda$. For example, the wriggly arrow at the middle level of the left-hand column is defined (on objects) like this. Given a 2-natural $x : \mathcal{Q}\Phi_\lambda \to \mathscr{K}_\lambda(A, UG_\lambda -)$ form the lax-natural composite

$$\mathcal{Q}\Phi_\lambda \xrightarrow{\ x\ } \mathscr{K}_\lambda(A, UG_\lambda -) \xrightarrow{\ T\ } \mathscr{K}_\lambda(TA, TUG_\lambda -) \overset{\mathscr{K}_\lambda(TA, g-)}{\rightsquigarrow} \mathscr{K}_\lambda(TA, UG_\lambda -)$$

where g is as above. This corresponds to a unique 2-natural $y : \mathcal{Q}^2\Phi_\lambda \to \mathscr{K}_\lambda(TA, UG_\lambda -)$, which we compose with the comultiplication $d : \mathcal{Q}\Phi_\lambda \to \mathcal{Q}^2\Phi_\lambda$ to obtain the map $yd : \mathcal{Q}\Phi_\lambda \to \mathscr{K}_\lambda(TA, UG_\lambda -)$ which is the image of our x. The case of the lower wriggly map in the left-hand column of Fig. 1 is similar.

The solid horizontal isomorphisms in Fig. 1 are instances of the adjointness of $\mathcal{Q}$ and $\mathcal{R}$, as in Lemma 4.1. The straight maps on the right-hand side are again just composition, and these obviously commute with the horizontal isomorphisms. Moreover, tracing through the definitions, we see that in order for the wriggly arrows to commute with the horizontal isomorphisms, the middle wriggly map on the right-hand side must be given by composing with the map

$$\mathscr{K}_\lambda\big(A, \mathcal{R}(UG)_\lambda\big) \xrightarrow{\ T\ } \mathscr{K}_\lambda\big(TA, [\mathbb{D}, T]\mathcal{R}(UG)_\lambda\big) \xrightarrow{\mathscr{K}_\lambda(TA, \mathcal{R}(g).k)} \mathscr{K}_\lambda\big(TA, \mathcal{R}(UG)_\lambda\big)$$

where $\mathcal{R}(g).k$ is the (strictly 2-natural) $[\mathbb{D}, T]$-algebra structure of $\mathcal{R}(G)$, as above. The lower wriggly map is similar. Therefore, the c-descent object of the right-hand column is exactly $[\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, T_\lambda\text{-Alg}_l(\mathbf{A}, \mathcal{R}(G)_\lambda -))$, as shown, and so we have the dotted isomorphism across the top. But this says exactly that to give a limit $\{\Phi_\lambda, \mathcal{R}(G)_\lambda\}$ is the same as to give a limit $\{\mathcal{Q}\Phi_\lambda, G_\lambda\}$. Thus (L', ℓ') is the latter limit, as desired.

Step 2: Φ_λ-limits lift. Since $T_\lambda\text{-Alg}_l(\mathbf{A}, G_\lambda -) \to \mathscr{K}_\lambda(A, UG_\lambda -)$, as a morphism in $M\text{-Alg}_c$, is strict and detects strictness, the following square

$$
\begin{array}{ccc}
[\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, T_\lambda\text{-Alg}_l(\mathbf{A}, G_\lambda -)) & \longrightarrow & M\text{-Alg}_c(\Phi_\lambda, T_\lambda\text{-Alg}_l(\mathbf{A}, G_\lambda -)) \\
\big\downarrow & & \big\downarrow \\
[\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, \mathscr{K}_\lambda(A, UG_\lambda -)) & \longrightarrow & M\text{-Alg}_c(\Phi_\lambda, \mathscr{K}_\lambda(A, UG_\lambda -))
\end{array}
$$

is a pullback in **Cat**. We can also write this, using the c-morphism classifier $Q^{\mathbb{D}}$ for $M\text{-Alg}_c$, as a pullback

$$
\begin{array}{ccc}
[\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, T_\lambda\text{-Alg}_l(\mathbf{A}, G_\lambda -)) & \xrightarrow{\ q^*\ } & [\mathscr{D}_\lambda, \mathbf{Cat}](Q^{\mathbb{D}}\Phi_\lambda, T_\lambda\text{-Alg}_l(\mathbf{A}, G_\lambda -)) \\
\big\downarrow & & \big\downarrow \\
[\mathscr{D}_\lambda, \mathbf{Cat}](\Phi_\lambda, \mathscr{K}_\lambda(A, UG_\lambda -)) & \xrightarrow[\ q^*\ ]{} & [\mathscr{D}_\lambda, \mathbf{Cat}](Q^{\mathbb{D}}\Phi_\lambda, \mathscr{K}_\lambda(A, UG_\lambda -))
\end{array}
\tag{11}
$$

in which the horizontal arrows are given by composition with $q : \mathcal{Q}\Phi \to \Phi$. As above, write $\mathbf{L}' = (L', \ell')$ for $\{\mathcal{Q}\Phi_\lambda, G_\lambda\}$, and $\mathbf{L} = (L, \ell)$ for the T-algebra which we are to show has the universal

property of $\{\Phi_\lambda, G_\lambda\}$. The map $q : \mathcal{Q}\Phi \to \Phi$ induces a morphism $q^* = \{q, UG_\lambda\} : L \to L'$. In the diagram

$$
\begin{array}{ccc}
T_\lambda\text{-Alg}_l(\mathbf{A}, \mathbf{L}) & \cdots\!\!\!\rightarrow & T_\lambda\text{-Alg}_l(\mathbf{A}, \mathbf{L}') \\
\downarrow & & \downarrow \\
\mathscr{K}_\lambda(A, L) & \xrightarrow[\mathscr{K}_\lambda(A, q^*)]{} & \mathscr{K}_\lambda(A, L')
\end{array}
\tag{12}
$$

all vertices except the top left are known to be isomorphic to the corresponding vertices in the previous square (11), and these isomorphisms are compatible with the edges of the square. We need to show that the top left vertices in the two squares are also isomorphic; this is equivalent to filling in the dotted arrow in the square (12) in such a way as to give a pullback.

The structure map $s : \Phi \to \mathcal{Q}\Phi$ induces a map $s^* = \{s, UG_\lambda\} : L' \to L$. By the description of the T-actions $T\{\Phi, UG\} \to \{\Phi, UG\}$ given at the end of the proof of Proposition 5.8, it is clear that these actions are strictly natural with respect to strict morphisms of $\mathcal{Q}$-coalgebras, and so in particular with respect to $s : \Phi \to \mathcal{Q}\Phi$. Thus s^* is a strict morphism $\mathbf{L}' \to \mathbf{L}$ of T-algebras. Since $q \dashv s$ with identity counit, $s^* : L' \to L$ is left adjoint to $q^* : L \to L'$ with identity counit. The unit $\eta : 1 \to q^*s^*$ induces a 2-cell

$$
\ell'.Tq^* \xrightarrow{\eta.\ell'.Tq^*} q^*.s^*.\ell'.Tq^* =\!\!= q^*.\ell.Ts^*.Tq^* =\!\!= q^*.\ell
$$

which makes q^* into a lax T_λ-morphism $(q^*, \bar{q}^*) : \mathbf{L} \to \mathbf{L}'$. Then composition with $(q^*, \bar{q}^*)$ gives the dotted arrow making the square (12) commute; it remains to show that the square is a pullback. Since the horizontal arrows are both given by composition with a morphism having a left adjoint with identity counit, they are both full embeddings. Therefore, the square being a pullback amounts to saying that a lax T-morphism $(f, \bar{f}) : \mathbf{A} \to \mathbf{L}'$ factorizes through $(q^*, \bar{q}^*) : \mathbf{L} \to \mathbf{L}'$ provided that f factorizes through q^*. But if $f = q^*g$, then $g = s^*f$, and we now define $(g, \bar{g})$ to be the composite $s^*(f, \bar{f})$. We must show that $(q^*, \bar{q}^*)s^*(f, \bar{f}) = (f, \bar{f})$.

Now $q^*.s^*.f = f$ by assumption. The 2-cell part of $(q^*, \bar{q}^*)s^*(f, \bar{f})$ is given by the top path around the square

$$
\begin{array}{ccc}
\ell'.Tq^*.Ts^*.Tq^*.Tg & \xrightarrow{\ \eta.1\ } & q^*.s^*.\ell'.Tq^*.Ts^*.Tq^*.Tg \\
\| & & \| \\
\ell'.Tq^*.Tg & & q^*.\ell.Ts^*.Tq^*.Tg \\
& & \| \\
\bar{f} \downarrow & & q^*.s^*.\ell'.Tq^*.Tg \\
& & \downarrow q^*.s^*.\bar{f} \\
q^*.g.a & \xrightarrow[\eta.q^*.g.a]{} & q^*.s^*.q^*.g.a
\end{array}
$$

which is equal to the bottom path. But $\eta.q^*.g.a$ is the identity by one of the triangle equations, so this bottom path is just $\bar{f}$. $\square$

Theorem 5.10. *If Φ is a w-rigged $\mathscr{F}$-weight, then for any $\mathscr{F}$-monad T on an $\mathscr{F}$-category $\mathbb{K}$, the forgetful functor $U_w : T\text{-}\mathbb{A}\mathrm{lg}_w \to \mathbb{K}$ creates Φ-weighted limits.*

Proof. Once again, we treat only the case $w = l$ of lax T-morphisms. By Proposition 5.6 we may suppose that $\mathbb{K}$ is complete.

Let $G : \mathbb{D} \to T\text{-}\mathbb{A}\mathrm{lg}_w$ be the diagram of which we wish to calculate the limit $\{\Phi, G\}$. We know that the limit $L = \{\Phi_\lambda, UG_\lambda\}$ in $\mathscr{K}_\lambda$ lifts to a limit $\mathbf{L} = (L, \ell) = \{\Phi_\lambda, G_\lambda\}$ in $(T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$; we want to show that it is also the limit $\{\Phi, G\}$ in $T\text{-}\mathbb{A}\mathrm{lg}_w$. This amounts to proving that the family of projections $p_{\lambda, \varphi(a)} : \mathbf{L} \to GD$, where $D \in \mathbb{D}$ and $a \in \Phi_\tau D$, are tight and jointly detect tightness in $T\text{-}\mathbb{A}\mathrm{lg}_w$. Tightness in $T\text{-}\mathbb{A}\mathrm{lg}_w$ has two aspects: strictness as a T-morphism, and tightness at the level of the underlying morphism in $\mathbb{K}$.

We know, by the tight part of the universal property of $\{\Phi, UG\}$, that the $Up_{\lambda, \varphi(a)} : L \to UGD$ are tight and jointly detect tightness in $\mathbb{K}$, so we only need to worry about strictness. The fact that the $p_{\lambda, \varphi(a)}$ are strict follows from commutativity of the diagrams (7) and (8) on page 326. It remains to show that they jointly detect strictness.

Suppose then that $\mathbf{A} = (A, a)$ is a T-algebra and $\mathbf{f} = (f, \bar{f}) : \mathbf{A} \to \mathbf{L}$ a lax T-morphism, with f tight in $\mathbb{K}$, whose composite with each $p_{\lambda, \varphi(a)}$ is strict. Then the composite 2-cell

$$\Phi_\tau \xrightarrow{\varphi} \Phi_\lambda J \xrightarrow{p_\lambda J} \mathscr{K}_\lambda(L, UG_\lambda J-) \underset{\mathscr{K}_\lambda(fa,1)}{\overset{\mathscr{K}_\lambda(\ell.Tf,1)}{\Rrightarrow}} \mathscr{K}_\lambda(TA, UG_\lambda J-)$$

is an identity, and so by adjointness the composite 2-cell

$$\mathrm{Lan}_J \Phi_\tau \xrightarrow{\bar{\varphi}} \Phi_\lambda \xrightarrow{p_\lambda} \mathscr{K}_\lambda(L, UG_\lambda -) \underset{\mathscr{K}_\lambda(fa,1)}{\overset{\mathscr{K}_\lambda(\ell.Tf,1)}{\Rrightarrow}} \mathscr{K}_\lambda(TA, UG_\lambda -)$$

is an identity. But $\bar{\varphi}$ is pointwise surjective on objects, so by Lemma 5.3, the 2-cell

$$\Phi_\lambda \xrightarrow{p_\lambda} \mathscr{K}_\lambda(L, UG_\lambda -) \underset{\mathscr{K}_\lambda(fa,1)}{\overset{\mathscr{K}_\lambda(\ell.Tf,1)}{\Rrightarrow}} \mathscr{K}_\lambda(TA, UG_\lambda -)$$

is an identity, and now finally by the universal property of the limit $L = \{\Phi_\lambda, UG_\lambda\}$ it follows that $\bar{f}$ is an identity. $\square$

5.4. All limits that lift are rigged

In the previous subsection we showed that all rigged limits lift; here we prove the converse.

Lemma 5.11. *Every representable $Y_D = \mathbb{D}(D, -)$ is w-rigged, for any w.*

Proof. We define the $\mathcal{Q}$-coalgebra structure of $Y_D = \mathbb{D}(D, -)$ to be the map $s : Y_D \to \mathcal{Q}Y_D$ which corresponds under the Yoneda lemma to the element $p(1_D) \in \mathcal{Q}Y_D(D)$. The coassociativity and counit axioms both assert the equality of maps with domain Y_D, so can be verified using the Yoneda lemma with the calculations

$$q.s.1_D = q.p.1_D = 1_D$$

for the counit and

$$\mathcal{Q}s.s.1_D = \mathcal{Q}s.p.1_D = p.s.1_D = p.p.1_D = \mathcal{Q}p.p.1_D = d.p.1_D = d.s.1_D$$

for coassociativity. Finally, observe that $\mathrm{Lan}_J \mathscr{D}_\tau(D, -) \cong \mathscr{D}_\lambda(D, -)$, and that this isomorphism is precisely the $\bar{\varphi}$ which must be pointwise surjective on objects in order that Y_D be w-rigged. $\square$

Theorem 5.12. *Suppose that $\Phi : \mathbb{D} \to \mathbb{F}$ is an $\mathscr{F}$-weight such that Φ-weighted limits lift to T-$\mathbb{Alg}_w$ for any $\mathscr{F}$-monad T. Then Φ is w-rigged.*

Proof. By duality, if Φ-weighted limits lift to $\mathscr{F}$-categories T-$\mathbb{Alg}_w$ of algebras for $\mathscr{F}$-monads T, then Φ-weighted *colimits* must lift to $\mathscr{F}$-categories W-$\mathbb{Coalg}_{\bar{w}}$ of *coalgebras* for $\mathscr{F}$-comonads W. (Note the reversal of sense from w to $\bar{w}$, as remarked in Section 2.3.) But by general enriched category theory, Φ itself is the Φ-weighted colimit in $[\mathbb{D}, \mathbb{F}]$ of the Yoneda embedding $Y : \mathbb{D}^{\mathrm{op}} \to [\mathbb{D}, \mathbb{F}]$, i.e. $\Phi = \Phi \star Y$.

Since the forgetful functor $\mathcal{Q}_{\bar{w}}$-$\mathbb{Coalg}_{\bar{w}} \to [\mathbb{D}, \mathbb{F}]$ is $\mathscr{F}$-fully-faithful on rigged weights, by Lemma 5.11 the Yoneda embedding lifts uniquely to a functor $\tilde{Y} : \mathbb{D}^{\mathrm{op}} \to \mathcal{Q}_{\bar{w}}$-$\mathbb{Coalg}_{\bar{w}}$. By assumption, the Φ-weighted colimit of Y, namely Φ itself, thereby acquires a $\mathcal{Q}_{\bar{w}}$-coalgebra structure. Thus it remains to show that $\bar{\varphi} : \mathrm{Lan}_J \Phi_\tau \to \Phi_\lambda$ is pointwise surjective on objects; we will also do this by lifting $\Phi \star Y$ to an appropriate category of coalgebras.

For any 2-functor $\Psi : \mathscr{D}_\lambda \to \mathbf{Cat}$, define $\widetilde{\Psi} : \mathbb{D} \to \mathbb{F}$ by $\widetilde{\Psi}_\lambda = \Psi$, $\widetilde{\Psi}_\tau = \Psi J$. Then for any $\Phi : \mathbb{D} \to \mathbb{F}$, any 2-natural transformation $\Phi_\lambda \to \Psi$ extends uniquely to an $\mathscr{F}$-natural transformation $\Phi \to \widetilde{\Psi}$. That is, $\widetilde{(-)}$ is right adjoint to $(-)_\lambda$.

As usual, we write in the case $w = l$. Let W be the $\mathscr{F}$-comonad $(-)^{\mathbf{N}}$ on $[\mathbb{D}, \mathbb{F}]$; that is, cotensoring with the discrete monoid of natural numbers. Then a W-coalgebra is an $\mathscr{F}$-weight Φ equipped with an endomorphism $e_\Phi : \Phi \to \Phi$. A tight arrow in W-$\mathbb{Coalg}_c$ is an $\mathscr{F}$-natural transformation commuting strictly with the endomorphisms, while a loose one is a map $f : \Phi_\lambda \to \Upsilon_\lambda$ equipped with a modification

$$\begin{array}{ccc}
\Phi_\lambda & \xrightarrow{\ f\ } & \Upsilon_\lambda \\
{\scriptstyle (e_\Phi)_\lambda}\Big\downarrow & {\overline{f}\nearrow} & \Big\downarrow{\scriptstyle (e_\Upsilon)_\lambda} \\
\Phi_\lambda & \xrightarrow[\ f\ ]{} & \Upsilon_\lambda
\end{array}$$

We have an evident $\mathscr{F}$-functor $i : [\mathbb{D}, \mathbb{F}] \to W$-$\mathbb{Coalg}_c$ that equips an $\mathscr{F}$-weight with its identity endomorphism.

Now, as before we have $\Phi = \Phi \star Y$ in $[\mathbb{D}, \mathbb{F}]$, with colimiting cocone $c : \Phi \to [\mathbb{D}, \mathbb{F}](Y, \Phi)$. We also have the composite functor $iY : \mathbb{D}^{\mathrm{op}} \to W\text{-}\mathbb{C}\mathrm{oalg}_c$ which lifts Y. Therefore, the forgetful $\mathscr{F}$-functor $W\text{-}\mathbb{C}\mathrm{oalg}_c \to [\mathbb{D}, \mathbb{F}]$ creates a colimit of iY, which must be given by a W-coalgebra structure e_Φ on Φ.

However, we also have the Φ-weighted cocone $ic : \Phi \to W\text{-}\mathbb{C}\mathrm{oalg}_c(iY, i\Phi)$, so there is a unique induced tight W-map $h : (\Phi, e_\Phi) \to (\Phi, 1) = i\Phi$. Since both the colimiting cocone and the cocone ic project to c in $[\mathbb{D}, \mathbb{F}]$, the map h must project to the identity of Φ, and hence $e_\Phi = 1_\Phi$ as well. Thus ic is actually itself colimiting, i.e. $i\Phi \cong \Phi \star iY$.

Now let

$$\Phi_\lambda \overset{f}{\underset{g}{\Rrightarrow}} \Downarrow\beta \quad \Psi$$

be a modification such that the composite

$$\Phi_\tau \overset{fJ.\varphi}{\underset{gJ.\varphi}{\Rrightarrow}} \Downarrow\beta J.\varphi \quad \Psi J$$

is an identity, where $\varphi : \Phi_\tau \to \Phi_\lambda J$ is the structure map of Φ. We equip $\widetilde{\Psi} \times \widetilde{\Psi}$ with the endomorphism (π_2, π_2) sending (x, y) to (y, y), making it a W-coalgebra. Then β defines a loose arrow $i\Phi \rightsquigarrow \widetilde{\Psi} \times \widetilde{\Psi}$ in $W\text{-}\mathbb{C}\mathrm{oalg}_c$:

$$
\begin{array}{ccc}
\Phi_\lambda & \overset{(f,g)}{\longrightarrow} & \Psi \times \Psi \\
{\scriptstyle 1}\downarrow & {\scriptstyle (\beta,1)}\nearrow & \downarrow{\scriptstyle (\pi_2,\pi_2)} \\
\Phi_\lambda & \underset{(f,g)}{\longrightarrow} & \Psi \times \Psi
\end{array}
$$

(the top-right composite being (g, g) and the bottom-right composite being (f, g)). We claim that in fact, this is a tight arrow, and therefore β itself is an identity.

Since $i\Phi = \Phi \star iY$ in $W\text{-}\mathbb{C}\mathrm{oalg}_c$, and (f, g) does extend to a tight map $\Phi \to \widetilde{\Psi} \times \widetilde{\Psi}$ in $[\mathbb{D}, \mathbb{F}]$, to show that β is tight in $W\text{-}\mathbb{C}\mathrm{oalg}_c$ it suffices to show that for every tight coprojection $iYD \overset{k}{\to} i\Phi$, the composite $iYD \overset{k}{\to} i\Phi \rightsquigarrow \widetilde{\Psi} \times \widetilde{\Psi}$ is tight. This amounts to saying that the composite

$$(YD)_\lambda \overset{f.k_\lambda}{\underset{g.k_\lambda}{\Rrightarrow}} \Downarrow\beta.k_\lambda \quad \Psi$$

is an identity. But by assumption and since k is tight, the composite

$$(YD)_\tau \quad \overset{(f.k_\lambda)J.y}{\underset{(g.k_\lambda)J.y}{\Longrightarrow}} \quad \Downarrow (\beta.k_\lambda)J.y \quad \Psi$$

is the identity, and therefore so is

$$\mathrm{Lan}_y(YD)_\tau \quad \overset{f.k_\lambda.\bar{y}}{\underset{g.k_\lambda.\bar{y}}{\Longrightarrow}} \quad \Downarrow \beta.k_\lambda.\bar{y} \quad \Psi$$

where $y : (YD)_\tau \to (YD)_\lambda J$ is the structure map of YD and $\bar{y} : \mathrm{Lan}_J(YD)_\tau \to (YD)_\lambda$ is its adjunct. But since YD is representable, $\bar{y}$ is an isomorphism, so this implies that $\beta.k_\lambda$ is also an identity, as desired.

We have proven that given any modification

$$\Phi_\lambda \quad \overset{f}{\underset{g}{\Longrightarrow}} \quad \Downarrow \beta \quad \Psi$$

if $\beta J.\varphi$, or equivalently $\beta.\bar{\varphi}$, is an identity, then β is an identity. By Lemma 5.3, this implies that $\bar{\varphi} : \mathrm{Lan}_J \Phi_\tau \to \Phi_\lambda$ must be pointwise surjective on objects. $\square$

Combining the results of this section and the previous one, we finally obtain our characterization theorem:

Theorem 5.13. *For an $\mathscr{F}$-weight Φ, the following are equivalent.*

  (i) *Φ is w-rigged, as in Definition 5.2.*
 (ii) *For any $\mathscr{F}$-monad T on an $\mathscr{F}$-category $\mathbb{K}$, the functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ creates Φ-weighted limits.*
(iii) *For any 2-monad T on a 2-category $\mathscr{K}$, the functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ (where $\mathbb{K}$ denotes the chordate $\mathscr{F}$-category on $\mathscr{K}$) creates Φ-weighted limits.*

5.5. Rigged colimits

We end this section by briefly considering colimits in categories of algebras, which are generally more subtle than limits. Even in the case of ordinary categories, colimits are not in general created by monadic functors. One thing one can say is that if a monad T preserves colimits with a given weight, then the category of T-algebras has colimits of that sort created by the forgetful functor. We now show that rigged $\mathscr{F}$-weights satisfy an analogous property for categories of algebras and weak morphisms. The analogous result for PIE-colimits in the 2-categorical context has been proven independently by John Bourke.

Theorem 5.14. *Let Φ be $\bar{w}$-rigged, and let T be an $\mathscr{F}$-monad on an $\mathscr{F}$-category $\mathbb{K}$ such that $\mathbb{K}$ has, and T preserves, Φ-weighted colimits. Then the forgetful functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ creates Φ-weighted colimits.*

Proof. First of all, we observe that just as in Proposition 5.6, we may assume that $\mathbb{K}$ is small; for otherwise we can pick a small full subcategory of it, closed under Φ-weighted colimits and the action of T, and containing the image of G and any other given object.

Now, if $\mathbb{K}$ is small, let $\widehat{\mathbb{K}} \subset [\mathbb{K}^{\mathrm{op}}, \mathbb{F}]$ be the subcategory consisting of those presheaves on $\mathbb{K}$ which preserve Φ-weighted limits (i.e. which take Φ-weighted colimits in $\mathbb{K}$ to limits in $\mathbb{F}$). Since $\mathbb{K}$ has all Φ-weighted colimits, by [10, 6.17] T extends to an $\mathscr{F}$-monad $\widehat{T}$ on $\widehat{\mathbb{K}}$ which has a right adjoint $\widehat{T}^*$. Therefore, as in Section 2.3, $\widehat{T}^*$ becomes an $\mathscr{F}$-comonad whose coalgebras are the same as $\widehat{T}$-algebras.

However, since Φ is $\bar{w}$-rigged, Φ-weighted colimits lift from $\widehat{\mathbb{K}}$ to $\widehat{T}^*\text{-}\mathbb{Coalg}_w$, hence also to $\widehat{T}\text{-}\mathbb{Alg}_w$. Moreover, the embedding $\mathbb{K} \hookrightarrow \widehat{\mathbb{K}}$ preserves Φ-weighted colimits. Thus, any $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_w$ has a colimit $\Phi * G$ in $\mathbb{K}$, which remains a colimit in $\widehat{\mathbb{K}}$ and thus lifts to $\widehat{T}\text{-}\mathbb{Alg}_w$. But the underlying presheaf of this colimit $\widehat{T}$-algebra is representable, hence it is a T-algebra and thus a Φ-weighted colimit of G. $\quad\square$

6. On rigged weights

Our goal in this section is to analyze the notion of rigged weight a little further, to clarify the relationship between the two parts of the definition and the connection to 2-categorical weights such as PIE-weights.

6.1. The structure of $\mathcal{Q}$-coalgebras

We begin by unpacking the notion of $\mathcal{Q}_{\bar{w}}$-coalgebra a little. Let $\mathbb{D}$ be an $\mathscr{F}$-category, and $Q_{\bar{w}} := Q_{\bar{w}}^{\mathbb{D}}$ the $\bar{w}$-transformation classifier for the monad on $[\mathscr{D}_\tau, \mathbf{Cat}]$ whose category of algebras is $[\mathscr{D}_\lambda, \mathbf{Cat}]$. Recall from Section 4.2 that $\mathcal{Q}_{\bar{w}}$ can be constructed as $(\mathcal{Q}_{\bar{w}}\Phi)_\tau = \Phi_\tau$ and $(\mathcal{Q}_{\bar{w}}\Phi)_\lambda = Q_{\bar{w}}(\Phi_\lambda)$, with the structure map being the composite

$$\Phi_\tau \xrightarrow{\ \varphi\ } \Phi_\lambda J \xrightarrow{\ p\ } Q_{\bar{w}}(\Phi_\lambda)J.$$

Therefore, the coaction $s : \Phi \to \mathcal{Q}_{\bar{w}}\Phi$ of a $\mathcal{Q}_{\bar{w}}$-coalgebra must be of the form

$$
\begin{array}{ccc}
\Phi & \qquad & \Phi_\tau \xrightarrow{\ \varphi\ } \Phi_\lambda J \\
\big\downarrow & & \big\| \qquad\qquad \big\downarrow {\scriptstyle sJ} \\
\mathcal{Q}_{\bar{w}}\Phi & & \Phi_\tau \xrightarrow[\ p.\varphi\ ]{} (Q_{\bar{w}}\Phi_\lambda)J
\end{array}
$$

It follows that Φ is a $\mathcal{Q}_{\bar{w}}$-coalgebra if and only if

(I) Φ_λ is a $Q_{\bar{w}}$-coalgebra, with coaction $s : \Phi_\lambda \to Q_{\bar{w}}\Phi_\lambda$, and
(II) $sJ.\varphi = p.\varphi$.

We can refine the second of these conditions a little further. Suppose that $w = l$, and recall that Q_c is colax-idempotent. Therefore, if $s : \Phi_\lambda \to Q_c\Phi_\lambda$ makes Φ_λ into a Q_c-coalgebra, then s is right adjoint to $q : Q_c\Phi_\lambda \to \Phi_\lambda$ with identity counit, and hence sJ is right adjoint to qJ with identity counit. On the other hand, $p : \Phi_\lambda J \to Q_c\Phi_\lambda J$ is left adjoint to qJ with identity unit, so we get a string of adjunctions $p \dashv qJ \dashv sJ$. We write $\alpha : 1 \to sq$ for the unit of the adjunction $q \dashv s$ and $\beta : p.qJ \to 1$ for the counit of the adjunction $p \dashv qJ$. Then the diagram

$$
\begin{array}{ccc}
p & \xrightarrow{\ \alpha J.p\ } & sJ.qJ.p \\[2pt]
\| & & \| \\[8pt]
p.qJ.sJ & \xrightarrow[\ \beta.sJ\ ]{} & sJ
\end{array}
$$

commutes, and we write $\tau : p \to sJ$ for the common value.

In the case $w = c$ there is an analogous $\tau : sJ \to p$, while for $w = p$ the 2-cell is invertible.

Lemma 6.1. *If Ψ is a $Q_{\bar{w}}$-coalgebra, then a morphism $\upsilon : \Upsilon \to \Psi J$ satisfies $sJ.\upsilon = p.\upsilon$ if and only if $\tau.\upsilon$ is an identity.*

Proof. If $\tau.\upsilon : p.\upsilon \to sJ.\upsilon$ is an identity, its source and target must be equal. On the other hand, by definition $\tau = \beta.sJ$, so if $sJ.\upsilon = p.\upsilon$ then

$$\tau.\upsilon = \beta.sJ.\upsilon = \beta.p.\upsilon,$$

and $\beta.p$ is an identity by one of the triangle identities for the adjunction between p and qJ. $\square$

In particular, this applies to $\varphi : \Phi_\tau \to \Phi_\lambda J$ whenever Φ is a $Q_{\bar{w}}$-coalgebra.

6.2. Canonical riggings

It is natural to ask, given a coalgebra for the comonad given by the 2-categorical relative w-morphism classifier $Q_{\bar{w}}^{\mathbb{D}}$, how can we extend it to a w-rigged $\mathscr{F}$-weight? In all the examples in Section 3.5, there was an obvious "canonical" choice of which projections to make tight. The following proposition says that this situation is generic.

Proposition 6.2. *If Ψ is a $Q_{\bar{w}}$-coalgebra, then the category of all $Q_{\bar{w}}$-coalgebras Φ with $\Phi_\lambda = \Psi$ (as $Q_{\bar{w}}$-coalgebras) is a preorder with a greatest element. Moreover, if any such Φ is w-rigged, so is the greatest one.*

Proof. We saw in Section 6.1 that to give a $Q_{\bar{w}}$-coalgebra Φ with $\Phi_\lambda = \Psi$ is precisely to give a pointwise full embedding $\varphi : \Phi_\tau \to \Psi J$ such that $sJ.\varphi = p.\varphi$, or equivalently such that $\tau.\varphi$ is an identity. Since full embeddings are monic and fully faithful, the category of such (for fixed Ψ) is a preorder. Its greatest element is the identifier of τ (note that any identifier is a full embedding).

Finally, if Φ is w-rigged with $\Phi_\lambda = \Psi$, and $\psi : \Psi_\tau \to \Psi J$ is the identifier of τ, then $\varphi : \Phi_\tau \to \Phi_\lambda = \Psi J$ factors through ψ via some $k : \Phi_\tau \to \Psi_\tau$. Thus the composite

$$\mathrm{Lan}_J\,\Phi_\tau \xrightarrow{\ \mathrm{Lan}_J(k)\ } \mathrm{Lan}_J\,\Psi_\tau \xrightarrow{\ \overline{\psi}\ } \Psi$$

is the pointwise surjective on objects $\overline{\varphi}$, so $\overline{\psi}$ must also be pointwise surjective on objects. $\quad\square$

Note that Lemma 6.1 implies that the identifier of τ is also the equalizer of sJ and p.

The canonical rigging constructed in Proposition 6.2 does, however, depend on the choice of $Q_{\bar{w}}$-coalgebra structure on Ψ. Since $Q_{\bar{w}}$ is $\bar{w}$-idempotent, any two such coalgebra structures are uniquely isomorphic, but they need not be identical. Here is an example which admits two distinct $Q_{\bar{w}}$-coalgebra structures, with correspondingly distinct canonical riggings. (This should be contrasted with the remark after Lemma 5.5 that an $\mathscr{F}$-weight with a *given* tight part can "be rigged" in at most one way.)

Example 6.3. Let $\mathscr{D}_\lambda$ have two objects a and b, with two morphisms $r : b \to a$ and $i : a \to b$ such that $ri = 1_a$, and hence $f := ir$ is idempotent. Let Φ_λ be constant at $\mathbb{1}$. Since a is initial in $\mathscr{D}_\lambda$ (or, equivalently, Φ_λ is the representable $\mathscr{D}_\lambda(a, -)$), for any 2-categorical diagram $G : \mathscr{D}_\lambda \to \mathscr{K}$, the object $G(a)$ is a Φ_λ-weighted limit of G.

Let $\mathscr{D}_\tau$ contain the identities together with the idempotent f. Then $Q_w\Phi_\lambda$ is constant at the free-living isomorphism $2_{\cong}$, and so Φ_λ has two distinct (but, of course, isomorphic) Q_w-coalgebra structures. One of the corresponding identifiers has $\Phi_\tau(a) = \mathbb{1}$ and $\Phi_\tau(b) = \emptyset$, while the other has $\Phi_\tau(a) = \emptyset$ and $\Phi_\tau(b) = \mathbb{1}$. Both resulting $\mathscr{F}$-weights are rigged.

More explicitly, suppose T is a 2-monad on a 2-category $\mathscr{K}$. Then a $\mathbb{D}$-diagram in $T\text{-}\mathbb{Alg}_w$ consists of a strict idempotent f of a T-algebra $\mathbf{B}$, together with a splitting of the underlying morphism f in $\mathscr{K}$, and a T-algebra structure $\mathbf{A}$ on the splitting making the section and retraction into *weak* T-morphisms (though their composite, being the idempotent, is strict).

On the one hand, clearly $\mathbf{A}$ is a limit of this diagram in $T\text{-}\mathbb{Alg}_w$, and its identity projection to itself is strict and detects strictness. But on the other hand, since f is a strict T-morphism, we can give the underlying object A a different T-algebra structure induced directly from $\mathbf{B}$, in which case the section i becomes strict and strictness-detecting. These two T-algebra structures on A are isomorphic in $T\text{-}\mathbb{Alg}_w$, by an isomorphism whose 1-morphism part is the identity 1_A, but they are generally not equal (i.e. this isomorphism is not generally a strict T-morphism).

On the other hand, a 2-categorical $Q_{\bar{w}}$-coalgebra may have no extension to a w-rigged $\mathscr{F}$-weight at all.

Example 6.4. Let $\mathbb{D}$ be locally discrete, with two objects a and b, and with morphisms generated by two tight morphisms $r, s : a \rightrightarrows b$ and a loose morphism $g : b \rightsquigarrow a$, subject to $sg = 1$ and $rgr = rgs$. Then $f := rg$ is a loose idempotent with $fr = fs$. We can write out all the homsets explicitly as follows:

$$\mathscr{D}_\lambda(a, a) = \{1_a, gr, gs, gfr\}, \qquad \mathscr{D}_\lambda(b, b) = \{1_b, f\},$$
$$\mathscr{D}_\lambda(a, b) = \{r, s, fr\}, \qquad \mathscr{D}_\lambda(b, a) = \{g, gf\}.$$

The only tight morphisms are the identities and r, s.

Let Φ_λ be constant at $\mathbb{1}$ and Φ_τ be constant at $\emptyset$. Then $\Phi_\lambda J : \mathscr{D}_\tau \to \mathbf{Cat}$ is the quotient of the representable $\mathscr{D}_\tau(a, -)$ by the equivalence relation setting $[r] = [s]$. Therefore, $\mathrm{Lan}_J(\Phi_\lambda J)$

is the quotient of $\mathscr{D}_\lambda(a, -)$ by the equivalence relation generated by $[r] = [s]$, which implies $[gr] = [gs]$ but no more relations.

Since for any w, $Q_w \Phi_\lambda$ is a type of codescent object of $\mathrm{Lan}_J(\Phi_\lambda J)$, it has the same objects as the latter. We can therefore define a morphism $\Phi_\lambda \to Q_w \Phi_\lambda$ which picks out $[gfr] = [gfs] \in Q_w \Phi_\lambda(a)$ and $[fr] = [fs] \in Q_w \Phi_\lambda(b)$. Verifying the coassociativity and counit axioms, and recalling that $\Phi_\tau \equiv \emptyset$ so that this extends to a morphism $\Phi \to Q_w \Phi$, we see that Φ is a Q_w-coalgebra for any w.

Of course, with $\Phi_\tau \equiv \emptyset$ and Φ_λ nontrivial, Φ is not rigged. In fact, however, $\Phi_\tau \equiv \emptyset$ is the *only* Φ_τ for which the above morphism $\Phi_\lambda \to Q_w \Phi_\lambda$ extends to an $\mathscr{F}$-natural transformation. This is because the inclusion $p : \Phi_\lambda J \to (Q_w \Phi)_\lambda J$ picks out instead $[1_a]$ and $[r] = [s]$, and so the identifier constructed above is itself empty.

More concretely, a Φ-weighted limit of a $\mathbb{D}$-shaped diagram is really just a splitting of the loose idempotent f. Splitting of general loose idempotents is flexible but not rigged, but in this case the additional existence of the tight morphisms r and s enables us to make the weight into a Q-coalgebra, or equivalently to show that the limit of a $\mathbb{D}$-diagram in $T\text{-}\mathbb{Alg}_w$ actually gets a strict T-algebra structure. But neither of the projections is necessarily a strict T-morphism, and so the weight cannot be rigged.

We can, however, identify conditions on the $\mathscr{F}$-category $\mathbb{D}$ which ensure that any $Q_{\bar{w}}$-coalgebra can be rigged. For a 2-category $\mathscr{C}$, we write $\mathscr{C}_0$ for its underlying 1-category, and similarly for 2-functors.

Proposition 6.5. *Suppose that* $\mathrm{Lan}_{J_0} : [(\mathscr{D}_\tau)_0, \mathbf{Set}] \to [(\mathscr{D}_\lambda)_0, \mathbf{Set}]$ *preserves coreflexive equalizers. Then the maximal extension of any* $Q_{\bar{w}}$-*coalgebra to a* $Q_{\bar{w}}$-*coalgebra is w-rigged, and moreover its structure map* $\bar{\varphi} : \mathrm{Lan}_J \Phi_\tau \to \Phi_\lambda$ *is pointwise bijective on objects.*

Proof. Let Φ be the maximal extension of Φ_λ to a $Q_{\bar{w}}$-coalgebra, and write $Q = Q_{\bar{w}} = (Q_{\bar{w}})_\lambda$. Then we have an equalizer diagram

$$\Phi_\tau \xrightarrow{\varphi} \Phi_\lambda J \overset{sJ}{\underset{p}{\rightrightarrows}} Q\Phi_\lambda J$$

in $[\mathscr{D}_\tau, \mathbf{Cat}]$. Moreover, the parallel pair (sJ, p) is coreflexive, since qJ is a common splitting. Thus, by assumption, the top row of the following diagram becomes an equalizer diagram in $[(\mathscr{D}_\lambda)_0, \mathbf{Set}]$ after composing with $\mathrm{ob} : \mathbf{Cat}_0 \to \mathbf{Set}$.

$$
\begin{array}{ccccc}
\mathrm{Lan}_J \Phi_\tau & \xrightarrow{\mathrm{Lan}_J \varphi} & \mathrm{Lan}_J(\Phi_\lambda J) & \overset{\mathrm{Lan}_J sJ}{\underset{\mathrm{Lan}_J p}{\rightrightarrows}} & \mathrm{Lan}_J(Q\Phi_\lambda J) \\
{\scriptstyle \bar{\varphi}} \downarrow & & {\scriptstyle \overline{p_{\Phi_\lambda}}} \downarrow & & \downarrow {\scriptstyle \overline{p_{Q\Phi_\lambda}}} \\
\Phi_\lambda & \xrightarrow{\quad s \quad} & Q\Phi_\lambda & \overset{Qs}{\underset{Qp}{\rightrightarrows}} & QQ\Phi_\lambda
\end{array}
$$

Since $\overline{p_{\Phi_\lambda}}$ and $\overline{p_{Q\Phi_\lambda}}$ exhibit their codomains as codescent objects of their domains (by construction of Q), they are pointwise bijective on objects. Moreover, the bottom row is also an equalizer diagram, since Φ_λ is a Q-coalgebra and Qp is the comultiplication of Q.

Thus, if we can show that each square in the diagram commutes, it will follow that $\bar{\varphi}$ is also pointwise bijective on objects. For this, it suffices to verify that the adjunct diagram commutes in $[\mathscr{D}_\tau, \mathbf{Cat}]$:

$$
\begin{array}{ccccc}
\Phi_\tau & \xrightarrow{\ \varphi\ } & \Phi_\lambda J & \underset{p}{\overset{sJ}{\rightrightarrows}} & Q\Phi_\lambda J \\
{\scriptstyle \varphi}\downarrow & & {\scriptstyle p_{\Phi_\lambda}}\downarrow & & \downarrow{\scriptstyle p_{Q\Phi_\lambda}} \\
\Phi_\lambda J & \xrightarrow[sJ]{} & (Q\Phi_\lambda)J & \underset{(Qp)J}{\overset{(Qs)J}{\rightrightarrows}} & (QQ\Phi_\lambda)J
\end{array}
$$

But now the left-hand square is just the equation $p.\varphi = sJ.\varphi$ which Φ must satisfy to be a $Q_{\bar{w}}$-coalgebra, while the two right-hand squares are naturality squares for p. □

The hypothesis of Proposition 6.5 holds in particular if $\mathbb{D}$ is inchordate, so that $(\mathscr{D}_\tau)_0$ is discrete. In this case, $Q_{\bar{w}}$ is just the 2-categorical $\bar{w}$-transformation classifier on $[\mathscr{D}_\lambda, \mathbf{Cat}]$, so we have:

Corollary 6.6. *If $\Psi : \mathscr{D} \to \mathbf{Cat}$ is a coalgebra for the 2-categorical $\bar{w}$-transformation classifier $Q_{\bar{w}}$, then it has a canonical extension to a w-rigged $\mathscr{F}$-weight, whose domain is the inchordate $\mathscr{F}$-category on $\mathscr{D}$, and for which $\bar{\varphi}$ is pointwise bijective on objects.*

One easy application of this result is to *cofree* $Q_{\bar{w}}$-coalgebras, i.e. weights of the form $Q_{\bar{w}}\Psi : \mathscr{D} \to \mathbf{Cat}$. Since a $Q_c\Psi$-weighted limit is simply an *oplax* Ψ-weighted limit, Corollary 6.6 implies that *oplax limits are canonically l-rigged*, and dually. The lifting of oplax limits to $T\text{-}\mathbb{A}\mathrm{lg}_l$, for T a 2-monad, was also proven directly in [17], and the lifting of pseudo limits to $T\text{-}\mathbb{A}\mathrm{lg}_p$ was shown in [2].

Note that in the cofree case, we have a split equalizer

$$
\Psi J \xrightarrow{\ p\ } (Q\Psi)J \underset{p_{Q\Psi}}{\overset{(Qp_\Psi)J}{\rightrightarrows}} (QQ\Psi)J
$$

with sections qJ and $(Qq)J$

so that the canonical rigging of $Q\Psi$ is just $p : \Psi J \to (Q\Psi)J$. In other words, the tight projections are the obvious "generating" ones of the oplax limit.

6.3. Tightly rigged weights

In Section 6.2 we were concerned with constructing a rigging of a weight that was known to be a Q-coalgebra, i.e. with deducing the second part of the definition of w-rigged weight from the first. We cannot hope to do the reverse in general, but there is one case in which we can.

Proposition 6.7. *If Φ is an $\mathscr{F}$-weight such that $\bar{\varphi} : \mathrm{Lan}_J \Phi_\tau \to \Phi_\lambda$ is pointwise bijective on objects, then Φ is p-rigged.*

Proof. Since $qJ.p$ is an identity, we have $\varphi = qJ.p.\varphi$, and hence by adjunction the following square commutes in $[\mathscr{D}_\lambda, \mathbf{Cat}]$.

$$
\begin{array}{ccc}
\mathrm{Lan}_J\, \Phi_\tau & \xrightarrow{\overline{p.\varphi}} & Q_{\bar{w}}\Phi_\lambda \\
{\scriptstyle \bar{\varphi}}\downarrow & & \downarrow{\scriptstyle q} \\
\Phi_\lambda & =\!=\!=\!= & \Phi_\lambda
\end{array}
$$

The left-hand map is bijective on objects (by assumption), and the right-hand map is fully faithful, since it is an equivalence in $\mathrm{Ps}(\mathscr{D}_\lambda, \mathbf{Cat})$. (Here, and the analogous assertion later on, is where we use the restriction to $w = p$.) Therefore, by orthogonality, there exists a unique diagonal $s : \Phi_\lambda \to Q_{\bar{w}}\Phi_\lambda$ satisfying $q.s = 1$ and $s\bar{\varphi} = \overline{p.\varphi}$. The former condition says that s is a section of q; the latter is equivalent to $sJ.\varphi = p.\varphi$, so that s is actually a morphism $\Phi \to \mathcal{Q}_{\bar{w}}\Phi$ of $\mathscr{F}$-weights.

Thus, for Φ to be a $\mathcal{Q}_{\bar{w}}$-coalgebra, it remains only to show that $Qp.s = Qs.s$. We claim that both $Qp.s$ and $Qs.s$ are diagonal fillers in a square of the following form:

$$
\begin{array}{ccc}
\mathrm{Lan}_J\, \Phi_\tau & \longrightarrow & QQ\Phi_\lambda \\
{\scriptstyle \bar{\varphi}}\downarrow & \nearrow & \downarrow{\scriptstyle Qq} \\
\Phi_\lambda & \longrightarrow & Q\Phi_\lambda
\end{array}
\qquad (13)
$$

This is equivalent to saying that

$$Qq.Qp.s = Qq.Qs.s \quad \text{and}$$
$$Qp.s.\bar{\varphi} = Qs.s.\bar{\varphi}.$$

The first equation is easy; both sides are equal to s since $q.p = 1 = q.s$. For the second, we consider the adjuncts and compute

$$
\begin{aligned}
2(Qs)J.sJ.\varphi = (Qs)J.p_\Phi.\varphi & \quad \text{(definition of } s) \\
= p_{Q\Phi}.sJ.\varphi & \quad \text{(naturality of } p) \\
= p_{Q\Phi}.p_\Phi.\varphi & \quad \text{(definition of } s) \\
= Qp_\Phi.p_\Phi.\varphi & \quad \text{(naturality of } p) \\
= Qp_\Phi.sJ.\varphi & \quad \text{(definition of } s).
\end{aligned}
$$

Thus, there does exist a square (13) in which $Qp.s$ and $Qs.s$ are both diagonal fillers. Since Qq is pointwise fully faithful and $\bar{\varphi}$ is bijective on objects, by orthogonality any such square has a unique diagonal filler; thus $Qp.s = Qs.s$ as desired. $\square$

We call an $\mathscr{F}$-weight **tightly rigged** if $\bar{\varphi}$ is pointwise bijective on objects. All of the rigged weights we have encountered so far are tightly rigged; Propositions 6.5 and 6.7 provide some

reasons why many rigged weights are tightly rigged. However, not every rigged weight is tightly rigged, or can even be made so by changing the tight part.

Example 6.8. Let $\mathscr{D}_\lambda$ be the set of natural numbers, regarded as a poset (hence a locally discrete 2-category) with the reverse of its usual ordering, and define the unique morphism $n \to m$ to be tight if either (1) $n = m$ or (2) n is even and m is odd. Let Φ_λ be constant at $\mathbb{1}$; then

$$Q_l \Phi_\lambda(n) = \begin{cases} \mathbb{1} & \text{if } n \text{ is odd,} \\ \mathbb{2} & \text{if } n \text{ is even,} \end{cases}$$

so we can make Φ_λ into a Q_l-coalgebra (or a Q_c- or Q_p-coalgebra).

Now for any Φ_τ and $\varphi : \Phi_\tau \to \Phi_\lambda J$ such that the resulting $\mathscr{F}$-weight Φ is a Q_w-coalgebra, Φ_τ can be supported only on the odd numbers (at each of which, it may be $\mathbb{1}$ or $\emptyset$). Then for any n, $\mathrm{Lan}_J \Phi_\tau(m)$ is a discrete category with one object for every odd $m \geqslant n$ such that $\Phi_\tau(m) = \mathbb{1}$. Hence, if Φ is to be rigged, then Φ_τ must be nonempty at arbitrarily large odd numbers; but $\bar{\varphi}$ cannot be bijective at n if Φ_τ is nonempty at more than one $m \geqslant n$. Thus, there are many choices of Φ_τ for which Φ is rigged, but none for which it is tightly rigged.

If we modify $\mathbb{D}$ by stipulating that $n \to m$ is tight if either (1) $n = m$ or (2) m is odd and $n - m$ is congruent to 0, 1, or 3 mod 4, then there are two incompatible choices of Φ_τ for which Φ is tightly rigged: we can take Φ_τ to be nonempty at exactly the numbers that are 1 mod 4, or at exactly those that are 3 mod 4. Neither of these is the maximal choice from Section 6.2, which would be nonempty at all odd numbers; in that case the weight is rigged but not tightly rigged. So although the canonical rigging is often tightly rigged, by Proposition 6.5, it is not always so, even if a tight rigging exists.

6.4. 2-Categories and PIE-limits

We now consider what the theory we have developed has to say about purely 2-categorical weights. Here the statements are simpler, since the distinction between rigged weights and Q-coalgebras evaporates. Specifically, we have the following.

Proposition 6.9. *Let $\Phi : \mathscr{D} \to \mathbf{Cat}$ be a $\mathbf{Cat}$-weight. Then Φ-weighted limits lift along $U_w : T\text{-}\mathrm{Alg}_w \to \mathscr{K}$, for any 2-monad T on a 2-category $\mathscr{K}$, if and only if Φ is a $Q_{\bar{w}}$-coalgebra, where $Q_{\bar{w}}$ is the 2-categorical $\bar{w}$-transformation classifier on $[\mathscr{D}, \mathbf{Cat}]$.*

Proof. If Φ is a $Q_{\bar{w}}$-coalgebra, then by Corollary 6.6 it has a canonical extension to a w-rigged $\mathscr{F}$-weight Ψ, so that Ψ-weighted limits lift to the $\mathscr{F}$-category $T\text{-}\mathbb{A}\mathrm{lg}_w$ for any T. Hence, in particular, Φ-weighted limits lift to the 2-category $T\text{-}\mathrm{Alg}_w = (T\text{-}\mathbb{A}\mathrm{lg}_w)_\lambda$ for any 2-monad T. We could also prove this by imitating the proof of Theorem 5.9 in the 2-categorical world.

For the converse, we seem to have no alternative to imitating (the easy part of) the proof of Theorem 5.12: if Φ-weighted limits lift to $T\text{-}\mathrm{Alg}_w$, then the Φ-weighted colimit $\Phi = \Phi * Y$ lifts to $Q_{\bar{w}}\text{-}\mathrm{Coalg}_{\bar{w}}$ (the diagram Y lying in $Q_{\bar{w}}\text{-}\mathrm{Coalg}_{\bar{w}}$ since representables are $Q_{\bar{w}}$-coalgebras and $Q_{\bar{w}}$ is $\bar{w}$-idempotent). Thus, Φ is a $Q_{\bar{w}}$-coalgebra. □

Note that the characterization is weaker than the $\mathscr{F}$-categorical version: since the 2-categorical U_w is not conservative, lifting of limits does not imply their creation.

Moreover, when $w = l$ or c, this 2-categorical version of the theorem is not very useful, since in these cases there seem to be few purely 2-categorical Q_w-coalgebras aside from the cofree ones. Furthermore, this version contains no information about which projections are strict and detect strictness, a detail which was important in [2] even when $w = p$.

However, when $w = p$, it does turn out that the Q_p-coalgebras are precisely the class of limits already known to lift to T-Alg for all 2-monads T, namely the PIE-limits (those constructible from Products, Inserters, and Equifiers). In the rest of this section we give a proof of this equivalence.

Let $\mathscr{D}$ be a 2-category and $\Phi : \mathscr{D} \to \mathbf{Cat}$ a 2-functor. We write $\mathscr{D}_0$ for the underlying ordinary category of $\mathscr{D}$, and $\mathrm{ob}\,\Phi_0 : \mathscr{D}_0 \to \mathbf{Set}$ for the composite $\mathscr{D}_0 \xrightarrow{\Phi_0} \mathbf{Cat}_0 \xrightarrow{\mathrm{ob}} \mathbf{Set}$. For any functor $F : \mathbf{C} \to \mathbf{Set}$ we write $\mathrm{el}(F)$ for its *category of elements* (aka "Grothendieck construction"). Recall the following theorem from [19].

Theorem 6.10. *A weight* $\Phi : \mathscr{D} \to \mathbf{Cat}$ *is a PIE-weight if and only if* $\mathrm{el}(\mathrm{ob}\,\Phi_0)$ *is a disjoint union of categories with initial objects.*

We require the following easy lemma. We write $\mathrm{ob}(\mathbf{C})$ for the set of objects of a category $\mathbf{C}$, regarded as a discrete category, with the obvious inclusion functor $J : \mathrm{ob}(\mathbf{C}) \to \mathbf{C}$.

Lemma 6.11. *For any functor* $F : \mathbf{C} \to \mathbf{Set}$, $\mathrm{el}(F)$ *is a disjoint union of categories with initial objects if and only if there exists a functor* $G : \mathrm{ob}(\mathbf{C}) \to \mathbf{Set}$ *and an isomorphism* $\mathrm{Lan}_J\, G \cong F$.

Proof. If G exists, then we have

$$F(c) = \sum_{\substack{c' \in \mathbf{C} \\ x \in G(c')}} \mathbf{C}(c', c)$$

and therefore

$$\mathrm{el}(F) = \sum_{\substack{c' \in \mathbf{C} \\ x \in G(c')}} c'/\mathbf{C},$$

and $c'/\mathbf{C}$ certainly has an initial object. Conversely, if $\mathrm{el}(F)$ is a disjoint union of categories with initial objects, we choose for each component $\mathbf{D}_i$ an initial object (c_i, x_i) and let $G(c) = \{x_i \mid c_i = c\}$. Then

$$1 \xrightarrow{(c_i, x_i)} \mathbf{D}_i \to \mathbf{C}$$

is a factorization of $1 \xrightarrow{c_i} \mathbf{C}$ as an initial functor followed by a discrete fibration. Since such factorizations are unique (see [23]), we must have $\mathbf{D}_i \cong (c_i/\mathbf{C})$, and therefore $F \cong \mathrm{Lan}_J\, G$. $\square$

Theorem 6.12. *A* $\mathbf{Cat}$-*weight* $\Phi : \mathscr{D} \to \mathbf{Cat}$ *is a* Q_p-*coalgebra if and only if it is a PIE-weight.*

Proof. In one direction the proof is obvious: PIE-weights are known to lift to T-Alg_p for any 2-monad T, hence by Proposition 6.9 they must be Q_p-coalgebras.

Alternatively, by Theorem 6.10 and Lemma 6.11 we have $\mathrm{ob}\,\Phi_0 \cong \mathrm{Lan}_J\,G$ for some G, which equivalently means we have a bijective-on-objects map $k : \mathrm{Lan}_J\,G \to \Phi$. Thus, if we define $\psi : \Phi_\tau \hookrightarrow \Phi J$ to be the full image of the adjunct $\bar{k} : G \to \Phi J$, then we have a tightly rigged $\mathscr{F}$-weight, whose domain is the inchordate $\mathscr{F}$-category on $\mathscr{D}$. By Proposition 6.7, it is a $\mathcal{Q}_p$-coalgebra, so its loose part, namely Φ, is a Q_p-coalgebra. Combining this argument with Proposition 6.9, we obtain a new proof that PIE-weights lift to T-Alg_p for any T.

For the converse, we invoke Proposition 6.5 for the inchordate $\mathscr{F}$-category on $\mathscr{D}$, and conclude that any Q_p-coalgebra Φ can be made into a w-rigged $\mathscr{F}$-weight for which $\mathrm{Lan}_J\,\Phi_\tau \to \Phi$ is pointwise bijective on objects. Defining $G := \mathrm{ob}(\Phi_\tau)_0$, we have $\mathrm{Lan}_{J_0}\,G \cong \mathrm{ob}\,\Phi_0$; hence by Theorem 6.10 and Lemma 6.11 Φ is a PIE-weight. $\quad\square$

6.5. Saturation

The **saturation** of a class $\mathcal{X}$ of $\mathscr{V}$-weights is the class of weights Φ such that every $\mathcal{X}$-complete $\mathscr{V}$-category is Φ-complete and every $\mathcal{X}$-continuous $\mathscr{V}$-functor is Φ-continuous. This notion was introduced in [1] under the name **closure**, but "saturation" is now standard. The main result of [1] is that $\Phi : \mathscr{D} \to \mathscr{V}$ lies in the saturation of $\mathcal{X}$ if and only if it lies in the closure of the representables under $\mathcal{X}$-colimits in $[\mathscr{D}, \mathscr{V}]$.

A class of weights is called **saturated** if it is its own saturation.

Theorem 6.13. *For any w, the class of w-rigged weights is saturated.*

Proof. Let $\mathscr{R}$ denote the class of w-rigged weights, and let Φ be an $\mathscr{F}$-weight such that every $\mathscr{R}$-complete $\mathscr{F}$-category is Φ-complete and every $\mathscr{R}$-continuous $\mathscr{F}$-functor is Φ-continuous. By Proposition 5.6, to show Φ is w-rigged it suffices to show that it lifts to T-$\mathbb{A}\mathrm{lg}_w$ for any $\mathscr{F}$-monad T on a *complete* $\mathscr{F}$-category $\mathbb{K}$. But in this case, T-$\mathbb{A}\mathrm{lg}_w$ is $\mathscr{R}$-complete and $U_w : T$-$\mathbb{A}\mathrm{lg}_w \to \mathbb{K}$ is $\mathscr{R}$-continuous. Hence, by assumption, T-$\mathbb{A}\mathrm{lg}_w$ is also Φ-complete and U_w is also Φ-continuous. But this is just to say that U_w lifts Φ-weighted limits; so by Theorem 5.12, Φ is w-rigged. $\quad\square$

Recall that essentially by definition, the PIE-weights are the saturation of the class consisting of products, inserters, and equifiers. However, we do not know of any manageable collection of weights which generates the w-rigged weights, even for $w = p$.

Acknowledgments

Appendix A. Alternative proof of the lifting theorem

Here, as promised, we give an alternative proof of the lifting theorem. This could replace all of Section 5.3 after Remark 5.7. We suppose throughout that $w = l$.

Suppose that rather than an individual $\mathcal{Q}^{\mathbb{D}}$-coalgebra Φ, we have a functor $\Psi : \mathbb{E}^{\mathrm{op}} \to \mathcal{Q}^{\mathbb{D}}$-$\mathbb{C}\mathrm{oalg}_c$ for some other small $\mathscr{F}$-category $\mathbb{E}$. Then Ψ has an underlying functor $\mathbb{E}^{\mathrm{op}} \to [\mathbb{D}, \mathbb{F}]$, which is equivalently an $\mathscr{F}$-profunctor $\mathbb{D} \nrightarrow \mathbb{E}$, and so (if we assume, as before, that $\mathbb{K}$ is complete) we have an induced functor

$$\{\Psi, -\} : [\mathbb{D}, \mathbb{K}] \to [\mathbb{E}, \mathbb{K}]$$

where for $M : \mathbb{D} \to \mathbb{K}$ and $E \in \mathbb{E}$ we have $\{\Psi, M\}E = \{\Psi E, M\}$. We would like to lift this to a functor $\mathbb{O}\mathrm{plax}(\mathbb{D}, T\text{-}\mathbb{A}\mathrm{lg}_l) \to \mathbb{O}\mathrm{plax}(\mathbb{E}, T\text{-}\mathbb{A}\mathrm{lg}_l)$, and by the same arguments as in Section 5.3, it suffices to construct a lax monad morphism from $\mathbb{O}\mathrm{plax}(\mathbb{D}, T)$ to $\mathbb{O}\mathrm{plax}(\mathbb{E}, T)$.

However, just as in Section 5.3, $\mathbb{O}\mathrm{plax}(\mathbb{D}, T)$ is a lifting of $[\mathbb{D}, T]$ to the Kleisli category $\mathbb{O}\mathrm{plax}(\mathbb{D}, \mathbb{K})$ of $\mathcal{R}^{\mathbb{D}}$ induced by a distributive law

$$k^{\mathbb{D}} : [\mathbb{D}, T]\mathcal{R}^{\mathbb{D}} \to \mathcal{R}^{\mathbb{D}}[\mathbb{D}, T],$$

and likewise for $\mathbb{O}\mathrm{plax}(\mathbb{E}, T)$ and $\mathcal{R}^{\mathbb{E}}$. Thus, by Corollary 2.2, it suffices to show that $\{\Psi, -\}$ is a lax monad morphism from $[\mathbb{D}, T]$ to $[\mathbb{E}, T]$ and also a colax monad morphism from $\mathcal{R}^{\mathbb{D}}$ to $\mathcal{R}^{\mathbb{E}}$, in such a way that the diagram (1) from Lemma 2.1 commutes.

Proposition A.1. *Let $\mathbb{K}$ be complete and $\mathbb{D}$ and $\mathbb{E}$ small, and let $\Psi : \mathbb{E}^{\mathrm{op}} \to \mathcal{Q}_c^{\mathbb{D}}$-$\mathbb{C}\mathrm{oalg}_c$ be an $\mathscr{F}$-functor. Then the $\mathscr{F}$-functor $F = \{\Psi, -\} : [\mathbb{D}, \mathbb{K}] \to [\mathbb{E}, \mathbb{K}]$ naturally has the structure of a morphism in $\mathrm{Mnd}_l(\mathrm{Mnd}_c(\mathscr{F}\mathbf{Cat}))$ from $k^{\mathbb{D}}$ to $k^{\mathbb{E}}$, and therefore lifts to a functor*

$$\overline{\{\Psi, -\}} : \mathbb{O}\mathrm{plax}(\mathbb{D}, T\text{-}\mathbb{A}\mathrm{lg}_l) \to \mathbb{O}\mathrm{plax}(\mathbb{E}, T\text{-}\mathbb{A}\mathrm{lg}_l).$$

Proof. The proof is mostly a straightforward generalization of Proposition 5.8; the one somewhat different thing is that we need F to be a colax monad morphism from $\mathcal{R}^{\mathbb{D}}$ to $\mathcal{R}^{\mathbb{E}}$, rather than merely a $\mathcal{R}^{\mathbb{D}}$-opalgebra. For this, recall that $\mathbb{L}\mathrm{ax}(\mathbb{E}^{\mathrm{op}}, \mathcal{Q}^{\mathbb{D}}$-$\mathbb{C}\mathrm{oalg}_c) \cong \mathbb{L}\mathrm{ax}(\mathbb{E}^{\mathrm{op}}, \mathcal{Q}^{\mathbb{D}})$-$\mathbb{C}\mathrm{oalg}_c$. Since Ψ is an object of the former, it is equivalently an object of the latter, i.e. the coalgebra structure maps $s_e : \Psi_e \to \mathcal{Q}^{\mathbb{D}}\Psi_e$ are (tight) lax $\mathscr{F}$-natural in e. Therefore, the composite

$$\left(F\mathcal{R}^{\mathbb{D}}\right)_e = \left\{\Psi_e, \mathcal{R}^{\mathbb{D}}-\right\} \cong \left\{\mathcal{Q}^{\mathbb{D}}\Psi_e, -\right\} \xrightarrow{s_e^*} \{\Psi_e, -\} = F_e \tag{14}$$

is (tight) oplax $\mathscr{F}$-natural in e. (The isomorphism is from Lemma 4.1.) Since $\mathcal{R}^{\mathbb{E}}$ coclassifies oplax $\mathscr{F}$-natural transformations, we have an induced tight and strict transformation

$$\chi : F\mathcal{R}^{\mathbb{D}} \to \mathcal{R}^{\mathbb{E}}F.$$

The unit and associativity axioms for the $\mathcal{Q}$-coalgebra structure of Ψ directly imply that χ makes F into a colax morphism of monads from $\mathcal{R}^{\mathbb{D}}$ to $\mathcal{R}^{\mathbb{E}}$.

The rest of the proof is basically the same as the proof of Proposition 5.8, so we leave it to the reader. In particular, the same argument shows that the induced T-algebra structures are the same as those satisfying (8). $\square$

We can also make this lifting functorial on morphisms between profunctors of the above sort.

Proposition A.2. *Let* $\Psi, \Upsilon : \mathbb{E}^{\mathrm{op}} \to \mathcal{Q}_c^{\mathbb{D}}\text{-}\mathbb{C}\mathrm{oalg}_c$ *be as in Proposition* A.1*, and let* $\alpha : \Psi \to \Upsilon$ *be a tight strict $\mathscr{F}$-natural transformation (whose components are thus tight strict $\mathcal{Q}$-morphisms). Then* $\alpha^* : \{\Upsilon, -\} \to \{\Psi, -\}$ *is a 2-cell in* $\mathrm{Mnd}_l(\mathrm{Mnd}_c(\mathscr{F}\mathbf{Cat}))$*, and hence induces a natural transformation*

$$\alpha^* : \overline{\{\Upsilon, -\}} \to \overline{\{\Psi, -\}}.$$

Proof. We must verify that α is a monad 2-cell for both the lax and the colax structures. For the lax monad morphism structures from $[\mathbb{D}, T]$ to $[\mathbb{E}, T]$, this follows from the existence of a lifting $\{\alpha, -\}$ at the level of strict algebras. For the colax monad morphism structures from $\mathcal{R}^{\mathbb{D}}$ to $\mathcal{R}^{\mathbb{E}}$, this follows because they are constructed out of the $\mathcal{Q}$-coalgebra structures on Ψ and Υ, and α consists of strict $\mathcal{Q}$-morphisms. $\square$

There is one further sort of functoriality we need, which involves profunctor composition. Let $\mathscr{F}\mathbf{Prof}$ denote the bicategory of small $\mathscr{F}$-categories and $\mathscr{F}$-profunctors. (Recall that composition of profunctors is defined as a coend.) Then for any complete $\mathscr{F}$-category $\mathbb{K}$, we have a pseudofunctor

$$[-, \mathbb{K}] : \mathscr{F}\mathbf{Prof}^{\mathrm{co}} \to \mathscr{F}\mathbf{Cat}, \tag{15}$$

which sends $\mathbb{D}$ to $[\mathbb{D}, \mathbb{K}]$ and $\Psi : \mathbb{D} \nrightarrow \mathbb{E}$ to $\{\Psi, -\} : [\mathbb{D}, \mathbb{K}] \to [\mathbb{E}, \mathbb{K}]$.

Our goal is to lift this to a pseudofunctor sending $\mathbb{D}$ to $\mathbb{O}\mathrm{plax}(\mathbb{D}, T\text{-}\mathbb{A}\mathrm{lg}_l)$. However, since we are only considering profunctors that are "pointwise $\mathcal{Q}$-coalgebras", we need a bicategory of such. This is the purpose of the following sequence of lemmas, analyzing how $\mathcal{Q}$ interacts with profunctor composition.

Lemma A.3. *For $\mathscr{F}$-profunctors* $\Upsilon : \mathbb{C} \nrightarrow \mathbb{D}$ *and* $\Psi : \mathbb{D} \nrightarrow \mathbb{E}$*, we have* $\mathcal{Q}_c^{\mathbb{C}}(\Psi \otimes_{\mathbb{D}} \Upsilon) \cong \Psi \otimes_{\mathbb{D}} \mathcal{Q}_c^{\mathbb{C}}\Upsilon$.

Proof. We can observe that $\mathcal{Q}_c^{\mathbb{C}}$ is constructed using colimits in $[\mathbb{C}, \mathbb{F}]$, and $\Psi \otimes_{\mathbb{D}} -$ is a weighted colimit; hence the two commute. Or, we can verify that both satisfy the same universal property. $\square$

Lemma A.4. *For $\mathscr{F}$-profunctors* $\Upsilon : \mathbb{C} \nrightarrow \mathbb{D}$ *and* $\Psi : \mathbb{D} \nrightarrow \mathbb{E}$*, we have* $\mathcal{Q}_c^{\mathbb{D}}\Psi \otimes_{\mathbb{D}} \Upsilon \cong \Psi \otimes_{\mathbb{D}} \mathcal{Q}_l^{\mathbb{D}^{\mathrm{op}}}\Upsilon$.

Proof. Again, we can prove this using the construction of both sides out of colimits, or show directly that they have the same universal property. It can also be regarded as a special case of (the dual of) Lemma 4.1. (Note the reversal of sense from l to c on the two sides of the isomorphism.) $\square$

Lemma A.5. *Given* $\Psi : \mathbb{E}^{\mathrm{op}} \to \mathcal{Q}_c^{\mathbb{D}}\text{-}\mathbb{C}\mathrm{oalg}_c$ *and* $\Upsilon : \mathbb{D}^{\mathrm{op}} \to \mathcal{Q}_c^{\mathbb{C}}\text{-}\mathbb{C}\mathrm{oalg}_c$*, the composite* $\Psi \otimes_{\mathbb{D}} \Upsilon$ *has the structure of a functor* $\mathbb{E}^{\mathrm{op}} \to \mathcal{Q}_c^{\mathbb{C}}\text{-}\mathbb{C}\mathrm{oalg}_c$.

Proof. Since $\mathbb{L}\mathrm{ax}(\mathbb{D}^{\mathrm{op}}, \mathcal{Q}_c^{\mathbb{C}}\text{-}\mathbb{C}\mathrm{oalg}_c) \cong \mathbb{L}\mathrm{ax}(\mathbb{D}^{\mathrm{op}}, \mathcal{Q}_c^{\mathbb{C}})\text{-}\mathbb{C}\mathrm{oalg}_c$, the structure map $\Upsilon \to \mathcal{Q}_c^{\mathbb{C}}\Upsilon$ is lax natural in $\mathbb{D}^{\mathrm{op}}$, so it may equivalently be regarded as a map $\mathcal{Q}_l^{\mathbb{D}^{\mathrm{op}}}\Upsilon \to \mathcal{Q}_c^{\mathbb{C}}\Upsilon$. Similarly, we have $\mathcal{Q}_l^{\mathbb{E}^{\mathrm{op}}}\Psi \to \mathcal{Q}_c^{\mathbb{D}}\Psi$, and the desired structure on $\Psi \otimes_{\mathbb{D}} \Upsilon$ ought to be a morphism

$$\mathcal{Q}_l^{\mathbb{E}^{\mathrm{op}}}(\Psi \otimes_{\mathbb{D}} \Upsilon) \to \mathcal{Q}_c^{\mathbb{C}}(\Psi \otimes_{\mathbb{D}} \Upsilon).$$

We can define this morphism to be the composite

$$\mathcal{Q}_l^{\mathbb{E}^{\mathrm{op}}}(\Psi \otimes_{\mathbb{D}} \Upsilon) \overset{\cong}{\to} \mathcal{Q}_l^{\mathbb{E}^{\mathrm{op}}}\Psi \otimes_{\mathbb{D}} \Upsilon \overset{s \otimes 1}{\longrightarrow} \mathcal{Q}_c^{\mathbb{D}}\Psi \otimes_{\mathbb{D}} \Upsilon$$

$$\overset{\cong}{\to} \Psi \otimes_{\mathbb{D}} \mathcal{Q}_l^{\mathbb{D}^{\mathrm{op}}}\Upsilon \overset{1 \otimes s}{\longrightarrow} \Psi \otimes_{\mathbb{D}} \mathcal{Q}_c^{\mathbb{C}}\Upsilon \overset{\cong}{\to} \mathcal{Q}_c^{\mathbb{C}}(\Psi \otimes_{\mathbb{D}} \Upsilon). \qquad (16)$$

The axioms follow straightforwardly from those for Ψ and Υ. $\square$

There is another way to obtain a $\mathcal{Q}$-coalgebra structure on $\Psi \otimes_{\mathbb{D}} \Upsilon$: we can apply Proposition A.1 to Ψ and the monad $(\mathcal{Q}_c^{\mathbb{C}})^{\mathrm{op}}$ on $[\mathbb{C}, \mathbb{F}]^{\mathrm{op}}$. This gives a lifting of the functor

$$\{\Psi, -\} : \big[\mathbb{D}, [\mathbb{C}, \mathbb{F}]^{\mathrm{op}}\big] \longrightarrow \big[\mathbb{E}, [\mathbb{C}, \mathbb{F}]^{\mathrm{op}}\big]$$

(which is just $(- \otimes_{\mathbb{D}} \Psi)^{\mathrm{op}}$) to a functor

$$\big[\mathbb{D}, \big(\mathcal{Q}_c^{\mathbb{C}}\text{-}\mathbb{Coalg}_c\big)^{\mathrm{op}}\big] \longrightarrow \big[\mathbb{E}, \big(\mathcal{Q}_c^{\mathbb{C}}\text{-}\mathbb{Coalg}_c\big)^{\mathrm{op}}\big].$$

Comparing (16) to (9), and recalling that Lemma A.4 is the dual of Lemma 4.1, we see that these two definitions agree.

It is also easy to see from the above definition that tight strict transformations (consisting of strict $\mathcal{Q}$-morphisms) induce similar tight strict transformations between composites of profunctors. We conclude:

Lemma A.6. *There is a bicategory $\mathcal{Q}\mathbf{Prof}$ whose objects are small $\mathscr{F}$-categories, and whose hom-categories are*

$$\mathcal{Q}\mathbf{Prof}(\mathbb{D}, \mathbb{E}) = \mathscr{F}\mathbf{Cat}\big(\mathbb{E}^{\mathrm{op}}, \mathcal{Q}_c^{\mathbb{D}}\text{-}\mathbb{Coalg}_c\big).$$

Proof. The unit profunctors lie in $\mathcal{Q}\mathbf{Prof}$ by Lemma 5.11, while composition is given by Lemma A.5. A computation shows that the associativity and unitality isomorphisms in $\mathscr{F}\mathbf{Prof}$ are strict $\mathcal{Q}$-morphisms. $\square$

The 2-cells in $\mathcal{Q}\mathbf{Prof}$ are tight strict $\mathcal{Q}$-morphisms. There is a forgetful functor $\mathcal{Q}\mathbf{Prof} \to \mathscr{F}\mathbf{Prof}$ which is bijective on objects and faithful on 2-cells; Lemma 5.5 implies that it is full on 2-cells between profunctors that are pointwise rigged.[1] We can now deduce the following strong functoriality statement.

Proposition A.7. *For a complete $\mathscr{F}$-category $\mathbb{K}$ and an $\mathscr{F}$-monad T on $\mathbb{K}$, the pseudofunctor $[-, \mathbb{K}] : \mathcal{Q}\mathbf{Prof}^{\mathrm{co}} \to \mathscr{F}\mathbf{Prof}^{\mathrm{co}} \to \mathscr{F}\mathbf{Cat}$ lifts to a pseudofunctor $\mathcal{Q}\mathbf{Prof}^{\mathrm{co}} \to \mathscr{F}\mathbf{Cat}$ sending $\mathbb{D}$ to $\mathbb{Oplax}(\mathbb{D}, T\text{-}\mathbb{Alg}_l)$.*

---

[1] Although we will not need it, we observe that the pointwise-rigged profunctors actually form a sub-bicategory of $\mathcal{Q}\mathbf{Prof}$. This follows from Theorem 6.13, which implies that rigged weights are closed under rigged colimits.

Proof. It suffices to lift $[-, \mathbb{K}]$ to a pseudofunctor

$$Q\mathbf{Prof}^{\mathrm{co}} \to \mathrm{Mnd}_l\big(\mathrm{Mnd}_c(\mathscr{F}\mathbf{Cat})\big), \tag{17}$$

since then we can apply the Kleisli-object-assigning functor $\mathrm{Mnd}_c(\mathscr{F}\mathbf{Cat}) \to \mathscr{F}\mathbf{Cat}$, followed by the functor $\mathrm{Mnd}_l(\mathscr{F}\mathbf{Cat}) \to \mathscr{F}\mathbf{Cat}$ which constructs $T\text{-}\mathbb{Alg}_l$ from an $\mathscr{F}$-monad T.

However, we have already constructed the lifting (17) on morphisms (Proposition A.1) and 2-cells (Proposition A.2), so it remains to verify its functoriality. Functoriality on 2-cells is immediate, so we need to check that for $\Psi : \mathbb{C} \nrightarrow \mathbb{D}$ and $\Upsilon : \mathbb{D} \nrightarrow \mathbb{E}$ in $Q\mathbf{Prof}$, the colax and lax monad morphism structures on $\{\Upsilon \otimes_{\mathbb{D}} \Psi, -\}$ are the composites of those on $\{\Upsilon, -\}$ and $\{\Psi, -\}$.

For the lax monad morphism structures from $[\mathbb{C}, T]$ to $[\mathbb{E}, T]$, this follows easily since all limits lift, functorially, to categories of strict algebras. And for the colax monad morphism structures from $\mathcal{R}^{\mathbb{C}}$ to $\mathcal{R}^{\mathbb{E}}$, it follows from the construction in Lemma A.5 of the Q-coalgebra structure on $\Upsilon \otimes_{\mathbb{D}} \Psi$. $\quad\square$

Now we need a supply of good profunctors to which to apply this functoriality. Let $H : \mathbb{D} \to \mathbb{E}$ be any $\mathscr{F}$-functor, and $H^{\bullet} : \mathbb{E} \nrightarrow \mathbb{D}$ the profunctor defined by $H^{\bullet}(d, e) = \mathbb{E}(H(d), e)$. Then $\{H^{\bullet}, -\} : [\mathbb{E}, \mathbb{K}] \to [\mathbb{D}, \mathbb{K}]$ is simply given by precomposition with H. Moreover, since $H^{\bullet} : \mathbb{D}^{\mathrm{op}} \to [\mathbb{E}, \mathbb{F}]$ is the composite of H^{op} with the Yoneda embedding, by Lemma 5.11 it lifts naturally to a morphism in $Q\mathbf{Prof}$.

Lemma A.8. *The lifted functor*

$$\overline{\{H^{\bullet}, -\}} : \mathbb{O}\mathrm{plax}(\mathbb{E}, T\text{-}\mathbb{Alg}_l) \to \mathbb{O}\mathrm{plax}(\mathbb{D}, T\text{-}\mathbb{Alg}_l)$$

is also given by precomposition with H.

Proof. By construction of the Q-coalgebra structure in Lemma 5.11. $\quad\square$

Corollary A.9. *For $H : \mathbb{D} \to \mathbb{E}$ and $\Psi : \mathbb{E}^{\mathrm{op}} \to Q_c^{\mathbb{C}}\text{-}\mathbb{C}\mathrm{oalg}_c$, the composite $H^{\bullet} \otimes_{\mathbb{E}} \Psi$ in $Q\mathbf{Prof}$ is naturally isomorphic to the composite functor*

$$\mathbb{D} \xrightarrow{H} \mathbb{E} \xrightarrow{\Psi} Q_c^{\mathbb{C}}\text{-}\mathbb{C}\mathrm{oalg}_c.$$

Proof. This follows from Lemma A.8, together with the observation after Lemma A.5 that composition in $Q\mathbf{Prof}$ can also be described as a lifted limit. $\quad\square$

Similarly, for $H : \mathbb{D} \to \mathbb{E}$ as above, we have a profunctor $H_{\bullet} : \mathbb{D} \nrightarrow \mathbb{E}$ defined by $H_{\bullet}(e, d) = \mathbb{E}(e, H(d))$, such that $\{H_{\bullet}, -\}$ is right Kan extension along H.

Lemma A.10. *Suppose that $H : \mathbb{D} \hookrightarrow \mathbb{E}$ is the inclusion of a full subcategory such that for any $E \in \mathbb{E} \setminus \mathbb{D}$,*

- *the weight $\mathbb{E}(E, H-) : \mathbb{D} \to \mathbb{F}$ is a Q-coalgebra, and*
- *if there exists a nonidentity tight morphism $E' \to E$ in $\mathbb{E}$, then $\mathbb{E}(E, H-)$ is rigged.*

Then $H_{\bullet}$ has an induced structure of a morphism in $Q\mathbf{Prof}$.

Proof. By Lemma 5.11 and the first assumption, $H_\bullet$ takes each object of $\mathbb{E}$ to a $\mathcal{Q}$-coalgebra. Since $\mathcal{Q}_\lambda$ is weakly idempotent, $H_\bullet$ then necessarily takes each loose morphism of $\mathbb{E}$ to a (loose) weak $\mathcal{Q}$-morphism. Finally, by Lemma 5.5 and the fact that all representables are rigged, the second assumption implies that $H_\bullet$ takes tight morphisms to (tight) strict $\mathcal{Q}$-morphisms. $\quad\square$

Corollary A.11. *If $H : \mathbb{D} \hookrightarrow \mathbb{E}$ satisfies the hypotheses of Lemma A.10 and $\Phi : \mathbb{D} \to \mathbb{F}$ is a $\mathcal{Q}^{\mathbb{D}}$-coalgebra, then $\mathrm{Lan}_H \Phi : \mathbb{E} \to \mathbb{F}$ is a $\mathcal{Q}^{\mathbb{E}}$-coalgebra. Moreover, if Φ is rigged, so is $\mathrm{Lan}_H \Phi$.*

Proof. If Φ is identified with a profunctor in $\mathcal{Q}\mathbf{Prof}$ from $\mathbb{D}$ to the unit $\mathscr{F}$-category, then $\mathrm{Lan}_H \Phi$ can be identified with $H_\bullet \otimes_{\mathbb{D}} \Phi$, with $\mathcal{Q}$-coalgebra structure from Lemma A.5. The second statement is straightforward using the fact that colimits in $\mathscr{F}$, including left Kan extensions, are obtained as the full-embedding reflections of colimits in $\mathbf{Cat}^2$. $\quad\square$

Corollary A.12. *If $H : \mathbb{D} \hookrightarrow \mathbb{E}$ satisfies the hypotheses of Lemma A.10, then we have an isomorphism $H^\bullet \otimes_{\mathbb{E}} H_\bullet \cong 1_{\mathbb{D}}$ in $\mathcal{Q}\mathbf{Prof}$.*

Proof. Since H is fully faithful, we have such an isomorphism in $\mathscr{F}\mathbf{Prof}$. And since the profunctor $1_{\mathbb{D}}$ is pointwise rigged (as observed after Lemma 5.11), it has a unique $\mathcal{Q}$-coalgebra structure; thus the isomorphism lies in $\mathcal{Q}\mathbf{Prof}$. $\quad\square$

Finally, recall that given any profunctor $\Psi : \mathbb{D} \nrightarrow \mathbb{E}$, its *collage* is the category $|\Psi|$ whose objects are the disjoint union of those of $\mathbb{D}$ and $\mathbb{E}$, and whose morphisms are

$$|\Psi|(d, d') = \mathbb{D}(d, d'),$$

$$|\Psi|(e, e') = \mathbb{E}(e, e'),$$

$$|\Psi|(e, d) = \Psi(e, d),$$

$$|\Psi|(d, e) = 0.$$

We can now prove the loose part of the universal property.

Theorem A.13. *Let $\Phi : \mathbb{D} \to \mathbb{F}$ be an $\mathscr{F}$-weight which is a $\mathcal{Q}_c$-coalgebra, let T be an $\mathscr{F}$-monad on a complete $\mathscr{F}$-category $\mathbb{K}$, and let $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_w$ be an $\mathscr{F}$-functor. Then the T-algebra structure on $\{\Phi, UG\}$ obtained from Proposition A.1 gives it the universal property of the limit $\{\Phi_\lambda, G_\lambda\}$ in the 2-category $(T\text{-}\mathbb{Alg}_w)_\lambda$.*

Proof. Let $\mathbf{L} = (L, \ell)$ be the T-algebra constructed in Proposition A.1. We must exhibit a Φ-weighted cone $\eta : \Phi \to T\text{-}\mathbb{Alg}_w(\mathbf{L}, G)$ such that for any loose Φ-weighted cone $\alpha : \Phi_\lambda \to T\text{-}\mathbb{Alg}_w(\mathbf{A}, G_\lambda)$ there is a unique factorization $\alpha' : \mathbf{A} \to \mathbf{L}$ (plus a similar unique factorization of 2-cells). We do this by first specifying a factorization α' for each α, then showing that this is natural in $\mathbf{A}$, in the sense that for any loose map $f : \mathbf{B} \to \mathbf{A}$ in $T\text{-}\mathbb{Alg}_w(\mathbf{A}, G_\lambda)$, the assigned factorization of the loose Φ-weighted cone

$$\Phi_\lambda \xrightarrow{\ \alpha\ } T\text{-}\mathbb{Alg}_w(\mathbf{A}, G_\lambda) \xrightarrow{\ T\text{-}\mathbb{Alg}_w(f, G_\lambda)\ } T\text{-}\mathbb{Alg}_w(\mathbf{B}, G_\lambda)$$

is the composite $\alpha' f : \mathbf{B} \to \mathbf{A}$. Finally we show that the specified factorization of the loose Φ-weighted cone $\eta_\lambda : \Phi_\lambda \to T\text{-}\mathbb{A}\mathrm{lg}_w(\mathbf{L}, G_\lambda)$ is the identity. This gives the uniqueness of the factorization.

The strategy is to define various auxiliary $\mathscr{F}$-categories, $\mathscr{F}$-functors out of which describe the various structures (weak cones, factorizations, etc.) involved in the previous paragraph. These are listed below.

- Let $\mathbb{E}$ be the collage of Φ. An $\mathscr{F}$-functor $\mathbb{E} \to \mathbb{A}$ consists of an $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{A}$, an object $A \in \mathbb{A}$, and a Φ-weighted cone $\Phi \to \mathbb{A}(A, G)$. We write $*$ for the object of $\mathbb{E}$ not in $\mathbb{D}$.
- Let $\mathbb{D}^l$ be the collage of the weight $\Lambda(\Phi)$, defined by $\Lambda(\Phi)_\lambda = \Phi_\lambda$ and $\Lambda(\Phi)_\tau = 0$. An $\mathscr{F}$-functor $\mathbb{D}^l \to \mathbb{A}$ consists of an $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{A}$, and object $A \in \mathbb{A}$, and a loose Φ-weighted cone; that is, a 2-natural $\Phi_\lambda \to \mathscr{A}_\lambda(A, G_\lambda)$. We write x for the object of $\mathbb{D}^l$ not in $\mathbb{D}$.
- Let $\mathbb{E}^l$ be the $\mathscr{F}$-category obtained from $\mathbb{E}$ by adjoining an object x and a loose morphism $x \rightsquigarrow *$. An $\mathscr{F}$-functor $\mathbb{E}^l \to \mathbb{A}$ consists of an $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{A}$, objects $A, B \in \mathbb{A}$, a Φ-weighted cone $\eta : \Phi \to \mathbb{A}(A, G)$, and a loose morphism $f : B \rightsquigarrow A$. (Of course there is then an induced loose Φ-weighted cone $\Phi_\lambda \to \mathscr{A}_\lambda(B, G_\lambda)$, and which factorizes through η by f.)
- Let $\mathbb{D}^{ll}$ be the $\mathscr{F}$-category obtained from $\mathbb{D}^l$ by adjoining an object y and a loose morphism $y \rightsquigarrow x$. An $\mathscr{F}$-functor $\mathbb{D}^l \to \mathbb{A}$ consists of an $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{A}$, objects $A, B \in \mathbb{A}$, a loose Φ-weighted cone $\Phi_\lambda \to \mathscr{A}_\lambda(A, G_\lambda)$, and a loose morphism $B \rightsquigarrow A$. (Once again this induces a second loose Φ-weighted cone which factorizes through the first by the morphism $B \rightsquigarrow A$.)
- Finally let $\mathbb{E}^{ll}$ be the $\mathscr{F}$-category obtained from $\mathbb{E}^l$ by adjoining an object y and a loose morphism $y \rightsquigarrow x$. An $\mathscr{F}$-functor $\mathbb{E}^{ll} \to \mathbb{A}$ consists of an $\mathscr{F}$-functor $G : \mathbb{D} \to \mathbb{A}$, objects $A, B, C \in \mathbb{A}$, a Φ-weighted cone $\eta : \Phi \to \mathbb{A}(A, G)$ and loose morphisms $C \rightsquigarrow B \rightsquigarrow A$.

There is a diagram of fully faithful $\mathscr{F}$-functors

$$
\begin{array}{ccccc}
\mathbb{D} & \xrightarrow{\ M\ } & \mathbb{D}^l & \overset{M^x}{\underset{M^y}{\rightrightarrows}} & \mathbb{D}^{ll} \\
{\scriptstyle H}\downarrow & & {\scriptstyle H^l}\downarrow & & \downarrow{\scriptstyle H^{ll}} \\
\mathbb{E} & \xrightarrow[\ N\ ]{} & \mathbb{E}^l & \overset{N^x}{\underset{N^y}{\rightrightarrows}} & \mathbb{E}^{ll}
\end{array}
$$

all of which are literal inclusions except for M^y and N^y; these each send x to y and fix all other objects. There is also an $\mathscr{F}$-functor $K : \mathbb{D}^l \to \mathbb{E}$ sending x to $*$ and satisfying $KM = H$; and there is an $\mathscr{F}$-functor $P : \mathbb{E}^l \to \mathbb{E}$ satisfying $PN = 1$ and $P(x) = *$.

We now want to apply Lemma A.10 to conclude that the profunctors $H_\bullet$, $(H^l)_\bullet$, and $(H^{ll})_\bullet$ all lie in $\mathcal{Q}\mathbf{Prof}$. In all three cases, the only object we have to worry about is $*$, and there are no nonidentity tight morphisms with this target, so the second condition of Lemma A.10 is vacuous.

For $H_\bullet$, the weight mentioned in the first condition is just Φ, which is assumed to be a $\mathcal{Q}$-coalgebra. In the other two cases, the weight in question is the left Kan extension of Φ to $\mathbb{D}^l$ or $\mathbb{D}^{ll}$, respectively. Thus, by Corollary A.11 it suffices to show that the functors M and $M^x M = M^y M$ satisfy the hypotheses of Lemma A.10. In both cases, the second condition is

again vacuous, while the weight we have to check for the first condition is $\Lambda(\Phi)$, and this is easily seen to inherit a $\mathcal{Q}$-structure from Φ.

Thus, $H_\bullet$, $(H^l)_\bullet$, and $(H^{ll})_\bullet$ all lie in $\mathcal{Q}\mathbf{Prof}$. In particular, for any diagram $G : \mathbb{D} \to T\text{-}\mathbb{Alg}_l$, we have an induced diagram $\overline{\{H_\bullet, G\}} : \mathbb{E} \to T\text{-}\mathbb{Alg}_l$. By Corollary A.12 and Lemma A.8, the restriction of $\overline{\{H_\bullet, G\}}$ to $\mathbb{D}$ is G; hence it is a Φ-weighted cone over G. We aim to show that it is a limit cone.

Given any $\Lambda(\Phi)$-weighted cone over G, seen as an $\mathscr{F}$-functor $F : \mathbb{D}^l \to T\text{-}\mathbb{Alg}_w$, we have a canonical diagram

$$\overline{\left\{\left(H^l\right)_\bullet, F\right\}} : \mathbb{E}^l \to T\text{-}\mathbb{Alg}_l.$$

As before, by Corollary A.12 and Lemma A.8, the restriction of $\overline{\{(H^l)_\bullet, F\}}$ along H^l gives us F back again; hence $\overline{\{(H^l)_\bullet, F\}}$ is actually a loose factorization of F through some Φ-weighted cone. Now we claim that the following diagram of profunctors commutes (up to isomorphism) in $\mathcal{Q}\mathbf{Prof}$.

$$
\begin{array}{ccc}
\mathbb{D}^l & \overset{M^\bullet}{\longrightarrow} & \mathbb{D} \\
{\scriptstyle (H^l)_\bullet}\downarrow & & \downarrow{\scriptstyle H_\bullet} \\
\mathbb{E}^l & \underset{N^\bullet}{\longrightarrow} & \mathbb{E}
\end{array}
\tag{18}
$$

If this is so, then since $M^\bullet$ and $N^\bullet$ are given by restriction, we will be able to conclude that the above Φ-weighted cone through which F factors is actually the putative limit cone $\overline{\{H_\bullet, G\}}$.

We leave to the reader the proof that (18) commutes in $\mathscr{F}\mathbf{Prof}$ (and we will do likewise for all future such assertions). For commutativity in $\mathcal{Q}\mathbf{Prof}$, it remains to check that the $\mathcal{Q}$-coalgebra structures coincide. But by Corollary A.9, for any $E \in \mathbb{E}$, the $\mathcal{Q}^{\mathbb{D}^l}$-coalgebra structure of $(N^\bullet \otimes (H^l)_\bullet)(E)$ is that induced by Lemma A.10 applied to H^l, restricted to $\mathbb{E}$. When $E \in \mathbb{D}$, this is the unique structure of a representable, while for $E = *$ we took it to be the left Kan extension of Φ (according to Corollary A.11). But this left Kan extension is exactly what $H_\bullet \otimes M^\bullet$ computes. Hence the $\mathcal{Q}$-coalgebra structures agree, and (18) commutes in $\mathcal{Q}\mathbf{Prof}$.

We have shown that any $\Lambda(\Phi)$-weighted cone over G factors through $\overline{\{H_\bullet, G\}}$ in a specified way; we next show the naturality of these specified factorizations. Let $\alpha : F \rightsquigarrow F'$ be a loose morphism of $\Lambda(\Phi)$-weighted cones over G. We can regard this as a diagram $F^{(2)}$ of shape $\mathbb{D}^{ll}$, and then form the $\mathbb{E}^{ll}$-diagram $\overline{\{(H^{ll})_\bullet, F^{(2)}\}}$. As before, Corollary A.12 and Lemma A.8 imply that restricting this diagram along H^{ll} gives us back $F^{(2)}$, so that it consists of loose factorizations of F and F' through some Φ-weighted cone, and moreover these factorizations commute with α. Now by an argument just like that given above for (18), we can conclude that the diagrams of profunctors

$$
\begin{array}{ccc}
\mathbb{D}^{ll} & \overset{(M^x)^\bullet}{\longrightarrow} & \mathbb{D}^l \\
{\scriptstyle (H^{ll})_\bullet}\downarrow & & \downarrow{\scriptstyle (H^l)_\bullet} \\
\mathbb{E}^{ll} & \underset{(N^x)^\bullet}{\longrightarrow} & \mathbb{E}^l
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathbb{D}^{ll} & \overset{(M^y)^\bullet}{\longrightarrow} & \mathbb{D}^l \\
{\scriptstyle (H^{ll})_\bullet}\downarrow & & \downarrow{\scriptstyle (H^l)_\bullet} \\
\mathbb{E}^{ll} & \underset{(N^y)^\bullet}{\longrightarrow} & \mathbb{E}^l
\end{array}
$$

commute in $\mathcal{Q}\mathbf{Prof}$. This implies that the Φ-weighted cone appearing in $\overline{\{(H^{ll})_\bullet, F^{(2)}\}}$ must be $\overline{\{H_\bullet, G\}}$, and the factorizations of F and F' through it must be those produced by $\overline{\{(H^l)_\bullet, -\}}$.

Thus, $\overline{\{(H^l)_\bullet, -\}}$ gives us a natural transformation from the identity functor of the 1-category of $\Lambda(\Phi)$-weighted cones over G to the functor constant at $\overline{\{H_\bullet, G\}}$. As is well-known, to conclude from this that $\overline{\{H_\bullet, G\}}$ is a terminal object of this category (and hence that factorizations through it are unique), it suffices to check that the component of this transformation at $\overline{\{H_\bullet, G\}}$ itself is the identity. This will follow if we can show that the following diagram of profunctors commutes in $\mathcal{Q}\mathbf{Prof}$.

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\;H_\bullet\;} \mathbb{E} \xrightarrow{\;K^\bullet\;} & \mathbb{D}^l \\
& & \downarrow{\scriptstyle (H^l)_\bullet} \\
& \mathbb{E} \xrightarrow{\;P^\bullet\;} & \mathbb{E}^l
\end{array} \tag{19}
$$

To show this, first note that for all $D \in \mathbb{D}$, $(H^l)_\bullet(D) \cong \mathbb{D}^l(D, -)$, while $(H^l)_\bullet(x) \cong \mathbb{D}^l(x, -)$. Since left Kan extension preserves representables, we also have

$$
\begin{aligned}
\big((H^l)_\bullet \otimes K^\bullet\big)(D) &\cong \mathbb{E}(D, -) \quad \text{and} \\
\big((H^l)_\bullet \otimes K^\bullet\big)(x) &\cong \mathbb{E}\big(K(x), -\big) = \mathbb{E}(*, -).
\end{aligned}
$$

But by definition we also have $P^\bullet(D) = \mathbb{E}(D, -)$ and $P^\bullet(x) = \mathbb{E}(P(x), -) = \mathbb{E}(*, -)$, so that

$$
\big((H^l)_\bullet \otimes K^\bullet\big)(E) \cong P^\bullet(E)
$$

as $\mathcal{Q}$-coalgebras for all $E \neq *$, and this remains so after composing with $H_\bullet$.

It remains to deal with $E = *$. By definition, $(H^l)_\bullet(*)$ is the left Kan extension of Φ to $\mathbb{D}^l$, with $\mathcal{Q}$-coalgebra structure as in Corollary A.11; which is to say $(H^l)_\bullet(*) \cong \Phi \otimes M^\bullet$. Thus, we also have

$$
\big((H^l)_\bullet \otimes K^\bullet\big)(*) \cong \Phi \otimes M^\bullet \otimes K^\bullet \cong \Phi \otimes H^\bullet
$$

and hence

$$
\big((H^l)_\bullet \otimes K^\bullet \otimes H_\bullet\big)(*) \cong \Phi \otimes H^\bullet \otimes H_\bullet \cong \Phi
$$

as $\mathcal{Q}$-coalgebras (using Corollary A.12). But $P^\bullet(*) = \mathbb{E}(*, -)$, and so also

$$
\big(P^\bullet \otimes H_\bullet\big)(*) \cong \mathbb{E}\big(*, H(-)\big) = \Phi
$$

as $\mathcal{Q}$-coalgebras, since the structure of $H_\bullet$ as a morphism in $\mathcal{Q}\mathbf{Prof}$ is induced by Lemma A.10 from the $\mathcal{Q}$-coalgebra structure of Φ. This shows that (19) commutes in $\mathcal{Q}\mathbf{Prof}$, and hence factorizations of $\Lambda(\Phi)$-weighted cones through $\overline{\{H_\bullet, G\}}$ are unique.

To complete the loose part of the universal property of $\overline{\{H_\bullet, G\}}$, we need to deal with 2-cells. Let $\mathbb{E}_2^l$ be the $\mathscr{F}$-category obtained from $\mathbb{E}$ by adjoining an object x and two loose morphisms $x \rightrightarrows *$ with a 2-cell between them. Thus a diagram $\mathbb{E}_2^l \to \mathbb{K}$ consists of $G : \mathbb{D} \to \mathbb{K}$,

a cone $\Phi \to \mathbb{K}(L, G)$, and a parallel pair of loose morphisms $A \rightrightarrows L$ with a 2-cell between them (inducing a transformation between two loose cones $\Phi_\lambda \to \mathcal{K}(A, G)$).

Let $\mathbb{D}_2^l$ be the full subcategory of $\mathbb{E}_2^l$ on all the objects except $*$; thus a diagram of shape $\mathbb{D}_2^l$ is a 2-cell between two $\Lambda(\Phi)$-weighted cones over the same diagram. We have another diagram of fully faithful functors:

$$
\begin{array}{ccc}
\mathbb{D} & \xrightarrow{\;M_2\;} & \mathbb{D}_2^l \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H_2^l} \\
\mathbb{E} & \xrightarrow[\;N_2\;]{} & \mathbb{E}_2^l
\end{array}
$$

We claim that H_2^l satisfies the hypotheses of Lemma A.10. The second condition is again vacuous, and the relevant weight for the first condition is the left Kan extension of Φ to $\mathbb{D}_2^l$ along M_2; thus it suffices for M_2 to satisfy the hypotheses of Lemma A.10. In this case the weight we have to check is $\Lambda(\Phi) \times 2$, which becomes a $\mathcal{Q}$-coalgebra by lifting the colimit $(- \times 2)$ to the category of $\mathcal{Q}$-coalgebras (as we can do for any comonad). Thus, by Lemma A.10, $(H_2^l)_\bullet$ lies in $\mathcal{Q}$**Prof**.

Now applying Corollary A.12 and Lemma A.8 again, $\{(H_2^l)_\bullet, -\}$ shows that any 2-cell between $\Lambda(\Phi)$-weighted cones factors through some specified Φ-weighted cone. This Φ-weighted cone will be $\overline{\{H_\bullet, G\}}$ if we can show that the following diagram of profunctors commutes in $\mathcal{Q}$**Prof**.

$$
\begin{array}{ccc}
\mathbb{D}_2^l & \xrightarrow{\;(M_2)^\bullet\;} & \mathbb{D} \\
{\scriptstyle (H_2^l)_\bullet}\downarrow & & \downarrow{\scriptstyle H_\bullet} \\
\mathbb{E}_2^l & \xrightarrow[\;(N_2)^\bullet\;]{} & \mathbb{E}
\end{array}
$$

As before, by restriction and the definition of the $\mathcal{Q}$-coalgebra structure of $(H_2^l)_\bullet$, we have that $((N_2)^\bullet \otimes (H_2^l)_\bullet)(*)$ is the left Kan extension of Φ along M_2 with structure as in Corollary A.11; but this is exactly what $(H_\bullet \otimes (M_2)^\bullet)(*)$ computes.

Thus, any 2-cell between $\Lambda(\Phi)$-weighted cones factors through $\overline{\{H_\bullet, G\}}$. The uniqueness of this factorization is automatic since $T\text{-}\mathbb{Alg}_l \to \mathbb{K}$ is faithful on 2-cells. $\quad\square$

As we saw in Section 5.3, the hypotheses of Theorem A.13 are not strong enough to conclude that the T-algebra structure induced on $\{\Phi, UG\}$ is actually the Φ-weighted $\mathscr{F}$-limit of G; for that we need Φ to be w-rigged.

Theorem A.14. *If Φ is a w-rigged $\mathscr{F}$-weight, then for any $\mathscr{F}$-monad T on an $\mathscr{F}$-category $\mathbb{K}$, the forgetful functor $U_w : T\text{-}\mathbb{Alg}_w \to \mathbb{K}$ creates Φ-weighted limits.*

Proof. We must show that the limit projections corresponding to Φ_τ are tight and detect tightness in $T\text{-}\mathbb{Alg}_l$. They are certainly tight, since $\overline{\{H_\bullet, G\}}$ is a Φ-weighted cone and not merely a $\Lambda(\Phi)$-

weighted one. To show that they detect tightness, we continue the pattern of argument from the proof of Theorem A.13.

Let $\mathbb{E}$ be the collage of Φ, as before, with $H : \mathbb{D} \to \mathbb{E}$ the inclusion, and let $\mathbb{E}^t$ be the $\mathscr{F}$-category obtained from $\mathbb{E}$ by adjoining an object z and a *tight* morphism $z \to *$. Thus an $\mathscr{F}$-functor $\mathbb{E}^t \to \mathbb{A}$ consists of a tight factorization of one Φ-weighted cone through another.

Let $V : \mathbb{E} \to \mathbb{E}^t$ be the inclusion (which in particular sends $*$ to $*$), and let $H^t : \mathbb{E} \to \mathbb{E}^t$ be the evident functor satisfying $H^t H = V H$ and $H^t(*) = z$. Then H^t satisfies the conditions of Lemma A.10: the one weight we have to worry about is the left Kan extension of Φ to $\mathbb{E}$, which is rigged by Corollary A.11. Thus, $(H^t)_\bullet$ lies in $\mathcal{Q}\mathbf{Prof}$.

Therefore, using Corollary A.12 and Lemma A.8 as before, we conclude that $\overline{\{(H^t)_\bullet, -\}}$ factors any Φ-weighted cone over G through some other specified Φ-weighted cone. To show that the latter cone is in fact $\overline{\{H_\bullet, G\}}$, it suffices to show that the following diagram of profunctors commutes in $\mathcal{Q}\mathbf{Prof}$:

$$
\begin{array}{ccc}
\mathbb{E} & \xrightarrow{H^\bullet} & \mathbb{D} \\
{\scriptstyle (H^t)_\bullet}\Big\downarrow & & \Big\downarrow{\scriptstyle H_\bullet} \\
\mathbb{E}^t & \xrightarrow[V^\bullet]{} & \mathbb{E}
\end{array}
$$

But since all the weights in question are now rigged, this is automatic from its commutativity in $\mathscr{F}\mathbf{Prof}$, by Lemma 5.5. $\quad\square$

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