A Topology for Automata: A Note

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Let $A = (Q, X, \delta)$ be an $X$-automaton, with $Q$ its state set. For a subset $B \subseteq Q$, the source of $B$ is defined as $sB = \{q \in Q \mid \delta(q, x) \in B \text{ for some } x \in X\}$. The source turns out to be a closure operator for $Q$ and defines a topology on $Q$, viz. $B \subseteq Q$ is closed iff $B = sB$. It is shown that many automata-theoretic concepts, e.g., separation, connectivity, retrievability, strong connectivity, etc., have standard topological analogs under this topology and many results concerning these concepts are direct consequences of this observation.

Connectivity and separation in the sense of automata theory and other related properties, e.g., strong connectivity and retrievability, have been studied, among others, by Bavel (1971a, 1971b), who also gives characterizations of such properties. In this note we observe that these notions have topological analogs through a functorial passage from automata theory to topology. This is not particularly difficult and may not add anything substantially new to our present knowledge about automata, but the use of topological machinery, in a large number of cases, reduces our efforts in proving automata-theoretic results considerably.

Parentheses and subscripts will be used only when necessary.

An $X$-automaton is a triple $A = (Q, X, \delta)$, where $Q$ is a set of states; $X$ is semigroup with identity $e$, called the input alphabet; and $\delta: Q \times X \to Q$ is the transition map satisfying $\delta(q, e) = q$, for all $q \in Q$ and for all $x, y \in X$, $\delta(q, xy) = \delta(\delta(q, x), y)$. A triple $B = (Q', X, \delta')$ is called a subautomaton of $A$ iff $Q' \subseteq Q$ and $\delta'$ is the restriction of $\delta$ to $Q' \times X$ (we shall use $\delta$ for its restriction $\delta'$ when no ambiguity arises). For a subset $Q' \subseteq Q$, the set of successors of $Q'$ is $\delta Q' = \{\delta(q', x) \mid q' \in Q' \text{ and } x \in X\}$. The automaton generated by $Q' \subseteq Q$ is $\langle Q' \rangle = (\delta Q', X, \delta)$; $Q'$, then, is a generating set of $\langle Q' \rangle$. A subautomaton $B = (Q', X, \delta)$ of $A$ is separated iff $\delta(Q - Q') \cap Q' = \emptyset$ and is connected iff it has no separated proper subautomaton.

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An automaton $A$ is strongly connected iff for all $p, q \in Q, p \in \delta q$, is retrievable iff for all $(q, x) \in Q \times X$, there exists $y \in X$ such that $\delta(q, xy) = q$; it is discrete iff for all $q \in Q, \delta q = \{q\}$. A subautomaton $B$ of $A$ is a block of $A$ iff $B$ is connected and separated. For a subset $B$ of $Q$, the source of $B$ is $\sigma B = \{q \in Q \mid \delta(q, x) \in B \text{ for some } x \in X\}$. If $(Q, X, \delta)$ and $(P, X, \gamma)$ are two $X$-automata, a function $f: Q \to P$ is a homomorphism iff for all $(q, x) \in Q \times X, f(\delta(q, x)) = \gamma(fq, x)$. When, in addition, the function $f$ is one–one and onto, $f$ is an isomorphism.

Topological concepts that we make use of are standard and can be found, e.g., in Kelley (1955).

Let $B, B_1$, and $B_2$ be subsets of $Q$. Bavel (1971b), observed that

(a) $\sigma \emptyset = \emptyset$,
(b) $\sigma(B_1 \cup B_2) = \sigma B_1 \cup \sigma B_2$,
(c) $B \subseteq \sigma B$,
(d) $\sigma \sigma B = \sigma B$.

These four properties are precisely those which make $\sigma$ a closure operator for $Q$ and thus equip $Q$ with a topology $tA$ for which a subset $B$ of $Q$ is closed iff $B = \sigma B$ and consequently a subset $C \subseteq Q$ is open for this topology just when $Q - C$ is closed.

Most observations made hereafter offer little or no resistance in their verification.

A topology is called saturated\(^1\) (Lorrain, 1969) iff any union of closed sets is itself closed or equivalently, iff any intersection of open sets is itself open. Our $tA$ is one such topology. Indeed, if $\{F_i \mid i \in I\}$ is any family of $tA$-closed sets and $F = \bigcup_{i \in I} F_i$, then $q \notin F$ means that for no $i \in I$, $q$ can be in $F_i = \sigma F_i$. Thus, for no $x \in X, \delta(q, x) \in \bigcup F_i = F$, showing that $q \notin \sigma F$. Hence $\sigma F = F$ and $tA$ is saturated.

It is not hard to check that a subset $U \subseteq Q$ is $tA$-open iff it contains all its successors under the transition map $\delta$, i.e., iff $U = \delta U$. Since $tA$ is saturated, it makes sense to talk of the “minimum” neighborhood of any subset $B$ of $Q$; it is the intersection of all open sets containing $B$ (which turns out to be just $\delta B$). In particular, the minimum neighborhood of a point $q \in Q$ is $\delta q$. Incidentally, these sets $\delta q, q \in Q$, form a basis for the topology $tA$. That $tA$ is saturated offers another convenience: All $tA$-closed

\(^1\) These are precisely the finitely generated spaces of Herrlich and Strecker (1973), which are topological spaces that can be expressed as a topological quotient of a disjoint union of finite topological spaces or equivalently, for which the closure of a subset equals the union of closures of each point of that subset.
sets are all the open sets for another topology on $Q$ which we denote by $tA^*$. The closure operator for $tA^*$ is precisely the successor operator $\delta$. Moreover, $tA$ and $tA^*$ are dual in the sense that $tA^{**} = tA$—a fact which seems to assign precise meaning to a statement by Bavel (1971b) viz. the operator $\delta$ is complementary to the operator $\sigma$. In principle, therefore, most results about the source operator $\sigma$ can have their analogs (the dual results) in terms of the successor operator $\delta$; this is amply demonstrated by comparing Bavel (1971a) and Bavel (1971b).

To each $X$-automaton $A = (Q, X, \delta)$, we have now associated a topological space $(Q, tA)$. If $A = (Q, X, \delta)$ and $B = (P, X, \gamma)$ are two $X$-automata and $f: A \to B$ is a homomorphism then $f: (Q, tA) \to (P, tB)$ is a continuous map. Indeed, if $F \subseteq P$ is any $tB$-closed set and $q \in \sigma_B(f^{-1}F)$ then for some $x \in X$, $\delta(q, x) \in f^{-1}F$, i.e., $f(\delta(q, x)) \in F = \sigma_BF$ i.e., $\gamma(fq, x) \in F$, showing that $fq \in \sigma_BF = F$. This shows that $q \in f^{-1}F$ and $f^{-1}F$ is closed. Thus, under the transformation $(Q, X, \delta) \mapsto (Q, tA)$, homomorphisms go to continuous maps. Actually this correspondence, in the language of category theory (Herrlich and Strecker, 1973), defines a faithful functor from the category of $X$-automata and their homomorphisms to the category of topological spaces and continuous functions.

A quick look at the definition of a subautomaton reveals that $(Q', X, \delta)$ is a subautomaton of $A = (Q, X, \delta)$ iff $Q'$ is $tA$-open and that it is a separated subautomaton iff $Q'$ is simultaneously closed and open (in short, clopen) —the latter fact in view of the following argument. Recall that $(Q', X, \delta)$ is a separated subautomaton iff $\delta(Q - Q') \cap Q' = \emptyset$. We must show that this implies closedness, and conversely. But to accomplish this we must observe that $\delta(Q - Q') \cap Q' = \emptyset$ iff $(Q - Q') \cap \sigma Q' = \emptyset$.

A connected automaton $A = (Q, X, \delta)$ is one which contains no separated proper subautomaton; viewed topologically, it is the one such that the associated topological space $(Q, tA)$ has no proper clopen subset. But this is precisely what is needed for $(Q, tA)$ to be connected in the sense of topology. The blocks of $A$ are just connected components of $(Q, tA)$. Most of Bavel's (1971a,b) results concerning separated subautomata, connected automata, or blocks, from a topological point of view, are either standard or can be obtained much more easily. For example, (a) A subautomaton $B = (Q', X, \delta)$ of $A$ is separated iff $\sigma Q' = Q'$, (b) for a subset $Q' \subseteq Q$, $\delta \sigma Q' = \sigma Q' = (\sigma Q', X, \delta)$ is a separated subautomaton of $A$ and $\sigma \delta Q' = \delta Q' = (\delta Q', X, \delta)$ is a separated subautomaton (Bavel, 1971b), or (c) Two blocks of a nonempty automaton are either identical or disjoint, (d) a nonempty automaton $A$ is not connected iff there exist two nonempty subsets, $Q_1$ and $Q_2$, of $Q$, such that $(\delta Q_1, X, \delta) \cap (\delta Q_2, X, \delta) = \emptyset$ and $(\delta Q_1, X, \delta) \cup (\delta Q_2, X, \delta) = A$.
(Bavel, 1971a), are all standard topologically, in view of the above considerations.

If $A = (Q, X, \delta)$ is a strongly connected automaton then the $tA$-closure of every state $q \in Q$ is the whole space $Q$ since $q$ is a successor of every other state of $Q$; put differently, $tA$ is an indiscrete topology. This property characterizes strongly connected automata and again standard topological results about such spaces help one recover known results about strongly connected automata. Discrete automata, on the other hand, correspond to discrete topological spaces.

A retrievable automaton $A = (Q, X, \sigma)$, as defined above, is one for which, given $q \in Q$ and $x \in X$, there exists $y \in X$ such that $\delta(q, xy) = q$. Thus, if $q_1$ and $q_2$ are two states of such an automaton and if $q_1 \in \sigma q_2$ then $\delta(q_1, x) = q_2$ for some $x \in X$ and so for some $y \in X$, $\delta(q_2, y) = q_1$, showing that $q_2 \in \sigma q_1$. Conversely, if for any automaton $A$, it is the case that $q_1 \in \sigma q_2 \Rightarrow q_2 \in \sigma q_1$ for any two states $q_1$ and $q_2$, then it is retrievable. A topological space $(S, T)$ is said to be an $R_0$-space or a symmetric space iff $s_1 \in \text{cl } s_2 \Rightarrow s_2 \in \text{cl } s_1$ for any two points $s_1, s_2$ of $S$. It follows now that $A = (Q, X, \delta)$ is retrievable iff $tA$ is an $R_0$-topology. $R_0$-spaces appear in Davis (1961), where they are also characterized. A characterization of $R_0$-spaces reads as follows. A topological space is $R_0$ iff each of its open sets contains the closure of each of its points. This agrees well with a characterization of retrievable automata by Bavel (1971b), viz., a nonempty automaton $A$ is retrievable iff $\sigma Q' = Q'$ for every subautomaton $B = (Q', X, \sigma)$ of $A$.

Since homomorphisms (isomorphisms) of automata correspond to continuous maps (homeomorphisms) between respective "state spaces," the following results of Bavel (1971b) from a topological point of view are standard: If $A = (Q, X, \delta)$ and $B = (P, X, \gamma)$ are two $X$-automata and $f: A \to B$ and $g: A \to B$ are homomorphism and isomorphism, respectively, then (a) $f\sigma Q' \subseteq \sigma fQ'$ and (b) $g\sigma Q' = \sigma gQ'$ for all subsets $Q'$ of $Q$. The case with a few other observations of Bavel (1971b) concerning isomorphisms, separation, connectivity, and blocks is also parallel.

In conclusion, it must be confessed that the association of topology with automata theory, as presented here, has not so far shown any significant results, but the possibility of its proving useful in the future cannot be ruled out. There is a possibility of a good trade-off between topology and automata theory. One direction of investigation has been pointed out in the present note: It consists of identifying automata-theoretic properties as known topological properties and then interpreting known results on these same topological properties in the context of automata theory. Thus, for example, one may start looking at various other connectivity and
reversibility properties of automata (see Bavel and Muller, 1970) from this point of view. Bavel (1971b) shows that an abelian\(^2\) automaton possesses the property that the state sets of any two of its disjoint subautomata have disjoint sources; in other words, its state set topology becomes extremally disconnected (a topology is extremally disconnected iff its disjoint open sets have disjoint closures or equivalently, iff the interior of every closed set is closed or equivalently, iff the closure of every open set is open). Bavel (1971b, Sect. 7) characterizes such finite state set topologies. Another obvious direction of investigation consists of using known topological results and properties to produce new automata-theoretic results. For example, it seems that in this context, quasi-compact topological spaces (i.e., spaces in which each open cover has a finite subcover) may be of interest. For saturated topological spaces, as observed by Lorrain (1969), quasi-compactness means that there exists a finite subset dense in the dual topology (a subset is called dense if its closure equals the whole space). If for an automaton \(A = (Q, X, \delta)\), its state set topology \(tA\) is quasi-compact then there must be a finite subset \(Q'\) of the state set \(Q\) such that \(\delta Q' = Q\); in other words every state \(q\) of such an automaton is of the form \(\delta(q', x)\) for \(q' \in Q'\) and \(x \in X\). Such automata may be called quasi-finite or finitely reachable for they generalize both finite automata and reachable automata.\(^3\) It may be noted that the only automata whose state set topologies are compact (i.e., quasi-compact with the property that distinct points have disjoint neighborhoods) are finite discrete automata.

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References


\(^2\) An automaton \(A = (Q, X, \delta)\) is abelian iff \(\delta(q, xy) = \delta(q, yx)\) for all \((q, x, y) \in Q \times X \times X\).

\(^3\) A reachable automaton \(A = (Q, X, \delta)\) is an automaton that has a distinguished state \(q_0\) (the initial state) of which every other state is a successor; equivalently it is one with a distinguished state \(q_0\), such that the mapping \(r: X \to Q\) defined by \(r(x) = \delta(q_0, x)\) is onto.

