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Connective Stability of Discontinuous Large Scale Systems

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The stability of discontinuous large scale systems under structural perturbations are studied in this paper. It is assumed that the discontinuous equations possess solutions in the sense of Filippov. The results obtained yield sufficient conditions for connective stability. The interconnected systems are treated in terms of their subsystems.

1. INTRODUCTION

In practice we frequently encounter systems which are most appropriately represented by discontinuous ordinary differential equations. Such equations are investigated qualitatively by Filippov [2], He [3], Michel and Porter [5, 6], and others. Many such systems often are complex and of a very large scale. It is possible to view such systems as consisting of several simpler subsystems which when interconnected in an appropriate fashion yield the original large scale systems. The behavior of such systems are studied by He [4], Michel and Porter [5, 6], etc.

In this paper, we develop the connective stability which was established for the continuous large scale systems by Siljak [10, 11] to the discontinuous large scale systems. Following He [4], the decomposition-aggregation method and comparison principle are used to derive sufficient conditions such that the trivial solution of the discontinuous large scale systems is connectively stable, uniformly, and connectively stable, respectively.

2. NOTATION

Let R^n be a real Euclidean space, $R_+ = [0, \infty)$, $0 \in D \subset R^n$, D is a bounded openset of R^n. Consider the discontinuous large scale system

$$\dot{x} = f(t, x), \tag{1}$$

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where $t \in R_+$, $f \in F(R_+ \times D, R^n)$, which denotes that f is defined and measurable almost everywhere (a.e.) in a domain $G \in R_+ \times D$, and for any closed bounded domain $Q \subset G \subset R_+ \times D$, it is assumed that there exists a summable function $m(t)$ such that $\|f(t, x)\| \leqslant m(t)$ a.e. in Q. In this case we say f satisfies the Filippov condition in $R_+ \times D$. The absolutely continuous solutions of (1) in the sense of Filippov [2], are denoted by $x(t; t_0, x_0)$ with $x(t_0) = x_0$. Furthermore, we assume that $f(t, 0) = 0$, $\forall t \in R_+$, and $x^* = 0$ is the unique equilibrium state of (1).

Let us decompose the state vector $x \in R^n$ into s vector components

$$x_i = (x_{i1}, x_{i2}, ..., x_{in_i})^T, \qquad i = 1, 2, ..., s,$$

where $x_i \in R^{n_i}$, $n = \sum_{i=1}^{s} n_i$. The scalar components f_i: $R_+ \times R^n \rightarrow R$ of the function $f(t, x)$ in (1) are further specialized as

$$f_i(t, x) \equiv \hat{f}_i(t, x_i, e_{i1} x_1, ..., e_{is} x_s), \tag{2}$$

where e_{ij} are elements of a given $s \times s$ interconnective matrix E (see [12]), $i \in S \overset{\text{def}}{=} \{1, 2, ..., s\}$. We formulate the following.

DEFINITION 1. The equilibrium $x^* = 0$ of (1) is connectively stable, if for each number $\varepsilon > 0$ there exists a number $\delta = \delta(t_0, \varepsilon)$, $t_0 \in R_+$, such that for the Filippov solutions $x(t; t_0, x_0)$ of (1) with $x(t_0) = x_0$, $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$, $\forall t \in J = [t_0, \infty)$ for all $E \in \bar{E}$, where $\bar{E}$ is the fundamental interconnection matrix of (1).

When for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ independent of t_0 such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$, $\forall t \in J$, or all $E \in \bar{E}$, then $x^* = 0$ is uniformly and connectively stable.

Assume that the system (1) can be decomposed into s interconnected subsystems described by the equations

$$\dot{x}_i = g_i(t, x_i) + h_i(t, x), \qquad i \in S, \tag{3}$$

where $x_i \in R^{n_i}$, $x = (x_1^T, x_2^T, ..., x_s^T) \in R^n$, $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$, g_i: $R_+ \times R^{n_i} \rightarrow R^{n_i}$ describe the decoupled subsystems

$$\dot{x} = g_i(t, x_i), \qquad i \in S \tag{4}$$

and the functions h_i: $R_+ \times R^n \rightarrow R^{n_i}$, $i \in S$ represent the interactions among the subsystems (3) which have the form

$$h_i(t, x) \equiv h_i(t, e_{i1} x_1, e_{i2} x_2, ..., e_{is} x_s), \qquad i \in S, \tag{5}$$

where again e_{ij} are elements of the $s \times s$ interconnection matrix E. Furthermore, we assume that for all $i \in S$, $g_i(t, 0) = 0$, $h_i(t, 0) = 0$, $\forall t \in J$.

When t is fixed, we define the essential supermum on a point $\bar{x}$ of a scalar function $u(t, x)$ as following

$$M_x\{u(t, \bar{x})\} = \lim_{\delta \to 0} \text{vrai} \max_{x \in U(\bar{x}, \delta)} u(t, x)$$

$$= \lim_{\delta \to 0} \inf_{\mu N = 0} \sup_{x \in U(\bar{x}, \delta) - N} u(t, x),$$

where $U(\bar{x}, \delta)$ denotes the δ neighbourhood of $\bar{x}$, and μN denotes the measure of set N (see [2]).

3. Main Results

THEOREM 1. *Suppose that for each subsystem* (3) *with* (5), $\forall i \in S$, *there exist functions* $\phi_{1i} \in K$ *class*, $v_i(t, x_i) \in C^1(J \times R^{n_i}, R)$, $\omega_i(t, y_i) \in F(J \times R, R)$ *with* $\omega_i(t, 0) \equiv 0$ *a.e.* $t \in J$, *and* $\eta_{ij}(t)$ *is Lebesgue integrable in* J, *such that for all* $E \in \bar{E}$ *the following conditions are satisfied*

(a) $\phi_{1i}(\|x_i\|) \leqslant v_i(t, x_i)$ *a.e.* $(t, x_i) \in J \times R^{n_i}$;

(b) $\partial v_i(t, x_i)/\partial t + (\partial v_i(t, x_i)/\partial x_i)\, g_i(t, x_i) \leqslant \omega_i(t, v_i(t, x_i))$ *a.e.* $(t, x_i) \in J \times R^{n_i}$;

(c) $(\partial v_i(t, x_i)/\partial x_i)\, h_i(t, e_{i1}x_1, e_{i2}x_2, ..., e_{is}x_s) \leqslant \sum_{j=1}^{s} \bar{e}_{ij}\eta_{ij}(t)\|x_i\|$ *a.e.* $(t, x) \in J \times R^{n_1} \times \cdots \times R^{n_s}$;

(d) *the trivial solution* $y^* = 0$ *of the comparison system*

$$\dot{y}_i = \omega_i(t, y_i) + \sum_{j=1}^{s} \bar{e}_{ij}\eta_{ij}(t)\|y_i\| \tag{6}$$

is stable (see [3]*).*

Then the trivial solution of large scale system (1) *with decomposition* (3) *is connectively stable.*

Proof. Since conditions (b) and (c) hold for all $E \in \bar{E}$, stability of $y^* = 0$ is connective. Therefore, if $0 < \varepsilon < \alpha_i$, $t_0 \in J$ are given, then for all $E \in \bar{E}$ and $\phi_{1i}(\varepsilon)$, there exists a positive number $\Delta = \Delta(t_0, \varepsilon)$ such that the right-hand maximum Filippov solution $\bar{y}_i(t; t_0, y_i^0)$ with $y_i(t_0) = y_i^0$ satisfies

$$\|\bar{y}_i(t; t_0, y_i^0)\| < \phi_{1i}(\varepsilon), \qquad t \in J. \tag{7}$$

provided $\|y_i^0\| < \Delta$.

For any $(t_0, x_i^0) \in R_+ \times R^n$, $x_i(t; t_0, x_i^0) \overset{\text{def}}{=} x_i$ denotes Filippov solution of system (3), along the solutions of (3) we have

$$\frac{dv_i(t, x_i(t; t_0, x_0))}{dt} = v_{it}(t, x_i(t; t_0, x_0)) + v_{ix_i}(t, x_i(t; t_0, x_i^0))\frac{dx_i}{dt} \tag{8}$$

a.e. $t \in J$. By [2, Lemma 2], we have

$$\begin{aligned} & v_{ix_i}(t, x_i(t)) \frac{dx_i(t)}{dt} \\ & \quad \leqslant M_{x_i}\{v_{ix_i}(t, x_i(t))[g_i(t, x_i + h_i(t, x)]\} \\ & \quad \leqslant M_{x_i}\{v_{ix_i}(t, x_i(t)\, g_i(t, x_i) + v_{ix_i}(t, x_i(t))\, h_i(t, e_{i1}x_1, ..., e_{is}x_s\} \\ & \quad \leqslant \text{vrai} \max_{x_i' \in U(x_i(t), \delta_i)} \left\{ v_{ix_i}(t, x_i'(t))\, g_i(t, x_i') + \sum_{j=1}^{s} \bar{e}_{ij}\eta_{ij}(t)\, \|x_i'(t)\| \right\} \end{aligned} \tag{9}$$

a.e. $t \in J$ and for any $\delta_i > 0$. By the definition of essential supremum and analogous to the method of Chen [1], we have

$$\frac{dv_i(t, x_i(t))}{dt} \leqslant M_{v_i}\left\{ \omega_i(t, v_i(t, x_i(t))) + \sum_{j=1}^{s} \bar{e}_{ij}\eta_{ij}(t)\, \|x_i(t)\| \right\}$$

a.e. $t \in J$. By [3, Lemma 1] we have

$$v_i(t, x_i(t)) \leqslant \bar{y}_i(t, t_0, y_i^0), \qquad t \in J.$$

We choose $y_i^0 = v_i(t_0, x_i^0)$. From (7) and condition (a) we have

$$\begin{aligned} \phi_{1i}(\|x_i(t; t_0, x_i^0\|) & \leqslant v_i(t, x_i(t; t_0, x_i^0)) \\ & \leqslant \bar{y}_i(t; t_0, y_i^0) < \phi_{1i}(\varepsilon), \qquad t \geqslant t_0. \end{aligned}$$

therefore, $\|x_i(t; t_0, x_i^0)\| < \varepsilon$ for $t \in J$ and all $E \in \bar{E}$, provided $\|x_i^0\| < \delta_i$, this completes the proof.

THEOREM 2. *Suppose for each subsystem* (3), $i \in S$, *there exists functions* $\phi_{1i}, \phi_{2i} \in K$ *class and* v_i, ω_i, η_{ij} *same as in Theorem* 1, *such that for all* $E \in \bar{E}$, *the following conditions are satisfied*

(a) $\phi_{1i}(\|x_i\|) \leqslant v_i(t, x_i) \leqslant \phi_{2i}(\|x_i\|)$ *a.e.* $(t, x_i) \in J \times R^n$;

(b) $\partial v_i(t, x_i)/\partial t + (\partial v_i(t, x_i)/\partial x_i)\, g_i(t, x_i) \leqslant \omega_i(t, v_i(t, x_i))$ *a.e.* $(t, x_i) \in J \times R^{n_i}$;

(c) $(\partial v_i(t, x_i)/\partial x_i)\; h_i(t, e_{i1}x_1, ..., e_{is}x_s) \leqslant \sum_{j=1}^{s} \bar{e}_{ij}\eta_{ij}(t)\; \|x_i\|$ *a.e.* $(t, x) \in J \times R^n$;

(d) *the trivial solution* $y^* = 0$ *of the comparison system* (6) *is uniformly stable.*

Then the trivial solution of discontinuous large scale system (1) *with* (3) *is uniformly and connectively stable.*

The proof of Theorem 2 is similar to that of Theorem 1, here we omit it.

4. Conclusions

The connective stability of discontinuous large scale systems were discussed. The comparison principle and decomposition-aggregation method were used to derive sufficient conditions such that the trivial solution of discontinuous large scale system (1) is connectively stable, uniformly and connectively stable, respectively. The method can be used to investigate other stability properties of the discontinuous large scale systems, and we can derive many corollaries by choosing various functions v_i and ω_i.

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