Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Pure and Applied Algebra 181 (2003) 297–308
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Cover-avoidance properties and the structure of finite groups

Guo Xiuyun[a,*,1], K.P. Shum[b,2]

[a] *Department of Mathematics, Shanxi University Taiyuan, Shanxi 030006, People's Republic of China*
[b] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T. Hong Kong, People's Republic of China (SAR)*

Received 6 November 2000; received in revised form 5 December 2001
Communicated by J. Huebschmann

Abstract

We call a subgroup A of a finite group G a *CAP*-subgroup of G if for any chief factor H/K of G, we have $H \cap A = K \cap A$ or $HA = KA$. In this paper, some characterizations for a finite group to be solvable are obtained under the assumption that some of its maximal subgroups or 2-maximal subgroups be *CAP*-subgroups. We also determine the p-solvability and p-nilpotency of finite groups by considering their *CAP*-subgroups.

MSC: 20D10; 20D20

1. Introduction

In 1962, Gaschütz [2] introduced a certain conjugacy class of subgroups of a finite solvable group which he called pre-Frattini subgroups. These subgroups have the property that they not only avoid the complemented chief factors of a finite solvable group G but also cover the rest of its chief factors. Thereafter, many authors studied this property, for example, Gillam [3] and Tomkinson [10]. In these papers, the main aim was

* Corresponding author. Department of Mathematics, Shanxi University, Taiyuan, Shanxi 030006, China. Tel.: 86-351-7010555.

E-mail addresses: gxy@sxu.edu.cn, xyguo@math.cuhk.edu.hk (X.Y. Guo), kpshum@math.cuhk.edu.hk (K.P. Shum).

[1] The research of the first author is supported by the grant of the National Natural Science Foundation of China and a research grant of Shanxi Province, People's Republic of China.

[2] The research of the second author is partially supported by a UGC(HK) grant #2160126 (1999/2001).

0022-4049/03/$ - see front matter
doi:10.1016/S0022-4049(02)00327-4

to find some kind of subgroups of a finite soluble group G having the cover and avoidance properties. However, the question arises whether we can obtain structural insight into a finite group when some of its subgroups have the cover and avoidance properties.

In 1993, Ezquerro [1] gave some characterization for a finite group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some Sylow subgroup of G have the cover and avoidance properties. In this paper, we will push further this approach and obtain some characterizations for a finite solvable group based on the assumption that some of its maximal subgroups or 2-maximal subgroups have the cover and avoidance properties. We will also investigate the p-solvability and p-nilpotency of a finite group provided some of its subgroups have the cover and avoidance properties.

Throughout this paper, all groups are supposed to be finite. We write $M < \cdot\, G$ to indicate that M is a maximal subgroup of the group G. Our notation and terminology is that of Robinson [7].

2. Basic definitions and preliminary results

We begin by listing some definitions and lemmas, which will be needed in the sequel. Some of these lemmas provide useful information concerning the solvability of finite groups. In particular, we will generalize a well-known result due to Schmidt.

Definition 2.1. Let A be a subgroup of a group G and H/K a chief factor of G. We will say that:

(1) A covers H/K if $H \leqslant KA$;
(2) A avoids H/K if $H \cap A \leqslant K$;
and that
(3) A has the cover and avoidance properties in G, in brevity, A is a *CAP*-subgroup of G, if A either covers or avoids every chief factor of G.

Let G be a group and p a prime number. Let
$\mathscr{F} = \{M \mid M < \cdot\, G\}$.
$\mathscr{F}_n = \{M \mid M \in \mathscr{F}$ and M is non-nilpotent$\}$
$\mathscr{F}_c = \{M \mid M \in \mathscr{F}$ whose index $|G:M|$ is composite$\}$
$\mathscr{F}^p = \{M \mid M \in \mathscr{F}$ and $N_G(P) \leqslant M$ for a Sylow p-subgroup P of $G\}$
$\mathscr{F}^{op} = \bigcup_{p\in\pi(G)-\{2\}} \mathscr{F}^p$
$\mathscr{F}^{pcn} = \mathscr{F}^p \cap \mathscr{F}_c \cap \mathscr{F}_n$
$\mathscr{F}^{ocn} = \mathscr{F}^{op} \cap \mathscr{F}_c \cap \mathscr{F}_n$.
These are families of subgroups of G.

Definition 2.2. $S^{pcn}(G) = \cap\{M \mid M \in \mathscr{F}^{pcn}\}$

if $\mathscr{F}^{pcn}$ is non-empty; otherwise $S^{pcn}(G) = G$.

$$S^{ocn}(G) = \cap\{M \mid M \in \mathscr{F}^{ocn}\}$$

if $\mathscr{F}^{ocn}$ is non-empty; otherwise $S^{ocn}(G) = G$.

We note that $S^{pcn}(G)$ and $S^{ocn}(G)$ are characteristic subgroups of G, and that, for any group G, the following inclusions $\Phi(G) \leqslant S^{ocn}(G) \leqslant S^{pcn}(G)$ always hold.

Lemma 2.3 (Schaller [9, Lemma 1.4]). *Let N be a normal subgroup of a group G and A a CAP-subgroup of G. Then AN is a CAP-subgroup of G.*

Lemma 2.4. *Let N be a normal subgroup of a group G such that $N \leqslant S^{pcn}(G)$. If p is the largest prime number in $\pi(N)$ then either G is solvable or N is p-closed. In both cases, N is always p-solvable. In particular, if p is the largest prime number dividing the order of $S^{pcn}(G)$, then $S^{pcn}(G)$ is p-solvable.*

Proof. We assume that G is not solvable and we will prove that N is p-closed. Trivially, one can easily see that the lemma holds for $p = 2$. Now, we may assume that p is an odd prime.

Let P_1 be a Sylow p-subgroup of N. Then, by Sylow's theorem, there exists $P \in Syl_p(G)$ such that $P_1 = P \cap N$. If $P_1 \lhd G$, then N is p-closed. Hence, we may assume that P_1 is not normal in G. In this case, there exists a maximal subgroup M of G such that $N_G(P) \leqslant N_G(P_1) \leqslant M$. Then, using the Frattini argument, we deduce that $G = NN_G(P_1)$. If $[G:M] = q$ is a prime number, then, by Sylow's theorem, we see that $q = 1 + kp$ and $q \| N|$. However, this contradicts p being the largest prime number in $\pi(N)$. Hence $[G:M]$ must be a composite number. If M is nilpotent, then by a result of Thompson [7, Theorem 10.4.2], we see that M is of even order. Now, let $M_{2'}$ be a Hall $2'$-subgroup of M. Then, by a result of Rose [8, Theorem 1], we have $M_{2'} \lhd G$ and therefore $P \lhd G$ since $P \leqslant N_G(P_1) \leqslant M$ and P is a characteristic subgroup of $M_{2'}$. It hence follows that $P_1 = P \cap N \lhd G$, a contradiction. This proves that $M \in \mathscr{F}^{pcn}$, and thereby we deduce that $G = NN_G(P_1) \leqslant M < G$, which is again a contradiction. Hence the proof is complete. □

Corollary 2.5. *Let p be the largest prime number dividing the order of the group G. If every maximal subgroup M of G in $\mathscr{F}^p \cap \mathscr{F}_c$ is nilpotent, then G is p-solvable.*

Lemma 2.6. *For any group G, $S^{ocn}(G)$ is solvable.*

Proof. We may assume that $S^{ocn}(G) \neq 1$ and also assume that N is a minimal normal subgroup of G satisfying $N \leqslant S^{ocn}(G)$. Then, it is clear that $S^{ocn}(G)/N \leqslant S^{ocn}(G/N)$. Thus, by induction, we see that $S^{ocn}(G/N)$ is solvable and consequently, $S^{ocn}(G)/N$ is solvable. If N is solvable, then $S^{ocn}(G)$ is solvable. Thus, we may assume that N is not solvable. In this case, we let p be the largest prime dividing the order of N and P_1 a Sylow p-subgroup of N such that $P_1 \leqslant P$, where P is a Sylow p-subgroup of G. Then there exists a maximal subgroup M of G such that $N_G(P) \leqslant N_G(P_1) \leqslant N_G(Z(J(P_1)))$ $\leqslant M$, where $J(P_1)$ is the Thompson subgroup of P_1. Using the Frattini argument, we obtain $G = NN_G(P_1) = NM$. If $[G:M] = q$ is a prime, then, by Sylow's theorem, we have $q = 1 + kp$ and $q \| N|$. This contradicts p being the largest prime number which divides the order of N. Hence $[G:M]$ must be a composite number. If M is nilpotent, then so is $N_G(Z(J(P_1)))$ and therefore $N_N(Z(J(P_1)))$ is nilpotent. Since we may assume that $p > 2$, by the Glauberman–Thompson theorem [4, Theorem 8.3.1], we see that N is p-nilpotent. However, since N is a minimal normal subgroup of G and $p \| N|$, we see that N is a p-group, a contradiction. Thus, M must be a non-nilpotent group and so $M \in \mathscr{F}^{ocn}$. However, this entails that $G = NN_G(P_1) \leqslant M < G$, again a contradiction. Hence the proof is complete. □

The next corollary is an immediate consequence of Lemma 2.6. We omit the details.

Corollary 2.7. *If every maximal subgroup M of a group G in $\mathscr{F}^{op} \cap \mathscr{F}_c$ is nilpotent, then G is solvable.*

Lemma 2.8. *Let N be a minimal normal subgroup and M a maximal subgroup of a group G. If M is solvable and $M \cap N = 1$, then G is solvable.*

Proof. We consider $core(M)$. If $core(M) = 1$, let T be a minimal normal subgroup of M. Since M is solvable, T is a p-group for some prime p. This leads to $T \cap N = 1$ and $N_G(T) = M$, and hence N is a p'-group so that $C_N(T) = 1$. By [6, Theorem 7.5], for every prime r dividing the order of N, there is a unique T-invariant Sylow r-subgroup R of N. On the other hand, for any $g \in M$, we have $(R^g)^T = R^{Tg} = R^g$. Thus, by the uniqueness of the Sylow r-subgroup R, we see immediately that the minimal normal subgroup N of G must be an r-group and therefore G is solvable. If $core(M) \neq 1$, we deduce that $M/core(M)$ is a maximal subgroup of $G/core(M)$ and $Ncore(M)/core(M)$ is a minimal normal subgroup of $G/core(M)$. Using the above arguments, we can show that, likewise, $G/core(M)$ is solvable. This proves that G is solvable. □

Using Lemma 2.8, we obtain the following characterization for solvable groups.

Corollary 2.9. *A group G is solvable if and only if there exists a maximal subgroup M of G such that M is a solvable CAP-subgroup of G.*

Proof. If the group G is solvable, then every maximal subgroup of G is a CAP-subgroup of G and M is of course solvable. Now, we assume that M is a solvable maximal subgroup of G such that M is also a CAP-subgroup of G. If $core(M) \neq 1$, then the quotient group $G/core(M)$ plainly satisfies the hypotheses of our corollary. By induction, we see that $G/core(M)$ is solvable, and therefore G itself is solvable. On the other hand, if $core(M) = 1$, let N be a minimal normal subgroup of G with $N \nleqslant M$. Since M is a CAP-subgroup of G, we have $M \cap N = 1$. Thus, by Lemma 2.8, we immediately see that G is solvable. The proof is complete. □

Remark 2.10. It is clear that Corollaries 2.7 and 2.9 generalize the well-known result due to Schmidt, which says that a group, all of whose proper subgroups are nilpotent must be solvable.

3. Main results

Let G be a solvable group. It is well known that every maximal subgroup of G and every Hall subgroup of G are CAP-subgroups of G. One could ask whether the converse holds. We shall give some characterizations for a group G to be solvable in terms of its CAP-subgroups. Some conditions on CAP-subgroups which lead to the p-solvability and p-nilpotency of the group G will also be explored.

We first characterize the solvable group G by its CAP-subgroups. We have the following theorems.

Theorem 3.1. *A group G is solvable if and only if every maximal subgroup M of G in $\mathscr{F}^{ocn}$ is a CAP-subgroup of G.*

Proof. We only need to show that if every maximal subgroup M of G in $\mathscr{F}^{ocn}$ is a CAP-subgroup of G then G is solvable. For this purpose, we suppose that the theorem is not true and let G be a counterexample of minimal order.

If $\mathscr{F}^{ocn} = \emptyset$, then $G = S^{ocn}(G)$ and so G is solvable by Lemma 2.6. Now we assume that $\mathscr{F}^{ocn} \neq \emptyset$. If G is a simple group, then $G/1$ is the only chief factor. Thus, for every maximal subgroup L of G in $\mathscr{F}^{ocn}$, we have either $G \leqslant L$ or $G \cap L \leqslant 1$ because L is a CAP-subgroup of G. But both cases above are impossible, hence G is not simple. Let N be a minimal normal subgroup of G. Then it is easy to see that our hypotheses is quotient closed, and so by the choice of G, we see that G/N is solvable. If G has two different minimal normal subgroups N_1 and N_2, then both G/N_1 and G/N_2 are solvable and so is $G/(N_1 \cap N_2)$. This implies that the group G is solvable. Hence we may assume that G has a unique minimal normal subgroup N.

Let L be a maximal subgroup of G in $\mathscr{F}^{ocn}$. Then, since L is a CAP-subgroup of G, we have either $N \leqslant L$ or $N \cap L \leqslant 1$. If $N \leqslant L$ for every maximal subgroup L of G in $\mathscr{F}^{ocn}$, then $N \leqslant S^{ocn}$. By Lemma 2.6, we see that N is solvable, and consequently G is solvable. So we may assume that there exists a maximal subgroup M of G in $\mathscr{F}^{ocn}$ such that $N \cap M \leqslant 1$. Hence $M \simeq G/N$ is solvable. Thus, using Lemma 2.8, we see that G is solvable. This proves Theorem 3.1. □

Theorem 3.2. *Let H_1 and H_2 be two Hall subgroups of a group G such that $G = H_1 H_2$. Then G is a solvable group if and only if H_1 and H_2 are both solvable CAP-subgroups of G.*

Proof. If G is a solvable group, then every Hall subgroup H of G is a CAP-subgroup of G, and consequently H is solvable because it is a subgroup of a solvable group. We now prove the converse. For this purpose, we let H_1 and H_2 be solvable CAP-subgroups of G and L/K any chief factor of G. If L/K is covered by H_1, then $L \leqslant KH_1$. Since $KH_1/K \simeq H_1/H_1 \cap K$ is solvable, we see that L/K is solvable and therefore L/K is abelian. Hence, the solvability of G will be implied by the following lemma, which is due to the referee. □

Lemma 3.3. *Let U and V be the Hall subgroups of a group G such that $G = UV$. If both U and V are CAP-subgroups of G, then every chief factor L/K of G is covered by U or by V.*

Proof. Let p be a prime number dividing the order of the chief factor L/K of G. Then $p \| U|$ or $p \| V|$ and we may choose a Sylow p subgroup P of G such that $P \leqslant U$ or $P \leqslant V$. Without loss of generality, we may assume that $P \leqslant U$. Since PK/K is a Sylow p-subgroup of G/K, we immediately see that the intersection of PK/K and L/K is non-trivial. This implies that K is a proper subgroup of the intersection of PK and L. Consequently, K is a proper subgroup of the intersection of UK and L. Since U is a *CAP*-subgroup of G, we see that L is contained in UK, and consequently U covers L/K. Thus, Lemma 3.3 is proved. □

Let M be a maximal subgroup of a group G. If L is a maximal subgroup of M, then L is called a 2-maximal subgroup of G. In the following we try to determine the solvability of a group by using the properties of the 2-maximal subgroups of a group. First we prove the following result.

Theorem 3.4. *If every 2-maximal subgroup of a group G is a CAP-subgroup of G, then G is solvable.*

Proof. Suppose that every 2-maximal subgroup of G is a *CAP*-subgroup of G. Then, we see that G is not simple because if G were simple then $G/1$ would be the only chief factor of G. In this case, the assumption that every 2-maximal subgroup of G is a *CAP*-subgroup implies that every 2-maximal subgroup of G must be 1. This means that every maximal subgroup of G is a cyclic group of prime order and therefore G is solvable. If G is not simple, we pick a minimal normal subgroup N of G. Now, N is not a maximal subgroup of G; for if this were the case every maximal subgroup of N would be a 2-maximal subgroup of G and consequently every maximal subgroup of N would be a *CAP*-subgroup of G. This implies that every maximal subgroup of N is the identity group and therefore N is solvable. On the other hand, by the maximality of N, we see that G/N is a cyclic group of prime order. This shows that G is solvable as well.

Now, we consider the quotient group G/N. By induction, we see that G/N is solvable. Furthermore, we may assume that $\Phi(G)=1$. Then, it follows that there exists a maximal subgroup M of G such that $G = MN$. If $N \cap M \neq 1$, let L be a maximal subgroup of M such that $N \cap M \leqslant L$. Thus, L is a 2-maximal subgroup of G with $L \cap N \neq 1$ and $N \nleqslant L$. However, this contradicts the fact that L is a *CAP*-subgroup of G. Hence $M \cap N = 1$ and therefore $M \simeq G/N$ is solvable. Now by Lemma 2.8, G is solvable. The proof is complete. □

We know that every maximal subgroup of a solvable group is a *CAP*-subgroup, but the following example illustrates that a 2-maximal subgroup of a solvable group is not necessary a *CAP*-subgroup.

Example 3.5. Let $G = A_4$, the alternating group of degree 4. Also let H be a Sylow 2-subgroup of G. Then $|H| = 2^2$ and H is a minimal normal subgroup of G. It is clear that every minimal subgroup of H is a 2-maximal subgroup of G but it is not a CAP-subgroup of G.

In spite of this example, we can prove the following lemma.

Lemma 3.6. *Let G be a solvable group. Then there exists a 2-maximal subgroup L of G such that L is normal in G and therefore L is a CAP-subgroup of G.*

Proof. Since G is solvable group, there exists a maximal subgroup M of G such that M is normal in G. Now let M/K be a chief factor of G. If $|M/K|$ is a prime, then K is a 2-maximal subgroup of G and we are done. Suppose that $|M/K| = p^\alpha$, where p is a prime and $\alpha > 1$ is a natural number. Consider the quotient group G/K. If G/K is a p-group, then it is clear that there exists a 2-maximal subgroup L/K of G/K such that L/K is normal in G/K Therefore, L is a 2-maximal subgroup of G and L is normal in G. If G/K is not a p-group, then, since $|G/M|$ is a prime, we may assume that $|G/K| = p^\alpha q$, where q is a prime and $q \neq p$. Let T/K be a Sylow q-subgroup of G/K. Then $|T/K| = q$ and it is easy to see that T/K is a maximal subgroup of G/K. It follows that K/K is a 2-maximal subgroup of G/K and therefore K is a 2-maximal subgroup of G. The proof is complete. □

Theorem 3.7. *A group G is solvable if and only if there exists a solvable 2-maximal subgroup L of G such that L is a CAP-subgroup of G.*

Proof. In view of Theorem 3.6, we only need to prove the sufficiency part. Suppose that L is a solvable 2-maximal subgroup of a group G and L is a CAP-subgroup of G. If $core(L) \neq 1$, then it is easy to see that the hypothesis of the theorem holds for the quotient group $G/core(L)$. An inductive argument shows that $G/core(L)$ is solvable, and therefore G is solvable. Because L is a CAP-subgroup of G, if G is simple, then $L = 1$. Thus G has a maximal subgroup M with prime order and therefore G is solvable [5, IV.7.4 Satz]. Now, we may assume that G is not simple and $core(L) = 1$. Let N be a minimal normal subgroup of G and consider the subgroup LN. Since L is a CAP-subgroup of G, we have $L \cap N = 1$. We consider the following cases separately.

Case I. $LN = G$. Let T be a minimal normal subgroup of L. Then $L \leqslant N_G(T) < G$. Since L is solvable, T is a p-group for some prime p. If $N_G(T) = L$, then N is a p'-group and hence $C_N(T) = 1$. By [6, Theorem 7.5], for every prime r dividing the order of N, there exists a unique T-invariant Sylow r-subgroup R of N. On the other hand, for any $g \in L$, we have $(R^g)^T = R^{Tg} = R^g$. Thus the uniqueness of R forces N to be an r-group and therefore $G = LN$ is solvable. If $L \neq N_G(T)$, then we have $L < N_G(T) < G$. In this case, $N_G(T)$ is a maximal subgroup of G and $N_G(T) \cap N$ is a minimal normal subgroup of $N_G(T)$ since L is a 2-maximal subgroup of G. Because $L \cap (N_G(T) \cap N) = 1$ and L is a maximal subgroup of $N_G(T)$, $N_G(T)$ is solvable by Lemma 2.8 and therefore $N_G(T) \cap N$ is an elementary abelian q-group for some prime q. Let $N_G(T)=M$ and $Q=M\cap N$. If $N_G(Q)=G$, then, by the minimality of N, we have $N=Q$ and therefore $G=LN\leqslant M$, a contradiction. If $N_G(Q)=M$, then Q is a Sylow q-subgroup of N for if otherwise, we have $N_G(Q)>M$, which is absurd. Therefore we have $N_N(Q)=Q=C_N(Q)$. By the well-known Burnside's theorem [7, Theorem 10.1.8], we see that N is q-nilpotent. However, because N is a minimal normal subgroup of G, N is a q-group and $N=Q\leqslant M=N_G(T)<G$, which contradicts $LN=G$.

Case II: $LN<G$. In this case, LN is a maximal subgroup of G. This implies that N is a minimal normal subgroup of LN since L is a 2-maximal subgroup of G. By Lemma 2.8, LN is solvable. In view of Lemma 2.3, LN is a *CAP*-subgroup of G. By Corollary 2.9, G is solvable. Thus, the proof is complete. □

We now discuss the p-solvability and p-nilpotency of a group G by assuming that some subgroups of G are *CAP*-subgroups.

Theorem 3.8. *Let G be a group and p the largest prime number dividing the order of G. If every maximal subgroup M of G in $\mathscr{F}^{pcn}$ is a CAP-subgroup of G, then G is p-solvable.*

Proof. If $\mathscr{F}^{pcn}=\emptyset$, then $G=S^{pcn}(G)$ and so G is p-solvable by Lemma 2.4. Now, we consider $\mathscr{F}^{pcn}\neq\emptyset$. If G is a simple group, then $G/1$ is the only chief factor of G. Thus, for every maximal subgroup L of G in $\mathscr{F}^{pcn}$, we have $G\leqslant L$ or $G\cap L\leqslant 1$ since L is a *CAP*-subgroup of G. But both cases are impossible, hence G can not be simple. In this situation, we can let N be a minimal normal subgroup of G. Now, it is easy to see that the hypotheses of our theorem are quotient closed. Thus an inductive argument shows that G/N is p-solvable. If N is a p'-group, then G is p-solvable. If N is not a p'-group, then, we can see that, for every $L\in\mathscr{F}^{pcn}$, we have $L\cap N\neq 1$ because L contains a Sylow p-subgroup of G. Since L is a *CAP*-subgroup of G, we have $NL=L$ or $N\cap L=1$. But the latter case is clearly impossible. Hence, we have $N\leqslant S^{pcn}$. Now, by Lemma 2.4, we see that N is p-solvable and therefore G is p-solvable. The proof of the theorem is complete. □

Theorem 3.9. *Let p be a prime dividing the order of the group G and P a Sylow p-subgroup of G. Then G is p-solvable if and only if P is a CAP-subgroup of G.*

Proof. Let G be a p-solvable group and H/K a chief factor of G. Then H/K is a p-group or a p'-group. If H/K is a p-group, then it is clear that $HP=KP$ since a Hall p'-subgroup of K is just a Hall p'-subgroup of H. If H/K is a p'-group, then it is clear that $H\cap P=K\cap P$. This shows that P is a *CAP*-subgroup of G.

Now, we assume that P is a *CAP*-subgroup of G. Let N be a minimal normal subgroup of G. It is easy to see that the quotient group G/N also satisfies the hypotheses of our theorem. By induction, we may assume that G/N is p-solvable. If N is a p'-group, then G is p-solvable. If N is not a p'-group, then, since P is a *CAP*-subgroup of G, we have $PN=P$ or $N\cap P=1\cap P$. But the latter case is impossible. Hence, it only remains the case where $PN=P$ and so N is a p-group. This shows that G is p-solvable. □

Corollary 3.10. *Let π be a set of primes and G a group. Then G is π-solvable if and only if, for every $p \in \pi$, there exists a Sylow p-subgroup P of G such that P is a CAP-subgroup of G.*

The following theorem gives conditions for a finite group to be p-nilpotent by considering some of its *CAP*-subgroups.

Theorem 3.11. *Let H be a normal subgroup of a group G and p the smallest prime number dividing the order of H. If all 2-maximal subgroups of every Sylow p-subgroup of H are CAP-subgroups of G and G is A_4-free, then H is p-nilpotent.*

To prove Theorem 3.11, the following lemma is crucial.

Lemma 3.12. *Let p be the smallest prime dividing the order of the group H and P a Sylow p-subgroup of H. If $|P| \leqslant p^2$ and H is A_4-free, then H is p-nilpotent.*

Proof. Let P be a Sylow p-subgroup of H. Then P is clearly an abelian group. If P is cyclic, then H is p-nilpotent by Robinson [7, Theorem 10.1.9]. Hence, we may assume that $|P| = p^2$ and P is an elementary abelian p-group. Let L be a maximal subgroup of H. Then, the order of the Sylow p-subgroups of L is not greater than p^2 and L is p-nilpotent by induction. Hence, we may assume that H is a minimal non-p-nilpotent group (that is, H is a non-p-nilpotent group but every maximal subgroup of H is p-nilpotent). Now, by Robinson [7, Theorems 10.3.3 and 9.1.9], we see that $H = PQ$, where P is normal in H and Q is a cyclic Sylow q-subgroup of H $(p \neq q)$. It follows that $1 \neq H/C_H(P)$, which is isomorphic to a subgroup of $Aut(P)$, is a q-group. Because $|Aut(P)| = (p^2-1)(p^2-p)$, we see that $q \mid p+1$ and therefore $p = 2$ and $q = 3$. It is now clear that $H/\Phi(Q)$ is isomorphic to A_4, which is a contradiction. This contradiction shows that H must be a p-nilpotent group. □

Proof of Theorem 3.11. We use induction on the order of H. First we let P be a Sylow p-subgroup of H with $|P| = p^{\alpha}$. Then we consider the following cases:

Case 1: $\alpha \leqslant 2$.

In this case, the result follows immediately from Lemma 3.12.

Case 2: $\alpha \geqslant 3$.

In this case, we let N be a minimal normal subgroup of G such that $N \leqslant H$ and P^* a Sylow p-subgroup of N such that $P^* \leqslant P$, where P is a Sylow p-subgroup of H. If $|P^*| \geqslant |P|/p^2$, then we may pick a subgroup P_1 of P^* such that P_1 is a 2-maximal subgroup of P. If $|P^*| < |P|/p^2$, we may pick a 2-maximal subgroup P_1 of P such that $P^* \leqslant P_1$. Since P_1 is a *CAP*-subgroup of G and $N/1$ is a chief factor of G, we have $N \cap P_1 = 1$ or $NP_1 = P_1$. It follows that N must be a p-group or a p'-group.

If $O_{p'}(H) \neq 1$, then we consider the quotient group $G/O_{p'}(H)$. It is easy to see that the hypotheses of the theorem still holds for the quotient group $G/O_{p'}(H)$. Thus, by induction, we see that $H/O_{p'}(H)$ is p-nilpotent and therefore H is p-nilpotent. So we may assume that $O_{p'}(H) = 1$, then we have $O_p(H) \neq 1$. Let N be a minimal normal subgroup of G such that $N \leqslant O_p(H)$. Now, we consider the quotient group G/N.

If the order of the Sylow p-subgroups of H/N is not greater than p^2, then, by Lemma 3.12, we know that H/N is p-nilpotent. If the order of the Sylow p-subgroups of H/N is greater than p^2, then, by induction we see that H/N is p-nilpotent. Now, let T/N be a normal p-complement of H/N. If $N \leqslant \Phi(H)$, then it is easy to see that H is p-nilpotent. So we may assume that $N \nleqslant \Phi(H)$. In this case, we see that there exists a maximal subgroup M of H such that $H = NM$. If N is a Sylow p-subgroup of H, then, by our hypotheses, every 2-maximal subgroup P_1 of N is a CAP-subgroup of G. This leads to $P_1 \cap N = 1$ or $N \leqslant P_1$, but this is impossible since $\alpha \geqslant 3$. Let P^+ be a Sylow p-subgroup of M. Then P^+N is a Sylow p-subgroup of H and $P^+ \neq P^+N \neq N$. It follows that $|P^+ \cap N| \leqslant p^{\alpha-2}$. If P^+ is a maximal subgroup of P^+N, we may pick a maximal subgroup P_1 of P^+ such that $P^+ \cap N \leqslant P_1$. If $|P^+| \leqslant p^{\alpha-2}$, we may pick a 2-maximal subgroup P_1 of P^+N such that $P^+ \leqslant P_1$. Since P_1 is a CAP-subgroup of G and $N/1$ is a chief factor of G, we can only have $P_1 \cap N = 1$. Hence $P^+ \cap N = 1$ and $|N| \leqslant p^2$. Since N is a Sylow p-subgroup of T, by Lemma 3.12, T as a normal p-complement, say K. Clearly, K is a normal p-complement of H. This completes the proof. □

As a direct consequence of Theorem 3.11, we have the following.

Corollary 3.13. *Let H be a normal subgroup of a group G. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of H are CAP-subgroups of G, then H is a Sylow tower group of supersolvable type.*

Using similar arguments as in the proof of Theorem 3.11, we also obtain the following.

Theorem 3.14. *Let H be a normal subgroup of a group G and p the smallest prime number dividing the order of H. If all maximal subgroups of every Sylow p-subgroup of H are CAP-subgroups of G, then H is p-nilpotent.*

We now further generalize Corollary 3.13.

Theorem 3.15. *Let $\mathscr{F}$ be the class of groups with Sylow tower of supersolvable type and H a normal subgroup of a group G such that $G/H \in \mathscr{F}$. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of H are CAP-subgroups of G, then G is in $\mathscr{F}$.*

To prove Theorem 3.15, the following lemma is crucial.

Lemma 3.16. *Let $\mathscr{F}$ be the class of groups with Sylow tower of supersolvable type. Also let H be a normal subgroup of a group G such that $G/H \in \mathscr{F}$. If G is A_4-free, and H is a q-group for some prime q with $|H| \leqslant q^2$, then G belongs to $\mathscr{F}$.*

Proof. Let $\mathscr{F}(p)$ be a class of π-groups where every prime in π is less than p for prime $p > 2$ and $\mathscr{F}(2) = \{1\}$. Then, it is clear that the formation $\mathscr{F}$ is locally defined by $\{\mathscr{F}(p)\}$. Of course, $\mathscr{F}$ is locally defined by $\{\mathscr{N}_p * \mathscr{F}(p)\}$, where $\mathscr{N}_p$ is the class of p-groups.

Since $G/H \in \mathscr{F}$, we only need to prove that $G/C_G(K_1/K_2) \in \mathscr{N}_q * \mathscr{F}(q)$ for every chief factor K_1/K_2 of G contained in H. If a chief factor K_1/K_2 of G contained in H is of order q, then $|Aut(K_1/K_2)| = q-1$ and therefore $G/C_G(K_1/K_2) \in \mathscr{N}_q * \mathscr{F}(q)$ since $G/C_G(K_1/K_2)$ is isomorphic to a subgroup of $Aut(K_1/K_2)$. So we may assume that H is a minimal normal subgroup of G and $|H| = q^2$. I this case, we have $|Aut(H)| = (q^2-1)(q^2-q)$. If $q > 2$, then, since every prime factor p of $|Aut(H)|$ (with $p \neq q$) must be less than q and $G/C_G(H)$ is a Sylow tower group (because G/H is a Sylow tower group and $H \leqslant C_G(H)$), we have $G/C_G(H) \in \mathscr{N}_q * \mathscr{F}(q)$. If $q=2$, then $|Aut(H)| = 3 \times 2$. Because G is A_4-free, we still have $G/C_G(H) \in \mathscr{N}_q * \mathscr{F}(q)$. This shows that $G \in \mathscr{F}$. □

Proof of Theorem 3.15. We first suppose that the theorem is not true and let G be a minimal counterexample. Then, by Corollary 3.13, we can see that H has a Sylow tower of supersolvable type. Let p be the largest prime number in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Then P must be a normal subgroup of G and every 2-maximal subgroup of P is a CAP-subgroup of G. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of H/P are CAP-subgroups of G/P and G/P is A_4-free. Thus, by the minimality of G, we have $G/P \in \mathscr{F}$.

Let N be a minimal normal subgroup of G such that $N \leqslant P$. If $N = P$, then, since every 2-maximal subgroup of $P = N$ is a CAP-subgroup of G, we have $|P| = |N| \leqslant p^2$. By Lemma 3.16, we see that $G \in \mathscr{F}$, a contradiction. Hence $N < P$. For the case $|P/N| \leqslant p^2$, we may apply Lemma 3.16. On the other hand, for $|P/N| > p^2$, we just make use of the choice of G. Thus, in both cases, we have $G/N \in \mathscr{F}$. Since $\mathscr{F}$ is a saturated formation, w may assume that N is the unique minimal normal subgroup of G contained in P and also $\Phi(P) = 1$. This shows that P is an elementary abelian p-group and therefore there exists $N_1 \leqslant P$ such that $P = N \times N_1$. If $|N| \geqslant p^3$, then, we can let P_1 be a 2-maximal subgroup of P such that $N_1 \leqslant P_1$. This implies that $N \cap P_1 \neq 1$, but by our hypotheses, we have $P_1 \cap N = 1$ or $NP_1 = P_1$ since $N/1$ is a chief factor of G and P_1 is a CAP-subgroup of G. Thus, it follows that $N \leqslant P_1$ and therefore $P = NN_1 \leqslant P_1$, which is a contradiction. Hence we must have $|N| \leqslant p^2$. Now, by Lemma 3.16 again, we see that $G \in \mathscr{F}$, a final contradiction. This proves Theorem 3.15. □

Acknowledgements

The authors would like to thank the referee and the editor, Professor J. Huebschmann, for their valuable suggestions and useful comments contributed to the final version of this paper. In particular, we thank Professor J. Huebschmann for helping us to polish the final version of this paper.

References

[1] L.M. Ezquerro, A contribution to the theory of finite supersolvable groups, Rend. Sem. Mat. Univ. Padova 89 (1993) 161–170.

[2] W. Gaschütz, Praefrattini gruppen, Arch. Math. 13 (1962) 418–426.
[3] J.D. Gillam, Cover-avoid subgroups in finite solvable groups, J. Algebra 29 (1974) 324–329.
[4] D. Gorenstein, Finite Groups, Harper and Row Publishers, New York, Evanston, London, 1968.
[5] B. Huppert, Endliche Gruppen, Vol. I, Springer, New York, Berlin, 1967.
[6] H. Kurzweil, Endliche Gruppen, Springer, New York, 1967.
[7] D.J.S. Robinson, A Course in the Theory of Groups, Springer, New York, Berlin, 1993.
[8] J. Rose, On finite insolvable groups with nilpotent maximal subgroups, J. Algebra 48 (1977) 182–196.
[9] K.U. Schaller, Über Deck–Meide–Untergruppen in endlichen auflösbaren Gruppen, Ph.D. Thesis, University of Kiel, 1971.
[10] M.J. Tomkinson, Cover-avoidance properties in finite soluble groups, Canad. Math. Bull. 19 (2) (1976) 213–216.