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# On graphs whose star sets are (co-)cliques

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In this paper we study graphs all of whose star sets induce cliques or co-cliques. We show that the star sets of every tree for each eigenvalue are independent sets. Among other results it is shown that each star set of a connected graph $G$ with three distinct eigenvalues induces a clique if and only if $G = K_{1,2}$ or $K_{2,\ldots,2}$. It is also proved that stars are the only graphs with three distinct eigenvalues having a star partition with independent star sets.



## 1. Introduction

We will consider only simple graphs, that is finite and undirected without loops or multiple edges. If $G$ is a graph with vertex set $\{1, \ldots, n\}$, the adjacency matrix of $G$ is an $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices $i$ and $j$, and 0 otherwise. The eigenvalues of $G$ are the eigenvalues of $A$. We denote the multiplicity of an eigenvalue $\lambda$ by $\mathrm{mult}(\lambda)$. We also denote the independence number of $G$ by $\alpha(G)$. We denote the complete $r$-partite graph $K_{t,\ldots,t}$ by $K_{t\times r}$.

A regular graph which is neither complete nor empty said to be *strongly regular* if every pair of adjacent vertices has a common number of neighbors and every pair of distinct non-adjacent vertices has a common number of neighbors.

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If $\lambda$ is an eigenvalue of $G$ of multiplicity $m$, then a *star set* for $\lambda$ in $G$ is a set $X$ of $m$ vertices such that $\lambda$ is not an eigenvalue of $G \setminus X$. The graph $G \setminus X$ is then called a *star complement* for $\lambda$ in $G$ (or, in [8], a $\lambda$-basic subgraph of $G$). Star sets and star complements exist for any eigenvalue and any graph; they need not be unique (see [4]). Let $G$ be a graph with distinct eigenvalues $\lambda_1 > \cdots > \lambda_s$. Then there is a partition of vertex set of $G$ into $s$ sets $X_1, \ldots, X_s$ such that, for every $i$, $1 \leqslant i \leqslant s$, $X_i$ is a star set for $\lambda_i$ (see [4, p. 151]). This partition is called a *star partition* of $G$.

Star sets and star partitions (not with this name) originated independently in 1993 in papers by Ellingham [8] and Rowlinson [12]. Seemingly, the term 'star complement' and 'star set' were first used by Rowlinson in [13]. Star sets and star complements were introduced as a way to study eigenspaces of graphs and also to investigate the graph isomorphism problem [3]. They have proved to be helpful ideas and a lot of research has been devoted to these topics. Mainly, they have been used to give bounds on the size of graphs and also to give characterizations of some well known graphs. For instance, it is used to prove that if $\lambda \neq -1, 0$ is an eigenvalue of a graph of order $n$, then $n \leqslant \binom{\ell}{2}$ in which $\ell = n - \mathrm{mult}(\lambda) + 1$.

Motivated by consideration of so-called 'ordering of graphs', Bell et al. [1] asked whether there exist graphs in which all star sets for all eigenvalues induce co-cliques or cliques. A *galaxy graph* is a graph all of whose star sets for all eigenvalues are independent sets. In this paper the question is partially answered. Namely, we prove that trees are galaxy graphs; a graph with three distinct eigenvalues is galaxy if and only if it is a star. All star sets of a graph with three distinct eigenvalues induce cliques if and only if it is either $K_{1,2}$ or a cocktail party graph $K_{2\times r}$.

The following result is known as the Reconstruction Theorem [4, Theorems 7.4.1 and 7.4.4].

**Theorem 1.** *Let $\lambda$ be an eigenvalue of a graph $G$ of multiplicity $m$ and $X$ be a set of $m$ vertices in $G$. Suppose that $A$ is the adjacency matrix of $G$ of the form*

$$A = \begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix},$$

*where $A_X$ is the adjacency matrix of the subgraph induced by $X$. Then $X$ is a star set for $\lambda$ in $G$ if and only if $\lambda$ is not an eigenvalue of $C$ and*

$$\lambda I - A_X = B^\top (\lambda I - C)^{-1} B.$$

We also make use of the following fact [4, Corollary 7.2.4].

**Lemma 1.** *Let $X$ be a star set for eigenvalue $\lambda$ of multiplicity $m$ and $S \subset X$. Then $\lambda$ is an eigenvalue of $G \setminus S$ of multiplicity $m - |S|$.*

## 2. Galaxy graphs

In this section we note that any tree is a galaxy graph. We also characterize graphs with three distinct eigenvalues which possess a star partition whose star sets are independent sets. Obviously, if all eigenvalues of a graph $G$ are simple, then $G$ is a galaxy graph. In [1] trees in certain families are shown to be galaxy graphs. These families include: any tree with zero as the only multiple eigenvalue; any subdivision of stars and all double stars (i.e. graphs consisting of an edge plus pendant edges at its ends).

Johnson and Sutton [10] studied star set of trees with a different terminology. They called a vertex 'downer' for the eigenvalue $\lambda$ if it belongs to a star set for the $\lambda$. They show that [10, Proposition 4.9] if one deletes two adjacent downer vertices for the eigenvalue $\lambda$ from a tree, then $\mathrm{mult}(\lambda)$ decreases by 1. In view of Lemma 1, this means that no star set of a tree contains two adjacent vertices. In other words, we have the following observation.

**Theorem 2.** *Any tree is a galaxy graph.*

**Theorem 3.** *Any star set for the eigenvalue* 0 *of a unicyclic graph is independent provided that the length of its cycle is not congruent to* 0 mod 4.

**Proof.** Let $G$ be a unicyclic graph that the length of its cycle is not congruent to 0 mod 4. We use induction on the order of $G$. If $G$ has no pendant vertex, it is a cycle and so $G$ has no zero eigenvalue. So assume that $G$ has a pendant vertex $u$ with a unique neighbor $v$. Since deletion of a pendant vertex along with its neighbor does not change the number of zero eigenvalue of a graph (see [2, Theorem 2.11]), any star set of $G \setminus \{u, v\}$ is a star set for $G$ too. On the other hand, since the multiplicity of eigenvalue zero in $G \setminus \{v\}$ is more than that of $G$, $v$ does not belong to a star set of $G$. Therefore any star set of $G$ is either a star set of $G \setminus \{u, v\}$ or of the form $X \cup \{u\}$, where $X$ is a star set of $G \setminus \{u\}$. Thus, by induction hypothesis, every star set of $G$ is a independent set. □

**Theorem 4.** *There is no connected strongly regular graph G with a independent star set for any eigenvalue of multiplicity at least* 2.

**Proof.** Let $G$ be a connected strongly regular graph with distinct eigenvalues $k > \lambda_2 > \lambda_3$. If $-\lambda_3 = k$, then $G$ is bipartite. Since the diameter of any strongly regular graph is 2, $G = K_{k,k}$. But $K_{k,k}$ has no independent star set for eigenvalue 0. So $-\lambda_3 < k$. If $\mathrm{mult}(\lambda_2) \neq \mathrm{mult}(\lambda_3)$, then one of them is at least $\frac{n}{2}$, where $n = |V(G)|$. By the Delsarte–Hoffman bound (see [2, p. 115]), the order of the largest co-clique of $G$ is at most

$$n\frac{-\lambda_{\min}(G)}{k - \lambda_{\min}(G)} = n\frac{-\lambda_3}{k - \lambda_3},$$

which is less than $\frac{n}{2}$, a contradiction. If $\mathrm{mult}(\lambda_2) = \mathrm{mult}(\lambda_3) = \frac{n-1}{2}$ (the 'half case'), then by Lemma 10.3.2 of [9], $k = \frac{n-1}{2}$ and $\lambda_3 = \frac{1-\sqrt{n}}{2}$. So $n\frac{-\lambda_3}{k-\lambda_3} < \frac{n-1}{2}$ and we are done. □

**Lemma 2.** *There is no connected graph with three distinct eigenvalues one of which is* $-1$.

**Proof.** Let $\lambda_1 > \lambda_2 > \lambda_3$ be the distinct eigenvalues of $G$. If $\lambda_2 = -1$, then $G$ has exactly one positive eigenvalue. Thus by [2, Theorem 6.7], $G$ is a complete multipartite graph. Since $G$ has no zero eigenvalue, $G$ must be a complete graph, a contradiction. So $\lambda_3 = -1$. By the Interlacing Theorem, $G$ contains no $K_{1,2}$ as an induced subgraph. This implies that $G$ is a complete graph, a contradiction. □

**Theorem 5.** *A connected graph with three distinct eigenvalues has a star partition whose cells are independent sets if and only if it is a star.*

**Proof.** By Theorem 2, every star set of a star induces a co-clique. For the converse, let $G$ be a connected graph of order $n$ with three distinct eigenvalues $\lambda_1 > \lambda_2 > \lambda_3$. If $n \leqslant 5$, then $G \in \{K_{1,2}, K_{1,3}, K_{2,2}, K_{1,4}, K_{2,3}\}$. The graphs $K_{2,2}$ and $K_{2,3}$ have no independent star set for the eigenvalue 0. So we may assume that $n \geqslant 6$. If $G$ has eigenvalue 0, then it has a single positive eigenvalue. Thus by [2, Theorem 6.7], $G$ is a complete multipartite graph. If two parts of $G$ have more than one vertex, then none of the star sets for the eigenvalue 0 induces a co-clique. So $G$ should be a $K_{1,\ldots,1,t}$ for some $t \geqslant 2$. This graph is the join of $K_{n-t}$ and $\overline{K}_t$ which has $-1$ as an eigenvalue (see [2, Theorem 2.8]) unless $t = n - 1$. So by Lemma 2, $G = K_{1,n-1}$. Thus we may assume that $G$ has no eigenvalue 0. Let $\lambda_2, \lambda_3$ have multiplicities $p, q$, so that $\alpha(G) \geqslant \max\{p, q\}$. On the other hand, $\alpha(G)$ does not exceed the number of non-negative or non-positive eigenvalues [2, Theorem 3.14], and so $\alpha(G) \leqslant \min\{p + 1, q\}$. Since $p + q = n - 1$, it follows that $\alpha(G) \in \{\frac{n-1}{2}, \frac{n}{2}\}$ which implies that $q \in \{\frac{n-1}{2}, \frac{n}{2}\}$. Let $q = \frac{n-1}{2}$. Let $X \cup Y \cup \{v\}$ be a star partition of $G$ in which $X$ and $Y$ are independent sets. It follows from the Interlacing Theorem [2, Theorem 0.10] that $\lambda_3$ is the smallest eigenvalue of $G \setminus \{v\}$ of multiplicity at least $\frac{n-3}{2}$. The graph $G \setminus \{v\}$ has also $\lambda_2$ as an eigenvalue of multiplicity at least $\frac{n-3}{2}$. Since $G \setminus \{v\}$ is bipartite, it has $-\lambda_3 = -\lambda_{\min}(G \setminus \{v\})$ as an eigenvalue with the same multiplicity as $\lambda_{\min}(G \setminus \{v\})$. Since $n$ is odd, $\frac{n-3}{2} \geqslant 2$ and we should have $\lambda_2 = -\lambda_3$ which is impossible. Therefore $q = \frac{n}{2}$. From the Interlacing Theorem it is seen that $G \setminus \{v\}$ has at most four distinct eigenvalues. Since the eigenvalues of bipartite graphs are symmetric around 0 and $G \setminus \{v\}$ is of odd order, it must have three distinct eigenvalues one of which is 0. Therefore $\lambda_2$ and $\lambda_3$ are eigenvalues of $G \setminus \{v\}$ with the equal multiplicity $\frac{n}{2} - 1$. Thus $\lambda_2 = -\lambda_3$ which implies $\lambda_1 = -\lambda_3 = \lambda_2$ (because the sum of eigenvalues of $G$ is 0), a contradiction. $\square$

## 3. Graphs all of whose star sets induce cliques

In this section we treat graphs with three distinct eigenvalues whose star sets induce cliques. We show that such graphs must be either $K_{1,2}$ or $K_{2\times r}$ for some $r$. We start this section with the following lemma which appears as Lemma 5.1.5 of [5] in column form.

**Lemma 3.** *Let $A$ be a real symmetric matrix of rank $r$. Then for any $r$ linearly independent rows of $A$, the corresponding principle submatrix of $A$ is invertible.*

**Lemma 4.** *Let $G$ be a graph of order $n$ with an eigenvalue $\lambda$ of multiplicity $m$ such that $\lambda \notin \{-1, m-1\}$. If each star set for $\lambda$ induce a clique, then $G$ has two disjoint cliques of order $m$.*

**Proof.** Let $X$ be a star set for $\lambda$. Then $A_X = J - I$ is an $m \times m$ matrix. Since $\lambda \notin \{-1, m-1\}$, $\lambda I - A_X$ is invertible. So we can extend the $m$ rows related to $\lambda I - A_X$ to $n - m = \operatorname{rank}(\lambda I - A)$ linearly independent rows of $\lambda I - A$. Hence by Lemma 3, there is an invertible $n - m \times n - m$ principal submatrix of $\lambda I - A$ which contains $\lambda I - A_X$. This means that the vertices corresponding to the remaining rows of $\lambda I - A$ form a star set for $\lambda$. This completes the proof. $\square$

**Lemma 5.** *The only connected strongly regular graph $G$ of order $n$ with a clique of order at least $\frac{n}{2}$ is $K_{2\times\frac{n}{2}}$.*

**Proof.** If $\overline{G}$, the complement of $G$, is connected, from the proof of Theorem 4, it is seen that $\alpha(\overline{G}) < \frac{n}{2}$. If $\overline{G}$ is disconnected, then $G = K_{t\times r}$ with $t \geqslant 2$ (see [9, Lemma 10.1.1]) for which the order of every clique of $G$ does not exceed $\frac{n}{3}$ unless $t = 2$. $\square$

**Lemma 6.** *A complete multipartite graph $G$ has a star partition all of whose cells induce cliques if and only if each part of $G$ contains at most two vertices.*

**Proof.** Let $V_1, \ldots, V_r$ be all the parts of $G$. Every star set of the eigenvalue 0 is of the form $X = \cup_{|V_i|\geqslant 2}(V_i \setminus \{v_i\})$ for some $v_i \in V_i$. Moreover, $X$ induce a clique if and only if no $V_i$ contains more than two vertices. This proves the necessity. To prove the sufficiency, note that the deleting a pair of vertices in the same part reduces the multiplicity of eigenvalue 0 (because in a complete $r$-partite graph of order $n$, $\operatorname{mult}(0) = n - r$) and also the multiplicity of at least a non-zero eigenvalue. So no two vertices in the same part can belong to a star set. Therefore every star set of $G$ induce a clique. $\square$

**Lemma 7.** *Let $G$ be a connected graph all of whose star sets induce cliques. If $G$ has three distinct eigenvalues, then the multiplicity of no eigenvalue of $G$ exceeds $\frac{n}{2}$.*

**Proof.** Let $\lambda$ be an eigenvalue of $G$ of multiplicity $m$. By Lemma 2, $\lambda \neq -1$. If $\lambda \neq m - 1$, then by Lemma 4, $m \leqslant \frac{n}{2}$. If $\lambda = m - 1$ and $m \geqslant \frac{n+1}{2}$, then by the Interlacing Theorem, the graph obtained by removing of each $\frac{n-1}{2}$ vertices has an eigenvalue $\lambda \geqslant \frac{n-1}{2}$. This implies that $G$ is a complete graph which is impossible. $\square$

Denote by $H_{n,t}$, for $1 \leqslant t \leqslant n - 1$, the graph obtained by joining a new vertex to $t$ vertices of the complete graph $K_n$.

**Lemma 8.** *The characteristic polynomial of the adjacency matrix of $H_{n,t}$ is*

$$(\lambda+1)^{n-2}(\lambda^3+(2-n)\lambda^2+(1-t-n)\lambda+t(n-t-1)). \tag{1}$$

*Moreover, $H_{n,t}$ $(n \geqslant 3)$ has at most one integer eigenvalue besides $-1$. Further, for every two positive integers $n$ and $t$ $(n \geqslant 3)$, there is at most one $s$ for which $H_{n,t}$ and $H_{n,s}$ have a common eigenvalue besides $-1$.*

**Proof.** The vertex set of $H_{n,t}$ has an 'equitable partition' (see [9, pp. 195–198]). It turns out that the characteristic polynomial of the following matrix:

$$\begin{pmatrix} 0 & t & 0 \\ 1 & t-1 & n-t \\ 0 & t & n-t-1 \end{pmatrix}$$

divides that of the adjacency matrix of $H_{n,t}$, see [9, Theorem 9.3.3]. On the other hand, by the Interlacing Theorem, $H_{n,t}$ has at least $n-2$ eigenvalues $-1$. Since $-1$ is not an eigenvalue of the above matrix, (1) is the the characteristic polynomial of the adjacency matrix of $H_{n,t}$. We have $n-1 < \lambda_{\max}(H_{n,t}) < n$ (see [11, pp. 52–53]). So $\lambda_{\max}(H_{n,t})$ is not integer and $H_{n,t}$ has its algebraic conjugate as eigenvalue. Thus $H_{n,t}$ has at most one integer eigenvalue besides $-1$. The last part of the theorem follows from the fact that if $H_{n,t}$ and $H_{n,s}$ for $s \neq t$ have a common eigenvalue $\lambda \neq -1$, then from (1) it seen that $\lambda = n-s-t-1$. $\square$

The proof of the following lemma is implicit in Section 4 of [6].

**Lemma 9.** *Let $G$ be a connected graph with three distinct eigenvalues. If $G$ has two vertex degrees, then the partition of the vertices according to their degrees induces two strongly regular subgraphs $G_1$ and $G_2$. Moreover, the vertices of $G_1$ (respectively, $G_2$) have the same number of neighbors in $G_2$ (respectively, $G_1$).*

**Theorem 6.** *Let $r$ be a positive integer. Apart from $K_{1,2}$ and $K_{2\times r}$, the vertices of a connected graph with three distinct eigenvalues cannot be partitioned into two cliques.*

**Proof.** Let $G$ be a connected graph of order $n$ with three distinct eigenvalues which is neither $K_{1,2}$ nor $K_{2\times r}$. We may assume that $G$ is not bipartite. Moreover let the vertices of $G$ be partitioned into two cliques $X, Y$ with $|Y| \geqslant |X|$. Suppose that $\rho, \lambda, \lambda'$ are the eigenvalues of $G$ with multiplicities $1, m_1, m_2$, respectively and $m_1 \geqslant m_2$. If $|X| < m_2$, then by the Interlacing Theorem, $\lambda$ is an eigenvalue of the subgraph induced by $Y$. By Lemma 2, $\lambda = |Y|-1$. By [6, Proposition 2] $\rho$ is an integer, and so $\lambda\prime$ is an integer. We have $|\lambda'| = \frac{\rho}{m_2} + \frac{m_1}{m_2}\lambda > \lambda$, whence $|\lambda'| \geqslant |Y|$. On the other hand $G$ has at most $\binom{n}{2}-1$ edges. Thus noting that $|Y| \geqslant \frac{n}{2}$, we have

$$\begin{aligned} n(n-1)-2 &\geqslant \sum \lambda_i^2(G) \\ &> 2|Y|^2+(n-2)(|Y|-1)^2 \\ &\geqslant n^2/2+(n-2)^3/4, \end{aligned}$$

which is impossible. Hence $|X| \geqslant \mathrm{mult}(\lambda)$ and so $\mathrm{mult}(\lambda) \leqslant \frac{n}{2}$. If $G$ has a non-integral eigenvalue, then by [6, Proposition 3], $n$ is odd and $\lambda_{\max}(G) = \frac{n-1}{2}$ which is a contradiction (because $G$ contains the clique $Y$ of order at least $\frac{n+1}{2}$). Therefore all eigenvalues of $G$ are integers. If $n$ is odd, then $|X| = \mathrm{mult}(\lambda) = \mathrm{mult}(\lambda') = \frac{n-1}{2}$. Choose $x \in X$ such that $Y \cup \{x\}$ does not induce a clique. By the Interlacing Theorem, both $\lambda$ and $\lambda'$ are eigenvalues of the subgraph induced by $Y \cup \{x\}$. But by Lemma 8, this subgraph has at most one integer eigenvalue other than $-1$, and we have a contradiction to Lemma 2 . So $n$ is even and $|X| = |Y| = \mathrm{mult}(\lambda) = \frac{n}{2}$. For any $x \in X$, by the Interlacing Theorem, $\lambda$ is an eigenvalue of the subgraph induced by $Y \cup \{x\}$. Again this subgraph has the form $H_{\frac{n}{2},k}$, for some $k$. By Lemma 8, there are at most two values $s, t$ for which $\lambda$ is an eigenvalue of both $H_{\frac{n}{2},s}$ and $H_{\frac{n}{2},t}$. Thus each $x \in X$ has either $s$ or $t$ neighbors in $Y$. Similarly, each $y \in Y$ has $s$ or $t$ neighbors in $X$. If $s = t$, then $G$ is strongly regular (see, e.g., [9, Lemma 10.2.1]) and by Lemma 5, $G = K_{2\times\frac{n}{2}}$, a contradiction. So $s \neq t$. Therefore $G$ has just two vertex degrees. By Lemma 9, the vertices of $G$ can be partitioned into two strongly regular subgraphs, say $H_1$ and $H_2$. By Lemma 5, $H_1 = K_{2\times r_1}$ and $H_2 = K_{2\times r_2}$ for some $r_1$ and $r_2$. Without loss of generality, $r_1 \geqslant r_2$. Let $X_1$ and $Y_1$ be the vertices of $H_1$ which belongs to $X$ and $Y$, respectively. Similarly, define $X_2$ and $Y_2$. Since the complements of $H_1$ and $H_2$ are bipartite regular graphs, we have $|X_1| = |Y_1|$ and $|X_2| = |Y_2|$. If $|X_1| = |X_2|$, from Lemma 9 it follows that the number of neighbors of any vertex of $H_1$ in $H_2$ should be the same as the number of neighbors of any vertex of $H_2$ in $H_1$ which implies that $G$ is a regular graph. So $G$ is a strongly regular graph and by Lemma 5, $G = K_{2\times\frac{n}{2}}$, a contradiction. So $|X_1| > |X_2|$. Since $|X_2 \cup Y_2| \leqslant \frac{n}{2} - 2$, by the Interlacing Theorem, $H_1$ has $\lambda$ and $\lambda'$ as eigenvalues. Note that $\lambda_{\max}(H_1) \notin \{\lambda, \lambda'\}$ since otherwise

$$\begin{aligned} n(n-1) - 2 &\geqslant \sum \lambda_i^2(G) \\ &> \lambda_1(G)^2 + (n/2-1)\lambda_{\max}^2(H_1) \\ &\geqslant (n/2+1)^2 + (n/2-1)n^2/4, \end{aligned}$$

which is impossible. Since $K_{2\times r_1}$ is a strongly regular graph, its eigenvalues are $-2$ 0, $2r_1 - 2$. So $\{\lambda, \lambda'\} = \{-2, 0\}$. If $\lambda = -2$, then, since $\mathrm{mult}(\lambda) = \frac{n}{2}$, we find $\lambda_{\max}(G) = n$, a contradiction. Thus $\lambda' = -2$ and then $\lambda_{\max}(G) = n - 2$. Hence $G$ has the same spectrum as $K_{2\times\frac{n}{2}}$. Since $K_{2\times\frac{n}{2}}$ is determined by its spectrum (see [7]), $G = K_{2\times\frac{n}{2}}$, a contradiction. □

**Theorem 7.** *Every star set of a connected graph with three distinct eigenvalues induce a clique if and only if it is either $K_{1,2}$ or $K_{2\times r}$ for some $r$.*

**Proof.** Let $G$ be a connected graph of order $n$ with three distinct eigenvalues. If $G$ is a complete bipartite graph and has more than two vertices in one of its parts, then $G$ has a star set for the eigenvalue 0 which does not induce a clique. Hence $G$ should be either $K_{1,2}$ or $K_{2,2}$. So we may assume that $G$ is not a complete bipartite graph. Two cases may be considered:

*Case* 1. Let not all eigenvalues of $G$ be integral. Then by [6, Proposition 3], $n$ is odd, $\lambda_1 = \frac{n-1}{2}$, and $\lambda_2, \lambda_3 = -\frac{1}{2} \pm \frac{\sqrt{b}}{2}$ for some positive integer $b$. If one of the $\mathrm{mult}(\lambda_2)$ or $\mathrm{mult}(\lambda_3)$ is more than $\frac{n-1}{2}$, then $G$ necessarily contains a clique of order at least $\frac{n+1}{2}$, so $\lambda_1 > \frac{n-1}{2}$, which is impossible. Thus $\mathrm{mult}(\lambda_2) = \mathrm{mult}(\lambda_3) = \frac{n-1}{2}$. Since $G$ has a star partition, it contains two disjoint cliques $X$ and $Y$ of order $\frac{n-1}{2}$. Let $v$ be the vertex missing from $X$ and $Y$. Let $v$ have a non-adjacent vertex $x \in X$. Choose $x \neq x' \in X$. We claim that the subgraph $H$ induced by $Y \cup \{x'\}$ has neither $\lambda_2$ nor $\lambda_3$ as an eigenvalue. By way of contradiction suppose that $H$ has $\lambda_2$ or $\lambda_3$ as an eigenvalue. Then $x'$ is adjacent to a vertex of $Y$. Since $\lambda_2$ and $\lambda_3$ are algebraically conjugate, both $\lambda_2$ and $\lambda_3$ are eigenvalues of $H$. We have $\lambda_1(H) = \frac{n-3}{2} + \epsilon$, for some $0 < \epsilon < 1$. By the interlacing theorem, $H$ has eigenvalue $-1$ of multiplicity $\frac{n-1}{2} - 2$. So the sum of all eigenvalues of $H$ is $0 = \frac{n-3}{2} + \epsilon - \frac{n-1}{2} + 2 - 1 = \epsilon$ (note that $\lambda_2 + \lambda_3 = -1$) which is impossible since $x$ and $v$ are not adjacent. So $\{v\} \cup (X \setminus \{x'\})$ is a star set for both $\lambda_2$ and $\lambda_3$, which is impossible. So $v$ is adjacent to all vertices. This implies that for each $z \in X$, $G \setminus \{z\}$ has two disjoint cliques of size $\frac{n-1}{2}$. Repeating the foregoing argument, we see that $z$ is adjacent to all other vertices. Thus every vertex of $X$ is adjacent to every vertex of $Y$. So $G$ should be a complete graph, a contradiction.

*Case* 2. All eigenvalues of $G$ are integers. If $n$ is even, then by Lemma 7, $G$ has an eigenvalue $\lambda$ of multiplicity $\frac{n}{2}$. We claim that $\lambda \neq \frac{n}{2} - 1$. Otherwise, $\lambda_3(G) = -\frac{n}{2} + t$, where $t < -1$. By the interlacing theorem, every induced subgraph of $G$ on $\frac{n}{2} + 2$ vertices has $\lambda_3$ as an eigenvalue which is impossible. Therefore, from Lemmas 2 and 4 it follows that $G$ contains two disjoint cliques $X$ and $Y$ of order $\frac{n}{2}$. So by Theorem 6, $G = K_{2\times\frac{n}{2}}$. □

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