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Five-Coloring Graphs on the Torus

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We prove that a graph on the torus is 5-colorable, unless it contains either K_6, the complete graph on six vertices, or C_3+C_5, the join of two cycles of lengths three and five, respectively, or K_2+H_7, the join of K_2 and the graph H_7 on seven vertices obtained by applying Hajos' construction to two copies of K_4, or a triangulation T_{11} with 11 vertices of the torus. This answers questions of Albertson, Hutchinson, Stromquist, and Straight. © 1994 Academic Press, Inc.

1. INTRODUCTION

Heawood [10] proved that a graph on S_g, the sphere with g handles added, can be colored in at most $h(g)=\lfloor\frac{1}{2}(7+\sqrt{48g+1})\rfloor$ colors (for $g\geqslant 1$). Ringel and Youngs (see [11]) proved that this is best possible. However, Dirac [5] proved that a graph on S_g can be colored in fewer than $h(g)$ colors unless it contains a complete graph on $h(g)$ vertices. In Dirac's terminology, $K_{h(g)}$ is the only $h(g)$-color-critical graph on S_g. More generally, Dirac [6] proved that, for each fixed nonnegative integer g and each fixed natural number $k\geqslant 8$, there are only finitely many k-color-critical graphs on S_g.

Gallai [8] proved that the vertices of degree $k-1$ in a k-critical graph induce a subgraph whose blocks are either odd cycles or complete graphs. This implies that a 7-critical graph G on S_g has at most $96(g-1)$ vertices. For if G has more than $96(g-1)$ vertices, then Euler's formula, combined with the fact that all vertices in a k-color-critical graph have degree $\geqslant k-1$, implies that G has a vertex v of degree 6, such that all neighbors of v have degree 6 and such that all faces incident with v are bounded by triangles. Hence v and its neighbors belong to a block in the subgraph induced by the vertices of degree 6. As that block is not an odd cycle, it is complete. Hence $G\supseteq K_7$, a contradiction because a 7-critical graph does not contain a 7-critical graph as a proper subgraph.

By similar arguments it follows that for each natural number $k\geqslant 7$ and for each natural number $m\geqslant 3$ there are only finitely many k-color-critical

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graphs on N_m, the sphere with m crosscaps added. Fisk (see [3]) gave a construction which implies that there are infinitely many 5-critical graphs on the torus and the projective plane and hence on each S_g, N_m, where $g \geqslant 1$, $m \geqslant 1$. We point out that there are infinitely many 5-vertex-critical 6-regular graphs on the torus. This raises the following general question.

Question 1. Let S be a fixed surface, $S = S_g$ $(g \geqslant 1)$ or N_m $(m \geqslant 2)$. Do there exist infinitely many 6-critical graphs on S?

A negative answer would imply the result in [16] that a graph on S with no short noncontractible cycles is 5-colorable. It would also imply a positive answer to the following question for $k = 5$.

Question 2. Let S be a fixed surface and let k be a fixed natural number. Does there exist a polynomially bounded algorithm for deciding if a given graph on S has a k-coloring?

For $k = 3$ this problem is NP-complete even for the sphere (see [9]). For $k \geqslant 6$ it is answered in the affirmative by the above results of Dirac and Gallai. So the cases $k = 4, 5$ remain.

In this paper we answer Question 1 in the negative for the torus. This special case of Question 1 was raised by Albertson and Hutchinson [2] who proved (using the 4-color theorem) that there exists precisely one 6-chromatic 6-regular graph T_{11} on the torus. The problem is also mentioned in [3, Question 3]. Our result (which is independent of the 4-color theorem) has several consequences as mentioned below.

It implies the conjecture ([2, Conjecture 1; 3, Question 1]) that every 6-chromatic graph on the torus must contain at least one noncontractible triangle. For, if G is one of the graphs in the abstract, then the number of triangles in G exceeds the number of face boundaries in any embedding of G on the torus. So G has a triangle, which is not a face boundary. Since G has no separating triangle, the above triangle is noncontractible.

This result can be applied to the conjecture of Straight [13] that every toroidal graph is 5-cocolorable. (In a k-cocoloring of a graph each color class is either a complete graph or a set of pairwise nonadjacent vertices). Straight's conjecture clearly implies the 4-color theorem. For, if G_0 is a counterexample to the latter, then the union of K_6 and five pairwise disjoint copies of G_0 is a counterexample to the former. On the other hand, the 4-color theorem combined with the result described in the abstract implies that every toroidal graph G has a 5-cocoloring. For, if G has no noncontractible 3-cycle, then G is 5-colorable by the main result of this paper. On the other hand, if G has a noncontractible 3-cycle, then we color its vertices by color 5 and color the remaining planar graph in colors 1, 2, 3, 4.

The result described in the abstract gives a positive answer to Question 2 for the torus and for $k=5$. Our proof also gives a polynomial time algorithm for actually describing the 5-coloring if it exists.

Finally, our result implies Tutte's 5-flow conjecture restricted to toroidal graphs. (For definition of 5-flows, see, e.g., [12], where Tutte's 5-flow conjecture is verified for graphs in the projective plane.) We sketch the proof in the toroidal case. Let G be a connected, 2-edge-connected (i.e., bridgeless) graph on the torus. If the (geometric) dual graph G^* of G is 5-colorable, then G has a 5-flow. If G^* contains one of the graphs in the abstract, then G is contractible to a graph H, which is a (geometric) dual graph of one of the graphs in the abstract. By inspection, H has a 5-flow. That 5-flow can be extended to a 5-flow of G. Finally, G^* may have a loop. Then G has an edge e such that $G-e$ is planar. But, it is an easy exercise to extend the dualized proof of the 5-color theorem to show that G has a 5-flow.

As mentioned earlier, examples of Fisk show that there are 5-chromatic graphs on the torus, which have no short noncontractible cycles. Stromquist asked if any such graph must contain one of Fisk's examples, which are all triangulations of the torus with precisely two vertices of odd degree. The answer is negative. For, if v_0 is a vertex of degree 4 in one of Fisk's examples and $v_1v_2v_3v_4$ is the cycle where v_1, v_2, v_3, v_4 are the neighbors of v_0, then we delete v_0 and add instead new vertices u_0, u_1, u_2, u_3, u_4 and the cycles $u_1u_2u_3u_4u_1$ and $v_1u_1v_2u_2v_3u_3v_4u_4v_1$ and the four edges from u_0 to $\{u_1, u_2, u_3, u_4\}$. The new graph is 5-chromatic and toroidal and does not contain one of Fisk's examples. So, the class of 5-critical graphs on the torus may be complicated.

2. Terminology

A *graph* G consists of a finite *vertex set* $V(G)$ and a set $E(G)$ of unordered pairs of vertices called *edges*. If the edge xy is present we say that x is *adjacent* to y or is *joined* to y and that x and y are *neighbors*. The *degree* of a vertex x is the number of neighbors. It is denoted $d_G(x)$ or just $d(x)$. If all vertices have degree r, then G is *r-regular* or just *regular*. If H is a subgraph of G, then $G(H)$, the subgraph of G *induced* by H, consists of H and all edges in G joining two vertices of H. We define $G-H=G(V(G)\setminus V(H))$. If v is a vertex in a graph G, then $N(v, G)$ or just $N(v)$ is the subgraph of G induced by the neighbors of v. If G and H are graphs, then $G \cup H$ is the disjoint union of G and H, unless it is clear from the context that they are both subgraphs of a given graph. The *join* $G+H$ of G and H is obtained from $G \cup H$ by adding all edges from G to H.

A walk in a graph G is a sequence $x_1x_2\cdots x_n$ of vertices such that x_i and x_{i+1} are neighbors for $i=1, 2, ..., n-1$. If all vertices are distinct, the walk is called a *path* or an *n-path* and is denoted P_n. The *n-cycle* C_n is obtained from P_n by adding $x_n x_1$. If C is a cycle in a graph and e is an edge joining two nonconsecutive vertices of C, then e is a *chord* of C. A cycle C in G is a *Hamiltonian cycle* if $V(C)=V(G)$. K_n is the complete graph with n vertices; i.e., all vertices of K_n have degree $n-1$. A complete subgraph in a graph G is also called a *clique* in G. The *clique number* of G is the maximum number of vertices of a clique in G.

The graph G' obtained from a graph G by identifying two vertices x, y into a vertex v is obtained by first deleting x, y, and then adding a new vertex v and all edges vz, where either zx or zy or both are present in G. The *multigraph* G'' obtained from G by identifying x and y is obtained from G' by adding an additional edge vz for every vertex z which in G is joined to both x and y. We refer to such an edge vz as a *double edge*. The operation of going from G' to G is called *splitting* v into x and y. If x and y are neighbors in G, then G' (respectively G'') is the graph (respectively multigraph) obtained by *contracting* the edge xy. We shall sometimes write $v=y$ and say that G' or G'' is obtained from G by *contracting* xy into y.

A *k-coloring* of a subgraph H of G is a map c: $V(H)\to\{1, 2, ..., k\}$ (the color set) such that any two neighboring vertices are mapped to distinct colors. If a graph G has a k-coloring, it is called *k-colorable*. G is called k-critical (respectively *k-vertex-critical*) if G is not $(k-1)$-colorable, but every proper subgraph (respectively every subgraph with fewer vertices) is $(k-1)$-colorable. G is *k-chromatic* if G is k-colorable but not $(k-1)$-colorable. Gallai [8] proved the following.

THEOREM 2.1. *If G is a k-critical graph with at most $2k-2$ vertices, then G is of the form $G=G_1+G_2$, where G_i is k_i-critical for $i=1, 2$ and $k_1+k_2=k$.*

It is easy to see that Theorem 2.1 remains true if "k-critical" is replaced by "k-vertex-critical."

Let H_7 be obtained by applying Hajos' construction to K_4. That is, H_7 is obtained from $K_4\cup K_4$ by deleting an edge xy in one of the K_4's, an edge uv in the other K_4. Then we add yv and identify x and u. Now H_7 is 4-critical. There is precisely one other 4-critical graph on seven vertices, namely M_7, which is obtained from a 6-cycle $x_1x_2\cdots x_6x_1$ by adding a new vertex v and the edges x_1x_3, x_3x_5, x_5x_1, vx_2, vx_4, vx_6; see [8, 17].

S_g is the sphere with g handles added. We shall here only consider the sphere S_0 and the torus S_1. S_1 is homeomorphic to a surface obtained by pasting triangles in the plane together. A graph embedded on a surface S is a graph on S such that the edges are polygonal arcs which do not intersect except at a common vertex. We shall speak of the *clockwise ordering* of the edges incident with a vertex v. A *face* of an embedded graph is an arcwise connected component of the surface minus the graph. We shall assume that each face is homeomorphic to a disc. *A facial walk* is obtained by turning sharp left at every vertex. If this walk has length k we speak of a *k-face*. A *facial cycle* is a cycle which is also a facial walk. If G is embedded in S_1 and H is a subgraph of G, then H is also embedded in S_1. We speak of the *induced embedding*. If C is a facial walk of H bounding a face which is homeomorphic to a disc, then that face is the *interior* of C and is denoted $\mathrm{int}(C, H)$ or just $\mathrm{int}(C)$. The *exterior* $\mathrm{ext}(C, H)$ is defined similarly. A vertex of G in $\mathrm{int}(C, H)$ is said to be *inside* C. If C is a cycle of G separating S_1 into components, one of which is homeomorphic to a disc, then C is *contractible*. Otherwise, C is *noncontractible*. A *triangulation* is an embedded graph such that every face is bounded by a triangle (3-cycle). A *near-triangulation* of the plane is a graph in the plane such that each face, except possibly one, which we call the *outer face*, is bounded by a 3-cycle. The outer face should be bounded by a cycle, the *outer cycle*.

The *excess* of a facial walk is the length of the walk minus 3. The *facial excess* of the embedding is the excess sum taken over all facial walks.

A graph is *planar*, respectively *toroidal*, if it can be embedded on S_0 or S_1, respectively.

Euler's formula implies:

LEMMA 2.2. (a) *If G is a multigraph with n vertices and e edges embedded in S_1 such that no face is bounded by a 2-cycle, then the facial excess equals $3n - e$.*

(b) *A necessary condition for a graph G to be toroidal is that $e \leqslant 3n$. If equality holds then each $N(v)$ has a Hamiltonian cycle. Moreover, it is possible to select a Hamiltonian cycle in each $N(v)$ such that every edge of G is covered twice by such Hamiltonian cycles.*

It is well known that K_7 is toroidal. Duke and Haggard [7] proved the following.

PROPOSITION 2.3. *The minimal nontoroidal graphs on eight vertices are obtained from K_8 by deleting the edges of one of the following three graphs: K_3, $K_{2,3}$, $K_2 \cup K_2 \cup P_3$.*

We use the fact that any graph which contains a subgraph which can be contracted to any of the three graphs above is nontoroidal.

LEMMA 2.4. *If K_6 is embedded on S_1, then every facial walk is a cycle.*

Proof. Suppose (*reductio ad absurdum*) that $W\colon x_1 x_2 \cdots x_m x_1$ is a facial walk, which is not a cycle. Then we can assume that $x_1 = x_i$ for some i, $1 < i \leqslant m$. As K_6 has no double edges, $4 \leqslant i \leqslant m-2$. If we cut S_1 along a simple polygonal curve from x_1 to x_i inside the face bounded by W, then the resulting topological space is connected (because x_2 and x_m are joined by an edge) and is therefore a cylinder. This cylinder and hence K_5 can be embedded in S_0, a contradiction. ▮

We use a special case of the genus additivity theorem [4].

LEMMA 2.5. *If G_1 and G_2 are nonplanar graphs having at most one vertex in common, then $G_1 \cup G_2$ is nontoroidal.*

We use the observation that, if a connected nonplanar graph is embedded on the torus, then each face is (homeomorphic to) a disc. Finally, we discuss the toroidal character of the graphs in the abstract. It is well known that K_7 triangulates the torus.

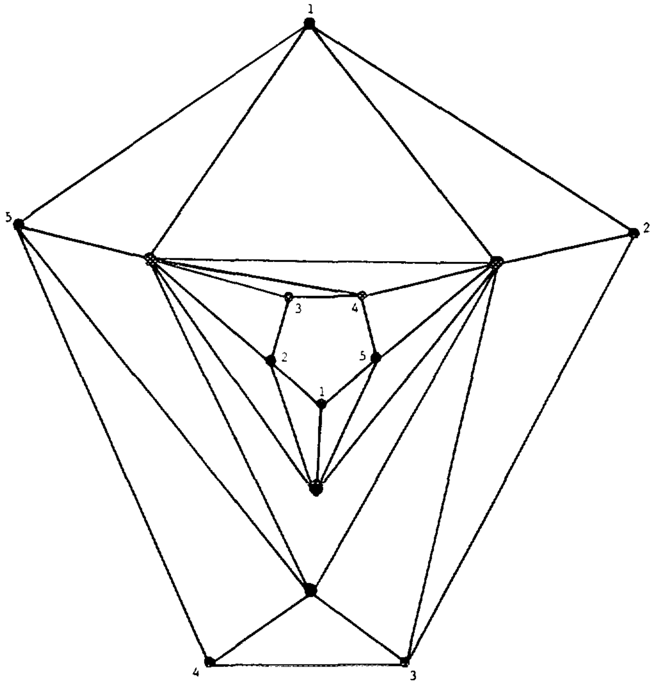

FIGURE 1

Figure 1 shows a graph in the plane. When the two faces bounded by pentagons are deleted and the pentagons are identified as indicated by the labeling, then the graph becomes $K_2 + H_7$ on the torus S_1. Let H_7^* be obtained from H_7 by adding an edge between two nonadjacent vertices, which are both joined to the vertex of degree 4 in H_7. Then Fig. 1 can be extended to a toroidal drawing of $K_2 + H_7^*$. If we add an edge to $K_2 + H_7$, such that the resulting graph G is not isomorphic to $K_2 + H_7^*$, then G is nontoroidal. For if we contract the new edge, we obtain a nontoroidal graph by Proposition 2.3. If we contract the new edge in $K_2+H_7^*$, we obtain a toroidal embedding of C_3+C_5, which is maximally toroidal. It is easy to see that $K_2+H_7^*$ is 5-vertex-critical. As T_{11} defined below is maximally toroidal, we obtain the curious corollary of Theorem 6.1: $K_2+H_7^*$ is the only graph on the torus, which is 5-vertex-critical, but not 5-critical. This answers [3, Question 2].

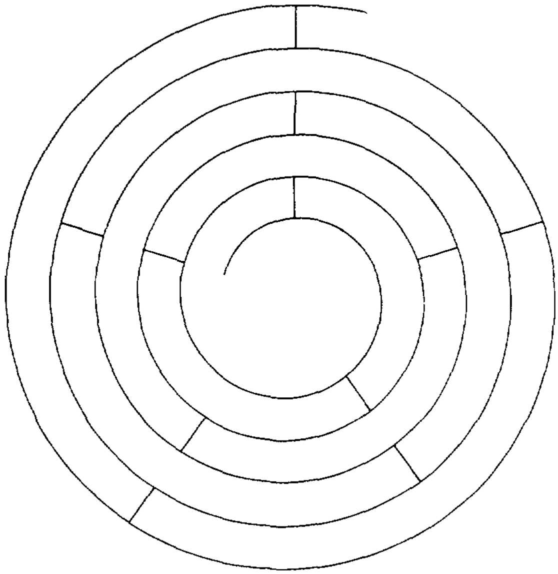

FIGURE 2

Figure 2 shows an infinite graph obtained from a spiral in the plane. This map can be face-colored in 4-colors but only in one way (except for a permutation of the colors). Now we fix a natural number $m \geqslant 4$. We call the faces in the spiral in Fig. 2, $\ldots F_{-1}, F_0, F_1, \ldots$. Then we identify F_i with F_{i+m} for $i \in \{\ldots, -1, 0, 1, \ldots\}$. This transforms the plane into the torus. We define T_m as the dual graph of the resulting toroidal graph (i.e., the vertices of T_m are the m faces of the toroidal graphs of Fig. 2, and two vertices of T_m are adjacent iff the corresponding faces are). Then T_m is 4-colorable if and only if m is divisible by four. For each $k \geqslant 1$, T_{4k+1} is 5-vertex-critical. T_{11} is 6-critical. T_{11} is called $H_{11,6,e}$ in [15] and J in [1, Fig. 1].

3. 5-Coloring 6-Regular Graphs on the Torus

LEMMA 3.1. *Let G be a planar graph with outer cycle S such that all vertices inside S have degree at least 6. Let q be the number of vertices inside S. If $q \geqslant 5$, then $|V(S)| \geqslant 11$.*

Proof. Put $|V(S)| = k$. By Euler's formula, G has at most $2k + 3q - 3$ edges. Hence $G - E(S)$ has at most $k + 3q - 3$ edges. On the other hand, the number of edges of $G - E(S)$ is at least $6q$ minus the number of edges in $G - V(S)$. Now $G - V(S)$ has q vertices and less than $3q - 7$ edges. For if $G - V(S)$ has $3q - 7$ or more edges, then $G - V(S)$ is either a triangulation or a triangulation with one edge missing and with outer cycle of length 3 or 4. But it is not possible that all vertices inside that cycle have degree at least 6. Hence $k + 3q - 3 \geq 6q - (3q - 8)$ which implies that $k \geq 11$. ∎

PROPOSITION 3.2. *Let G be a 6-regular graph on the torus S_1. If G contains a vertex v, such that $\{v\} \cup N(v)$ induces a nonplanar graph, then $G = K_7$ or G is obtained from K_8 or K_9 by deleting the edges of a 1-regular or 2-regular subgraph. In particular, G is 5-colorable unless $G = K_7$.*

Proof. By Euler's formula, G triangulates S_1. Let $S: v_1 v_2 \cdots v_6 v_1$ be the facial cycle of $G - v$ having v in its interior. As $\{v\} \cup N(v)$ induces a nonplanar graph, we can choose the notation such that S has two chords $v_1 v_i$, $v_j v_k$, where $1 < j < i < k \leq 6$. The two 3-cycles $vv_1 v_i v$ and $vv_j v_k v$ are noncontractible because G is 6-regular. Hence none of these triangles separate S_1. We consider the induced embedding of the subgraph G' consisting of v, S, the edges incident with v, and $v_1 v_i$, $v_j v_k$. Then G' has six 3-faces and one 10-face. By Lemma 3.1, G has at most four vertices not in G'. However, it is easy to see that we cannot add chords to S and then add four or three vertices to G' inside the 10-face and obtain a 6-regular graph. So $|V(G)| = 7$, 8, or 9. ∎

THEOREM 3.3. *Let G be a 6-regular graph on the torus S_1. Then G is 5-colorable unless $G = K_7$ or $G = T_{11}$.*

Proof. By Proposition 3.2, we can assume that, for each vertex v, $\{v\} \cup N(v)$ induces a planar graph. The structure of G is now completely described in [15]. By [15, Theorem 3.2], G is a dual graph of one of the graphs described in [15, Theorem 3.1]. It is now easy to face-color the graphs of [15, Theorem 3.1] in five colors such that neighboring faces obtain different colors unless $G = T_{11}$. ∎

The triangulation T_{11} was found by Albertson and Hutchinson [2], who used the 4-color theorem to prove that it is the only 6-regular, 6-chromatic graph on the torus. Our proof does not depend on the 4-color theorem. Also, it easily generalizes to an analogous result for the Klein bottle, and it can also be extended to characterize the 6-regular graphs on the torus, which are not 4-colorable. These are all of the form $H_{k,m,e}$ in [15].

4. 5-Coloring Planar Graphs with Precolored Outer Cycle

Proposition 4.1. *Let G be a planar graph with outer cycle $S: x_1 x_2 \cdots x_k x_1$, $k \leqslant 6$. Let c be a 5-coloring of $G(S)$. Then c can be extended to a 5-coloring of G if and only if none of* (i), (ii), (iii) *below hold*:

(i) *S has five colors. G has a vertex joined to all five colors of S.*

(ii) *$k = 6$, and S has precisely four colors. $G - S$ contains two adjacent vertices each joined to all four colors of S.*

(iii) *$k = 6$, and S has precisely three colors. $G - S$ contains three pairwise adjacent vertices each of which is joined to all three colors of S.*

Proof. If either (i), (ii), or (iii) holds, then clearly G is not 5-colorable. Assume now that none of (i), (ii), (iii) holds. We prove by induction on $|V(G)|$ that c can be extended to a 5-coloring of G. For $|V(G)| \leqslant 3$, there is nothing to prove so assume that $|V(G)| \geqslant 4$.

Let v be a vertex joined to say m vertices of S, where m is largest possible. Since (i) does not hold, the coloring c can be extended to v.

If $m \geqslant 4$, we apply the induction hypothesis to each face of $G(S \cup \{v\})$. So assume that $m \leqslant 3$.

If $m = 3$, then we easily complete the proof by induction, unless v is joined to three consecutive vertices of S, say x_1, x_2, x_3. We apply the induction hypothesis to $S' = x_1 v x_3 x_4 \cdots x_1$ and its interior. We cannot have (i) as $m \leqslant 3$.

Assume that G has two adjacent vertices v_1, v_2 joined to the same four colors of S'. As $m = 3$ both v_1 and v_2 are joined to v. Uncoloring v, there are two choices for each of $c(v)$, $c(v_1)$, $c(v_2)$. It is possible to obtain the 5-coloring unless each of v, v_1, v_2 are joined to the same three colors of S. That is, (iii) holds, a contradiction.

Assume next that (iii) holds for $G - x_2$; i.e., $G - (S \cup \{v\})$ has three pairwise adjacent vertices v_1, v_2, v_3 such that each of them is joined to the same three colors of S'. If v is joined to only one of v_1, v_2, v_3, say v_3, then we can color v_1, v_2, v_3, v in that order. So we can assume that v is joined to v_1, v_2 and that G has the edges $v_2 x_3$, $v_2 x_4$, $v_3 x_4$, $v_3 x_5$, $v_3 x_6$, $v_1 x_6$, $v_1 x_1$. By (iii) we can assume that $c(x_4) = 1$, $c(x_5) = 2$, $c(x_6) = 3$. Since also v_1 is joined to all three colors of S', $c(v) \in \{1, 2\}$. Since v_2 is joined to all three colors of S', $c(v) \in \{2, 3\}$. Hence $c(v) = 2$. But when we colored v we had two choices. Therefore we can assume that $c(v) \neq 2$. This contradiction completes the case $m = 3$.

Assume finally that $m \leqslant 2$. By Lemma 3.1, $G - S$ has a vertex v of degree $\leqslant 5$ (in G). We can assume that G has no separating triangle, since otherwise we complete the proof by induction. Hence v has two neighbors u_1, u_2, which are not adjacent and which are not on S. Now delete

v and identify u_1, u_2. If the resulting graph G' is 5-colorable, then so is G. So we can assume that G' satisfies either (i), (ii), or (iii). But then $m \geqslant 3$. This contradiction completes the proof. ∎

COROLLARY 4.2. *Let G be a 6-vertex-critical graph on some surface. Let H be a connected induced subgraph of G (i.e., $H = G(H)$). If each facial walk of H has length 3 or 4, then $H = G$.*

Proof. If $H \neq G$, then H is 5-colorable. Any 5-coloring of H can be extended to a 5-coloring of G by Proposition 4.1. But G is not 5-colorable, so $H = G$. ∎

5. CHROMATIC PROPERTIES OF $C_3 + C_5$ AND $K_2 + H_7$

LEMMA 5.1. *Let G be a copy of $C_3 + C_5$ or $K_2 + H_7$. Let $z_0 z_1 z_2 z_3 z_0$ be a 4-cycle in G such that z_0 and z_2 are neighbors. Let G_1 be obtained from G by splitting z_0 into two nonadjacent vertices x, y such that G_1 contains the edges $z_1 y$, $z_3 x$, $z_2 y$, $z_2 x$ and such that z_2 is the only vertex joined to both x and y. Then G_1 has 5-colorings c, c', c'' such that*

(a) $\{c(x), c(z_3)\} \neq \{c(y), c(z_1)\}$.

(b) $c'(y) = c'(z_3)$ *or* $c'(x) = c'(z_1)$.

(c) *Either* $c''(y)$, $c''(z_1)$, $c''(z_2)$, $c''(z_3)$, $c''(x)$ *are all distinct or* $c''(y)$, $c(z_1)$, $c''(z_2)$, $c''(z_3)$ *are not distinct.*

Proof of (a). $G_1 - x$ is a proper subgraph of G and is therefore 5-colorable. If x is joined to only 3-colors, then we have at least two choices for $c(x)$ and we can obtain the desired coloring. So assume that x is joined to at least four colors. In particular x (and similarly y) has degree $\geqslant 4$ in G_1. Then z_0 has degree $\geqslant 7$ in G, and so z_0 is joined to all other vertices of G. If $G = C_3 + C_5$, then we can assume that some z_i $(i = 1, 2, 3)$ is in the C_5. Then we 5-color $G - z_0$ such that at least one vertex distinct from z_i has the same color as z_i. This 5-coloring can be extended to the desired 5-coloring of G. (Note that, if $c(z_1) = c(z_3)$, then x and y cannot be joined to the same four colors because that would result in a 5-coloring of G.)

We now assume that $G = K_2 + H_7$. Again both x and y have degree $\geqslant 4$ in G_1 (and z_0 has degree 8 in G). So we can assume that $N(y)$ and $N(x)$ have vertex sets $\{z_1, z_2, w_1, w_2, w_3\}$ and $\{z_2, z_3, u_1, u_2\}$, respectively. As $G - \{z_0, z_3\}$ is 4-colorable, there exists a 5-coloring c of $G_1 - x$ such that y and z_3 are the only vertices of color 1. Also we can assume that x is joined to four colors; i.e., u_1, u_2, z_2 have colors 2, 3, 4, respectively. If $c(z_1) \neq 5$, we color x by 5 and complete the proof. So assume that $c(z_1) = 5$.

If y is not joined to colors 2, 3, we change $c(y)$ to 2 or 3 and complete the proof. So we can assume that $c(w_1)=2$, $c(w_2)=3$. We now interchange colors of w_1 and y (and change $c(w_3)$ to 1 if $c(w_3)=2$). This new coloring can be extended to the desired 5-coloring unless z_3 is joined to w_1 (or to w_3 if $c(w_3)=2$). Similarly, z_3 is joined to w_2 (or to w_3 if $c(w_3)=3$). So z_3 is joined to at least two of w_1, w_2, w_3, say w_1 and w_2. Also, z_3 is joined to u_1 since otherwise we change $c(u_1)$ to 1 and color x by 2. Similarly, z_3 is joined to u_2. So $d(z_3) \geqslant 6$. We have previously proved that in any 4-coloring of $G-\{z_0, z_3\}$, u_1, u_2, z_1, z_2 are forced to have distinct colors. Hence one of these vertices has degree 8. (That vertex must be one of z_1, z_2.) So z_3 is the unique vertex in G of degree 6. But if we 4-color $G-\{z_0, z_3\}$ and then delete the vertex in $G-\{z_0, z_3\}$ which is joined to all other vertices, then we obtain a 3-coloring of H_7 minus the vertex z_3 of degree 4 (in H_7). In that 3-coloring the four neighbors of z_3 (which are u_1, u_2, w_1, w_2) must have three distinct colors. This contradiction proves (a).

Proof of (b). At least one of x, y (say x) has degree $\leqslant 4$ in G_1. As in (a) we can 5-color $G_1 - x$ such that y and z_3 are vertices of color 1. This proves (b).

Proof of (c). We can assume that x has degree $\geqslant 5$ in G_1, since otherwise we use the proof of (b). If y has degree 2 in G_1, then we first 5-color G_1 and then change the color of y if necessary. So we can assume that y has degree $\geqslant 3$ in G_1. Hence z_0 is joined to all other vertices of G. Let w be a vertex in $N(y)-\{z_1, z_2\}$. As $G-\{w, z_0\}$ is 4-colorable, $G_1 - y$ has a 5-coloring c'' such that x and w are the only vertices of color 5. Extend c'' to 5-coloring (which we also call c'') of G_1. Then c'' has the desired property. ∎

LEMMA 5.2. *Let G be a copy of $C_3 + C_5$. Let S: $z_0 z_1 z_2 z_0$ be a 3-cycle in G and let u_1 be a vertex in $G-S$ joined to z_0. Let G' be obtained from G by splitting z_0 into two nonadjacent vertices x and y such that u_1 and at most one more vertex u_0 in G' is joined to both x and y and such that $y z_1 z_2 x$ is a path in G'. Let G'' be obtained from G' by adding a vertex v_0 and joining v_0 to x, y, u_1, z_1, z_2. If G'' is toroidal and non-5-colorable, then $G'' \supseteq C_3 + C_5$.*

Proof. Assume that $G'' \not\supseteq C_3 + C_5$. If one of x, y has the same neighbors in G' as z_0 does in G, then clearly $G' \not\supseteq C_3 + C_5$, a contradiction. So assume that z_0 has two neighbors in G such that one is a neighbor of x but not y, and the other is a neighbor of y but not x.

We can assume that each of x, y has degree at least 5 in G'', and hence z_0 has degree $\geqslant 6$ in G. For if x, say, has degree $\leqslant 4$ in G'', then

$G'' - \{x, v_0\}$ is a proper subgraph of $C_3 + C_5$ and has therefore a 5-coloring. Any such 5-coloring can be extended to a 5-coloring of G'' by first coloring v_0 and then x.

G consists of 5-cycle $p_1 p_2 p_3 p_4 p_5 p_1$ and a 3-cycle $q_1 q_2 q_3 q_1$ and all 15 edges $p_i q_j$ $(1 \leqslant i \leqslant 5, 1 \leqslant j \leqslant 3)$. As $d_G(z_0) \geqslant 6$, $z_0 \in \{q_1, q_2, q_3\}$.

Consider first the case where z_0, z_1, z_2 are q_3, q_1, q_2, respectively. If both u_0 and u_1 are in $\{p_1, p_2, p_3, p_3, p_5\}$, then we color y, z_1, z_2, x by 2, 1, 2, 1, respectively. The remaining vertices in G'' are colored 3, 4, 5. If $u_1 = p_1$ and $u_0 = z_1$, we color y, z_1, z_2, x, u_1 by 2, 1, 2, 3, 4, respectively. As y has degree $\geqslant 5$ in G'', some vertex in $\{p_2, p_3, p_4, p_5\}$ can obtain color 3. The remaining vertices are colored by 4, 5.

Consider next the case where z_0, z_1, z_2 are q_1, p_1, p_2, respectively. We consider the subcase where u_0 is not in $\{z_1, z_2\}$. Then we color y, z_1, z_2, x, u_0, u_1 by 2, 1, 2, 1, 3, 4, respectively. This coloring can be extended to a 5-coloring of G'' (coloring v_0 last), except in the following three cases (or cases which are equivalent to one of them by permuting colors or using automorphisms of G'' or renaming u_0 and u_1): $u_0 = q_2$ and $u_1 = p_4$ (in which case we color q_3 by the same color as x or y and recolor either z_1 or z_2 by 4 and color the remaining vertices 5) or $u_0 = p_3$ and $u_1 = p_4$ (in which case we color q_3 by 1 or 2 and recolor z_1 or z_2 by 4 as in the previous case; then we color p_5, q_2 by 3, 5, respectively). Finally, if $u_1 = p_3$ and $u_0 = p_5$, then color q_3 by 1 or 2 and recolor one of z_1, z_2 by 3 and we recolor p_3, p_4, p_5, q_2 by 4, 3, 4, 5, respectively.

Consider now the subcase where z_0, z_1, z_2 are q_1, p_1, p_2, respectively, and $u_0 \in \{p_1, p_2\}$, say $u_0 = p_2$. If $u_1 \in \{p_3, p_4, p_5\}$, then we color y, z_1, z_2, x by 2, 4, 3, 1, and we color u_1 by 3 unless $u_1 = p_3$, in which case we color it by 4. Then we color one of q_2, q_3 by 1 or 2. If both q_2, q_3 can be colored 1, 2, then it is easy to complete the coloring. So we can assume that q_2, q_3 are colored by 2, 5, respectively, and that both q_2, q_3 are adjacent to x. (If they are both adjacent to y, the proof is similar.) Since y has degree at least 4 in G', at least one vertex in $\{p_3, p_4, p_5\} \setminus \{u_1\}$ is joined to y and is colored 1. Now it is easy to complete the 5-coloring (by possibly interchanging the color of z_1 and z_2).

If z_0, z_1, z_2 are q_1, p_1, p_2, respectively, and $u_0 = p_2$, $u_1 = q_2$, then we color y, z_1, z_2, x, q_2 by 2, 1, 3, 1, 4, respectively. If q_3 can be colored 2, then we color p_3, p_4, p_5, v_0 by 5, 3, 5, 5, respectively. So assume that q_3 is adjacent to y. Then we color q_3 by 5. If we can color $\{p_3, p_4, p_5\}$ by colors $\{1, 2, 3\}$, then color v_0 by 5. So assume that $\{p_3, p_4, p_5\}$ cannot be colored $\{1, 2, 3\}$. Then p_3, p_4 are joined to the same vertex in $\{x, y\}$. That vertex must be x, since x has degree at least 4 in G'. Then we can assume that p_5 is adjacent to y, since otherwise we color p_3, p_4, p_5 by 2, 3, 2, respectively. (Although we shall not need it, we mention that the graph we are now considering is not 5-colorable.) We now show that G'' is nontoroidal. For, if we contract v_0x into x and p_5p_4 into p_4, then the only missing edges in the resulting graph are p_1p_3, p_3y, p_2p_4, xq_3 and, hence G'' is nontoroidal by Proposition 2.3.

Finally we consider the case where z_0, z_1, z_2 are q_1, q_2, p_1, respectively. If $u_0 \notin \{z_1, z_2\}$, then we color $y, z_1, z_2, x, p_2, p_3, p_4, p_5, q_3$ by 2, 1, 2, 1, 3, 4, 3, 4, 5, respectively. If $u_0 = p_1$, then we color y, z_1, z_2, x by 2, 1, 3, 1, respectively. If q_3 is not joined to y, then we color q_3 by 2 and the vertices $p_2, \ldots, p_5$ by 4 and 5. If q_3 is adjacent to y, then we color q_3 by 5. The assumption that x has degree at least 4 in G' implies that some vertex in $\{p_2, \ldots, p_5\}$ can be colored 2. The others are colored by 3, 4. So we assume that $u_0 = q_2 = z_1$. We color y, z_1, z_2, x, u_1 by 2, 3, 2, 1, 4 and we try to extend this coloring. If q_3 can be colored 1, then we color $p_2, p_3, \ldots$ by 4 and 5. So we assume that q_3 is joined to x. If $u_1 = p_3$, then we recolor z_2 by 4 and color q_3 by 2. As y has degree at least 4 in G', it is joined to at least one of p_4, p_5 which we color 1. The remaining vertices of $\{p_1, \ldots, p_5\}$ are colored 5. If $u_1 = q_3$, then we color one of p_2 or p_5 by 1 if possible, and we complete the coloring by using 5 for two vertices in $\{p_2, p_3, p_4, p_5\}$. So assume that both p_2 and p_5 are joined to x. Since y has degree at least 4 in G', it is joined to both of p_3, p_4. We claim that G'' is nontoroidal. For if G'' has an embedding on the torus, then that embedding is a triangulation (because G'' has 10 vertices and 30 edges). Consider the induced embeddings of $G'' - p_2$, $G'' - p_5$, and $G'' - v_0$, respectively. The face of $G'' - p_2$ containing p_2 is bounded by a Hamiltonian cycle R_1 of $N(p_2)$. Similarly, we let R_2, R_3 denote Hamiltonian cycles of $N(p_5)$ and $N(v_0)$, respectively. Each of R_1, R_2, R_3 contains xp_1. So xp_1 is in three facial triangles, a contradiction.

Finally, we consider the subcase, where z_0, z_1, z_2, u_0, u_1 are q_1, q_2, p_1, q_2, p_2, respectively. We color y, z_1, z_2, x, u_1, q_3 by 2, 3, 2, 1, 4, 5, respectively. We can assume that q_3 is joined to x since otherwise we recolor q_3 by 1 and complete the coloring. We can assume that p_5 is joined to x since otherwise we color p_5, p_4 by 1, 4 and complete the coloring. Now we color p_5 by 4. The coloring can be completed unless p_3 and p_4 are both joined to the same vertex in $\{x, y\}$. That vertex must be y as y has degree at least 4 in G'. If we contract v_0y into y and p_4p_5 into p_4, then the only missing edges in the resulting graph are q_3y, xp_3, p_3p_1, p_2p_4. So this graph (and hence also G'') is nontoroidal by Proposition 2.3. ∎

LEMMA 5.3. *Let G, G', and let G'' be as in Lemma 5.2, except that now G is a copy of $K_2 + H_7$. If G'' is toroidal and non-5-colorable, then $G'' \supseteq K_2 + H_7$.*

Proof. The proof is like that of Lemma 5.2. In the present lemma, the embedding part is easier in that we can replace the condition that G'' is toroidal by the weaker condition that $N(x)$ is Hamiltonian for each vertex x in G. We leave the tedious details for the reader. ▮

6. The 5-Color Theorem

Theorem 6.1. *Let G be a graph on the torus. Then G is 5-colorable if and only if G does not contain K_6 or $C_3 + C_5$ or $K_2 + H_7$ or T_{11}.*

Proof. The "only if" part is trivial. We prove the "if" part by contradiction. Assume that G_0 is a counterexample such that

(i) $|V(G_0)|$ is minimum.

By Theorem 3.3, G_0 has a vertex v_0 of degree $\leqslant 5$. We choose G_0 and v_0 such that

(ii) the clique number of $N(v_0)$ is maximum subject to (i).

(iii) The number of largest complete subgraphs in $N(v_0)$ is maximum subject to (i) and (ii).

(iv) The number of edges in $N(v_0)$ is maximum subject to (i), (ii), (iii).

(v) $|E(G_0)|$ is minimum subject to (i) – (iv).

We now derive a number of properties of G_0 and finally reach a contradiction.

(1) G_0 is 6-vertex-critical.

Proof of (1). For any vertex v of G_0, $G_0 - v$ is not a counterexample to Theorem 6.1. Hence $G_0 - v$ is 5-colorable.

(2) Each vertex of G_0 has degree at least 5.

Proof of (2). Every 6-vertex-critical graph has minimum degree at least 5.

By (2), we have

(3) v_0 has degree 5.

As G_0 does not contain K_6 we conclude that

(4) $N(v_0)$ is not a complete graph.

Let x, y be any pair of nonadjacent vertices in $N(v_0)$. Let G_{xy} denote the graph obtained from $G_0 - v_0$ by identifying x and y. Clearly, G_{xy} is toroidal. If G_{xy} has a 5-coloring, then it is easy to modify that 5-coloring to a 5-coloring of G_0. Hence

(5) G_{xy} is not 5-colorable.

Then (i) implies that

(6) G_{xy} contains either K_6 or C_3+C_5 or K_2+H_7 or T_{11}.

Let G'_{xy} be a copy of K_6 or C_3+C_5 or K_2+H_7 or T_{11} in G_{xy}. Let G''_{xy} be the multigraph in G_{xy} induced by G'_{xy}. Then

(7) G_0 consists of v_0, $N(v_0)$, the five edges from v_0 to $N(v_0)$, and the union of all the graphs obtained from G_{xy} by splitting the contracted vertex into x and y, where the union is taken over all pairs of nonadjacent vertices x, y in $N(v_0)$.

Proof of (7). The subgraph described in (7) is not 5-colorable. For if it had a 5-coloring, then some two nonadjacent vertices x, y in $N(v_0)$ would have the same color. This would result in a 5-coloring of G_{xy}, a contradiction. Now (7) follows by the minimality properties (i), (v) in Theorem 6.1.

(7) implies that G_0 has at most 106 vertices and, hence, what remains is a finite problem.

(8) $|V(G_0)| \geqslant 10$. If equality holds, then G_0 has a vertex which is adjacent to all other vertices.

Proof of (8). Assume that $|V(G_0)| \leqslant 10$. Then Theorem 2.1 implies that G_0 is of the form H_1+H_2, where H_i is k_i-vertex-critical, $k_1 \leqslant k_2$, $k_1+k_2=6$. If $k_1=k_2=3$, then $G_0=K_6$ or $G_0=C_3+C_5$, a contradiction. So $k_1 \leqslant 2$ and, hence, G_0 has a vertex adjacent to all other vertices. Suppose now (*reductio ad absurdum*) that $|V(G_0)| \leqslant 9$. If $k_1=1$, then $|V(H_2)| \leqslant 8$ and, hence, H_2 is of the form $H'_2+H''_2$, where $H'_2=K_2$ or K_1. So we can assume that $k_1=2$ and that H_2 is 4-vertex-critical. The only 4-vertex-critical graphs with $\leqslant 7$ vertices are K_4, K_1+C_5, H_7, and M_7 (see the remark following Theorem 2.1). By assumption, $G_0 \not\supseteq C_3+C_5 = K_2+(K_1+C_5)$ and $G_0 \not\supseteq K_2+H_7$. Hence $G_0 \supseteq K_2+M_7$. Now G_0 has nine vertices and at (least) 27 edges. Hence, $G_0=K_2+M_7$ triangulates S_1. But K_2+M_7 has a vertex v such that $N(v)$ is nonhamiltonian. This contradiction proves (8).

(9) If x and y are two nonadjacent vertices of $N(v_0)$, then G'_{xy} is a K_6 or T_{11}.

Proof of (9). Suppose (*reductio ad absurdum*) that $G'_{xy}=C_3+C_5$ or $G'_{xy}=K_2+H_7$. Let z_0 be the vertex in G'_{xy} corresponding to $\{x, y\}$ in G_0. Let $xu_1u_2 \cdots u_k yz_1z_2 \cdots z_m x$ be the facial walk in the subgraph of G_0-v_0 induced by $(V(G'_{xy})\backslash\{z_0\}) \cup \{x, y\}$ bounding the face containing v_0. We can assume that $1 \leqslant k \leqslant m$, $\{x, y\} \cap \{u_1, ..., u_k\} = \emptyset$ and that G_0 is drawn on the torus such that $k+m$ is minimum. We obtain G'_{xy} by deleting one edge in each double edge of G''_{xy}. As G'_{xy} is a $C_3 + C_5$ or $K_2 + H_7$, G'_{xy} has precisely one face bounded by a 4-cycle by Lemma 2.2. All other faces are bounded by 3-cycles. As G''_{xy} is obtained from G'_{xy} by adding edges, G''_{xy} has at most one face bounded by a 4-cycle. All other faces are bounded by 3-cycles or 2-cycles. Hence $k \leqslant 2$ and $m \leqslant 3$. Also, the vertices $y, z_1, ..., z_m, x$ are distinct. ($z_1 \neq y$ because yz_1 is an edge in G_0, and $z_1 \neq x$ because xy is not an edge in G_0. Also $z_2 \neq x, y$ because G_0 does not have multiple edges.) Finally, all vertices of G_{xy} are either in G'_{xy} or inside one of the cycles $R_1 : v_0 x u_1 \cdots u_k y v_0$ or $R_2 : v_0 y z_1 \cdots z_m x v_0$ by the proof of Corollary 4.2. Let q_i be the number of vertices inside R_i $(i = 1, 2)$.

Consider first the case $m = 3$. Then G''_{xy} has no 2-cycle except possibly xu_1y if $k = 1$. (For any other 2-cycle would be of the form xwy and it would be nonfacial, contradicting Lemma 2.2.) It follows that all vertices $x, u_1, ..., u_k, y, z_1, ..., z_k$ are distinct except that possibly z_2 equals one of u_1, u_2.

By Proposition 4.1 and Corollary 4.2, $q_1 \leqslant 1$ and $q_2 \leqslant 3$. If $q_2 = 3$, then by Lemma 2.2, $N(v_0)$ contains no C_3. But, since G'_{xy} is a $C_3 + C_5$ or a $K_2 + H_7$, it is easy to find a vertex v_1 of degree 5 (not only in G'_{xy} but also in G_0) such that $N(v_1)$ has a C_3 (contracting (ii) in Theorem 6.2) unless $G'_{xy} = C_3 + C_5$, $k = 2$, and z_1, z_2, z_3, u_1, u_2 are the vertices of the C_5. But in that special case it is easy to 5-color G_0: We color x, y by 1, 2, respectively. As G''_{xy} has no double edges when $k = 2$, we can color the other two vertices of the C_3 in $C_3 + C_5$ by 1 and 3. Then we color z_3 by 2 and the remaining four vertices in the C_5 in $C_3 + C_5$ by 4, 5. This coloring can be extended to a 5-coloring of G_0. This gives a contradiction when $q_2 = 3$.

If $m = 3$ and $q_2 = 2$, then we obtain a contradiction as in the case $q_2 = 3$, unless the interior of R_2 consists of vertices w_1, w_2 and the edges w_1w_2, w_1v_0, w_1y, w_1z_1, w_1z_2, w_2z_2, w_2z_3, w_2x, w_2v_0, and v_0 is also joined to $z_2 \in \{u_1, u_2\}$. In this case $q_1 = 0$, by Corollary 4.2, and G_0 is 5-colorable, by Lemma 5.1(a). This contradiction shows that $q_2 < 2$ when $m = 3$.

If $m = 3$ and $q_2 = 1$ and the vertex inside R_2 is called w_1, then w_1 is joined to at least five vertices of R_2. If w_1 is joined to both x and y, then $k = 2$ and v_0 is joined to x, w_1, y, u_2, u_1. Then $N(v_0)$ contains at most one C_3 (if $z_2 \in \{u_1, u_2\}$). But it is easy to find a vertex v of degree 5 in G'_{xy} and in G_0 such that $N(v)$ has at least two $C_3 - s$ because $\{z_1, z_2, z_3, u_1, u_2\}$ has less than five vertices. This contradiction shows that w_1 is not joined to both x and y. In fact, the above argument implies that the notation can be chosen such that w_1 is joined to v_0, y, z_1, z_2, z_3, and v_0 is joined to z_3 and to z_2 which is in $\{u_1, u_2\}$. In particular, $q_1 = 0$. Any 5-coloring of $G_0 - \{v_0, w_1\}$ satisfying the conclusion of Lemma 5.1(c) can be extended to a 5-coloring of G. This contradiction proves that $q_2 = 0$ if $m = 3$.

If $m=3$ and $q_2=0$, then v_0 has at least one neighbor in $\{z_1, z_2, z_3\}$. But then v_0 is joined to both z_1 and z_3. For if v_0 is not joined to z_3 but to z_i ($i=2$ or 1) then we can add the edge $z_i x$, contradicting either the maximality property (iv) in Theorem 6.1 or the minimality of $k+m$. (Note that if v_0 is joined to z_2 and $z_2=u_1$, then $q_1=0$, since otherwise we add $z_2 x$ and delete xu_1 which brings us back to the case $q_2=1$. But then v_0 is joined to z_1 and we can add $z_1 x$.) Now the 5-coloring of G_0-v_0 in Lemma 5.1(b) can be extended to a 5-coloring of G_0. This contradiction shows that $1 \leqslant k \leqslant m \leqslant 2$. Possibly $\{z_1, z_2\} \cap \{u_1, ..., u_k\} \neq \emptyset$.

Now we 5-color G_0 minus the interior of the walk $W=yz_1z_2xu_1 \cdots u_k y$. By Proposition 4.1, the interior of W either contains just one vertex (which must be v_0) joined to all five colors, or it contains two vertices v_0, v_0' and the edges $v_0 y$, $v_0 z_1$, $v_0 z_2$, $v_0 x$, $v_0 v_0'$, $v_0' x$, $v_0' u_1$, $v_0' u_2$, $v_0' y$. Assume first that the interior of W contains both v_0 and v_0'. Since G_{xy} is a C_3+C_5 or K_2+H_7 with one additional vertex (namely v_0'), it is easy to find a vertex v_1 of degree 5 in G_0 such that $N(v_1)$ has at least one triangle. Moreover, if $N(v_0)$ has at least one triangle, then v_1 can be chosen such that $N(v_1)$ has at least two triangles. Hence $N(v_0)$ has at least two triangles by the maximality property (iii) in Theorem 6.1. By Lemma 2.2(a), G_0 cannot contain both edges xz_1, yz_2. Hence $k=2$ and $\{z_1, z_2\} \cap \{u_1, u_2\} \neq \emptyset$. We cannot have $\{z_1, z_2\} = \{u_1, u_2\}$ because then G''_{xy} would have two double edges contradicting Lemma 2.2(a). So we can assume that $u_1 = z_1$ and $u_2 \neq z_2$. By Proposition 4.1, z_2 and u_2 have the same color in any 5-coloring of $G_0 - \{v_0, v_0'\}$. So $G_1 = (G_0 - \{v_0, v_0'\}) \cup \{z_2 u_2\}$ is not 5-colorable. Hence G_1 contains a subgraph G_1' which is one of the graphs K_6, C_3+C_5, K_2+H_7, T_{11}, by the minimality of G_0. If z_2u_2 is contained in a facial cycle $z_2u_2qz_2$ of G_1', then either $qz_2xu_1u_2q$ or $qz_2z_1yu_2q$ is a contractible 5-cycle with more than one vertex in its interior, contradicting Proposition 4.1. So z_2u_2 is not in a facial 3-cycle of G_1'. Hence $G_1' = K_6$. This K_6 contains z_2 and u_2 but none of x, y because G''_{xy} has only one double edge; x or y has degree $\geqslant 6$ in G_0, since otherwise x, y, v_0, v_0' are inside a walk of length 6. So $G_1' = K_6$ can be obtained from G'_{xy} by first deleting a vertex of degree $\geqslant 6$ (and some more vertices) and then adding an edge. But this is impossible as $G'_{xy} = C_3 + C_5$ or $K_2 + H_7$.

There only remains the case where W has one vertex, namely v_0, in its interior. If $k=1$, then there is at most one vertex distinct from u_1, say u_0, which is joined to both x and y, by Lemma 2.2. If $k=2$ we can interchange between $\{z_1, z_2\}$ and $\{u_1, u_2\}$. In any case we can assume that $N(v_0)$ has vertex set $\{x, y, z_1, z_2, u_1\}$ and we can apply Lemmas 5.2, 5.3 to 5-color G_0. This contradiction proves (9).

(10) If x and y are two nonadjacent vertices of $N(v_0)$, then G'_{xy} is a K_6.

Proof of (10). Suppose (*reductio ad absurdum*) that $G'_{xy} = T_{11}$. We define the facial walk W: $yz_1z_2 \cdots z_m xu_1u_2 \cdots u_k y$ as in the proof of (9). As T_{11} is a triangulation, $1 \leqslant k \leqslant m \leqslant 2$. As x and y have degree $\geqslant 5$ in G_0 and they correspond to a vertex of degree 6 in T_{11}, it follows that there are at least two vertices of G_0 inside W. Using Proposition 4.1 we conclude that $k = m = 2$ and that there are precisely two vertices v_0, v'_0 inside W with neighbor sets $\{x, z_1, z_2, y, v'_0\}$ and $\{x, y, u_1, u_2, v_0\}$, respectively. Clearly, G_{xz_1} contains no 6-regular graph with 11 vertices. As $T_{11} \not\supseteq K_5$, G_{xz_1} does not contain K_6 either. This contradiction proves (10).

(11) Each vertex v in $N(v_0)$ has degree at least 2 in $N(v_0)$ or else $N(v_0) \supseteq K_4$.

Proof of (11). Suppose v has degree $\leqslant 1$ in $N(v_0)$. Then $N(v_0)$ has a vertex u such that u is not in $N(v)$, but $G_0 \cup \{uv\}$ is toroidal. The maximality property (iv) in Theorem 6.1 implies that $G \cup \{uv\}$ contains a subgraph M which is isomorphic to K_6 or $C_3 + C_5$ or $K_2 + H_7$ or T_{11}. If M does not contain v_0, then in any 5-coloring of $G - v_0$, u and v must have the same colors (because M is not 5-colorable). But then G_{uv} is 5-colorable, contradicting (5). Hence M contains v_0. As T_{11} is 6-regular, $M \neq T_{11}$. If $M = K_6$ or $C_3 + C_5$, then each vertex of $N(v_0)$ has degree $\geqslant 3$ in $N(v_0) \cup \{uv\}$ and, hence, degree $\geqslant 2$ in $N(v_0)$. So we can assume that $M = K_2 + H_7$ and that v has degree 2 in $N(v_0) \cup \{uv\}$. Hence u has degree 8 in M. By Proposition 4.1, G_0 consists of $M - uv$ and one more vertex joined to u, v and three more vertices in M. But then it is easy to find a vertex v_1 of degree 5 in G_0 such that $N(v_1) \supseteq K_4$. Then also $N(v_0) \supseteq K_4$ by the maximality property (ii). This proves (11).

We let v_0, x, y and the facial walk W: $xu_1u_2 \cdots u_k yz_1z_2 \cdots z_m x$ in $G_0 - v_0$ $(1 \leqslant k \leqslant m)$ and R_1, R_2 be as in (9). As $G'_{xy} = K_6$, G''_{xy} is obtained from K_6 by adding additional edges. It follows that $m \leqslant 5$ and that $m + k \leqslant 7$, by Lemma 2.2. Also, the vertices $x, y, z_1, ..., z_m$ are distinct and $x, y, u_1, ..., u_k$ are distinct by Lemma 2.4. Possibly, $\{z_1, ..., z_m\} \cap \{u_1, ..., u_k\} \neq \emptyset$. Also note that $G(\{z_1, ..., z_m\}) = K_m$ and $G(\{u_1, ... u_k\}) = K_k$. Let z_0 be the vertex in G_{xy} obtained by identifying x and y.

(12) Every vertex s in $G_{xy} - (G'_{xy} \cup N(v_0))$ is joined to four vertices of $G'_{xy} - z_0$ which induce a K_4 in G_0.

Proof of (12). By (7), s is contained in a graph of the form $G'_{x'y'}$, where x' and y' are nonadjacent vertices in $N(v_0)$. As $G'_{x'y'} = K_6$, s is in a $K_5 \subseteq G_0$. That K_5 has four vertices in G'_{xy} because it is not possible to add r vertices inside R_1 or R_2 (or any other cycle) and join them completely to $5 - r$ vertices of the same cycle unless $r = 1$.

(13) $G_{xy} - (G'_{xy} \cup N(v_0))$ has at most one vertex s.

Proof of (13). Suppose (*reductio ad absurdum*) that s, t are two distinct vertices in $G_{xy} - (G'_{xy} \cup N(v_0))$. By (12), each of s, t is joined to four vertices of some facial cycle of G'_{xy}. If both s and t are inside R_2 then the two $K_4 - s$ in $G(R_2)$ that s and t are joined to have at most one vertex in common unless $m = 5$ and the notation can be chosen such that the two $K_4 - s$ in (12), which s and t (respectively) are joined to, have vertex sets $\{z_1, z_2, z_3, z_4\}$, $\{z_4, z_5, y, z_1\}$ or $\{y, z_1, z_2, z_3\}$, $\{z_3, z_4, z_5, y\}$. In the former case we get a contradiction to Lemma 2.2 and in the latter case $\{y, z_1, z_2, z_3, z_4, z_5\}$ induces a K_6 in G, contradicting the initial assumption of the proof of Theorem 6.1. So we can assume that $t \notin \mathrm{int}(R_2)$. Now $N(t) \subseteq G'_{xy}$. For otherwise t would have a neighbor which is in a face of the graph induced by $(G'_{xy} - z_0) \cup \{x, y, v_0, t\}$ bounded by a 3-cycle of a 4-cycle. (If $t \in \mathrm{int}(R_1)$ and $k = 3$, then $N(t) \cap G(R_1) = K_4$ which implies that $u_1 y$ or $u_3 x$ is a double edge in G'_{xy}. Then $m = 3$, $s \in \mathrm{int}(R_2)$, and $d(v_0) \leqslant 4$, a contradiction.) As G_0 minus the vertices in that face is 5-colorable, G_0 is also 5-colorable by Proposition 4.1. So $N(t) \subseteq G'_{xy}$. If t has degree $\geqslant 6$ in G_0, then all facial cycles of the graph induced by $(G'_{xy} - z_0) \cup \{x, y, v_0, t\}$ are 3-cycles or 4-cycles (except possibly R_1 and R_2 which may be 5-cycles) and since that graph does not contain s we again obtain a contradiction by Proposition 4.1. So t has degree 5. If $t \notin \mathrm{int}(R_1)$, then $k \leqslant m \leqslant 3$ by Lemma 2.2(a). And if $t \in \mathrm{int}(R_1)$, then $3 = k \leqslant m \leqslant 4$. Also, $N(t)$ contains a K_4. Hence $N(v_0)$ contains a K_4, by the maximality property (ii) of $N(v_0)$. At most one of x, y is in this K_4 in $N(v_0)$. That K_4 contains at most one vertex q in $\mathrm{int}(R_1) \cup \mathrm{int}(R_2)$. The fifth neighbor of v_0 is either x or y. The subgraph of G_0 induced by $(G'_{xy} - z_0) \cup \{x, y, v_0, t, q\}$ (or $(G'_{xy} - z_0) \cup \{x, y, v_0, t\}$ if q does not exist) has a 5-coloring. That 5-coloring can be extended to a 5-coloring of G_0 by Proposition 4.1. This contradiction proves (13).

(14) If the vertex s in (12), (13) does not exist for any choice of x, y, then $|V(G_0)| = 11$.

Proof of (14). If s does not exist, then $G'_{xy} - z_0$ contains at least four vertices not in $N(v_0)$, by (8). On the other hand G'_{xy} is a K_6, so it has at most five vertices not in $N(v_0)$. So $10 \leqslant |V(G_0)| \leqslant 11$. Suppose (*reductio ad absurdum*) that $|V(G_0)| = 10$. Let $N(v_0) = \{x, y, t_1, t_2, t_3\}$, and the remaining vertices of $G_0 - v_0$ are called w_1, w_1, w_3, w_4. We can assume that $V(G'_{xy}) = \{z_0, t_1, w_1, w_2, w_3, w_4\}$. By (8) we can assume that t_1 has degree 9 in G_0. As $G(\{t_1, w_1, w_2, w_3, w_4\})$ is a K_5, $N(v_0)$ contains no K_4 by Lemma 2.5. Hence each of $w_1, ..., w_4$ has degree >5 by the maximality property (ii) of $N(v_0)$. Also, if $v_0 u$, $v_0 v$ are consecutive in the clockwise ordering around v_0, then uv is an edge of G_0 by (the proof of) (11). So we can assume that $N(v_0)$ contains the path xt_2t_3y or xt_2yt_3 (say the latter, since otherwise we rename the vertices in $N(v_0)$) and all four edges from t_1 to $N(v_0)-t_1$. As $\{v_0\}\cup N(v_0)$ induces a planar graph, by Lemma 2.5, $N(v_0)$ contains no additional edges. There are at least 12 edges from $\{w_1, w_2, w_3, w_4\}$ to $N(v_0)$. Then an easy count shows that G_0 has at least $12+6+7+5=30$ edges. Hence G_0 triangulates the torus and each vertex w_i has degree 6 in G_0. Any vertex in G_0 which has degree 5 can play the role of v_0. So we have proved the following: G_0 has a vertex of degree 9, and if v is a vertex of degree 5 in G_0, then the four vertices not joined to v have degree 6. So the vertices of degree 5 induce a complete graph K_1, K_2, or K_3. Since G_0 has precisely 30 edges, the vertices of degree 5 must induce a K_3. It follows that G_0 is a triangulation of the torus with one vertex of degree 9, six vertices of degree 6, and three vertices of degree 5 which induce a K_3. Suppose $a_1a_2a_3a_1$ is the 3-cycle whose vertices have degree 5. As $G_0-\{a_1, a_2, a_3\}$ has 7 vertices and 18 edges it is nonplanar, and, hence, $a_1a_2a_3a_1$ is contractible. Since $G_0-\{a_1, a_2, a_3\}$ is connected, $a_1a_2a_3a_1$ is facial. Let $b_1b_2\cdots b_6b_1$ be the unique facial 6-walk in $G_0-\{a_1, a_2, a_3\}$. Then we can assume that G_0 contains the walk $a_1b_2a_2b_4a_3b_6a_1$ and the edges a_1b_1, a_2b_3, a_3b_5. Without loss of generality we can assume that b_2 has degree 9. Hence $b_2=b_5$, and b_1, b_2, b_3, b_4, b_6 are distinct. The two vertices c_1, c_2 in $G_0-\{a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_6\}$ are neighbors and are both joined to b_1, b_2, b_3, b_4, b_6. As b_1, b_3, b_4, b_6 have degree 6, G_0 contains the edge b_1b_3. Now we contract a_2b_3 into b_3 and a_3b_4 into b_4. The only missing edges in the resulting graph are a_1c_1, a_1c_2, b_1b_4, b_3b_6. Hence that graph and G_0 are nontoroidal by Proposition 2.3. This contradiction proves (14).

(15) The vertex s in (12), (13) does exist for some two nonadjacent vertices x, y in $N(v_0)$.

Proof of (15). Suppose (*reductio ad absurdum*) that s does not exist. Then $|V(G_0)|=11$ by (14). For any nonadjacent vertices x', y' in $N(v_0)$, $G_{x',y'}$ is a K_6 whose vertices are $\{x, y\}$ and the five vertices in $G_0-(N(v_0)\cup\{v_0\})$ which we denote by $w_1, ..., w_5$. As in the proof of (14) we show that $N(v_0) \not\supseteq K_4$, and if v_0x', v_0y' are consecutive in the ordering around v_0, then $x'y'$ is an edge of G_0.

Hence $N(v_0)$ contains a 5-cycle which we denote by Q: $q_1q_2q_3q_4q_5$. As $N(v_0) \not\supseteq K_4$, we conclude that w_i is joined to at least two vertices of Q for $i=1, 2, ..., 5$. Counting the edges mentioned until now we obtain 30. As G has at most 33 edges, we conclude that if Q has q chords, then there are at most $13-q$ edges from Q to $\{w_1, ..., w_5\}$. For any two nonadjacent vertices x', y' in Q and for each $i\in\{1, ..., 5\}$, w_i is joined to at least one of x', y'. So if $q=0$, w_i is joined to at least three vertices of Q. This implies the presence of at least 15 edges from Q to $\{w_1, ..., w_5\}$, a contradiction.

Also, Q cannot have two chords having no end in common, since that would imply the existence of two disjoint nonplanar subgraphs of G_0, contradicting Lemma 2.5. So we can assume that Q has the chord q_1q_3 and possibly q_1q_4 and no other chord. No vertex w_i $(1 \leqslant i \leqslant 5)$ is joined to both q_2 and a vertex in $\{q_4, q_5\}$, because then $G_0(\{v_0, w_i\} \cup N(v_0))$ would be nonplanar, contradicting Lemma 2.5. So, if G_0 contains q_1q_4 we conclude that, for each $i \in \{1, 2, ..., 5\}$, w_i is joined to either $\{q_2, q_3\}$ or to $\{q_4, q_5\}$ (and possibly to q_1, too). As each vertex in G_0 has degree $\geqslant 5$, we can assume that each of w_1, w_2, w_3 is joined to q_4 and q_5, and that each of w_4, w_5 is joined to q_3 and q_2. But this results in a nontoroidal graph. For if G_0 is on the torus, then $\{w_4, w_5, q_3, q_2\}$ is a K_4 inside a face of $G_0(\{w_1, w_2, w_3, q_4, q_5\})$ which is a K_5. That face is homeomorphic to a disc. Hence one of w_4, w_5, q_3, q_2 is inside the C_3 induced by the three others. But any vertex of w_4, w_5, q_3, q_2 is joined by an edge or a path $q_2q_1q_5$ to $G_0(\{w_1, w_2, w_3, q_4, q_5\})$. This is a contradiction, so assume that q_1q_4 is not present. Then we can assume that q_2 is joined to w_1, w_2 because q_2 has degree at least 5. Each vertex in $\{w_1, ..., w_5\}$ is joined to two consecutive vertices of $q_1q_3q_4q_5q_1$. So each of w_1, w_2 is joined to each of q_1, q_2, q_3, and each of w_3, w_4, w_5 is joined to each of q_4 and q_5. But then $G_0(\{v_0, w_1, w_2, q_1, q_2, q_3\})$ and $G_0(\{q_4, q_5, w_3, w_4, w_5\})$ are nonplanar, contradicting Lemma 2.5. This proves (15).

(16) The vertex s defined in (12), (13) has degree 5, and $N(v_0)$ contains a K_4.

Proof of (16). The last statement follows from (12) and the maximality property (ii) of $N(v_0)$. So let us assume that s has degree at least 6. By (12) s is joined to a K_4 in $G'_{xy} - z_0$. Now s cannot have five neighbors in $G'_{xy} - z_0$ because then G_0 would contain a K_6. It is easy to see that s is inside W. So either s has precisely four neighbors on R_1 or R_2 (which induce a K_4) and at least two neighbors t_1, t_2 not in W, or else s has precisely five neighbors on R_1 or R_2, one of which is either x or y and at least one neighbor t_1 not in W. By (13), v_0 is joined to t_1 (and to t_2 if t_2 exists). Note that s is not joined to both x and y, since then t_1 would be inside syv_0xs, contradicting Corollary 4.2. As $k+m \leqslant 7$, and $k \leqslant m$, s is inside R_2 (and $m \geqslant 4$).

Suppose now that t_2 exists. By Corollary 4.2, the notation can be chosen such that s is joined to z_1. Assume without loss of generality that t_1 is inside $st_2v_0yz_1s$. By Proposition 4.1, t_1 is the only vertex inside $t_2v_0yz_1st_2$. Now s has no neighbor t_3 inside $st_2v_0xz_m \cdots s$, because then t_3 would be in $N(v_0)$ and t_2 would be inside $st_1v_0t_3s$, contradicting Corollary 4.2. By Proposition 4.1, there is no vertex inside $st_2v_0xz_m \cdots s$. By Lemma 2.2 at least one of z_2, z_3 is not joined to both x and y. (Since $m \geqslant 4$, at most one vertex of G'_{xy} is joined to both x and y.) So any 5-coloring c of $G'_{xy} - z_0$ can be extended to x, y such that $\{c(x), c(y)\} \cap \{c(z_2), c(z_3)\} \neq \varnothing$. We extend this 5-coloring to s. (This is possible because $G_0 \not\supseteq K_6$. Note that s is joined to none of x, y.) Then we give either t_1 or t_2 a color in $\{c(x), c(y)\} \cap \{c(z_2), c(z_3)\}$. (Note that none of t_1, t_2 are joined to z_2 or z_3.) Then we color v_0 and finally we color the uncolored vertex in $\{t_1, t_2\}$. (This is possible because we have at least two choices for $c(v_0)$.) This 5-coloring can be extended to a 5-coloring of G_0 by Proposition 4.1 (because no vertex inside $st_2v_0xz_m \cdots s$ is joined to s), a contradiction which proves that t_2 does not exist. Hence s has five neighbors on R_2. As $G_0 \not\supseteq K_6$, s is joined to one of x, y, say x.

Consider first the case where s is not joined to z_1. Then $m = 5$ and s is joined to x, z_5, z_4, z_3, z_2. There exists a 5-coloring c of G minus the interior of $yz_1z_2sxv_0y$. We must have $c(x) \neq c(y)$ because $G'_{xy} = K_6$. Also $c(s) = c(z_1)$ because z_1, z_2, z_3, z_4, z_5 induce a K_5. As c cannot be extended to a 5-coloring of G_0 it follows by Proposition 4.1 and the assumption that t_2 does not exist that there is only one vertex inside $yz_1z_2sxv_0y$, namely t_1, and that $V(G_0) = \{z_1, z_2, z_3, z_4, z_5, y, x, v_0, s, t_1\}$. By (8), one of these vertices has degree 9. That vertex must be one of z_1, z_2. But this contradicts Lemma 2.2, so we can assume that s is joined to z_1.

By Proposition 4.1, t_1 is the only vertex inside yz_1sxv_0y, and t_1 is joined to all vertices of that cycle. As v_0 has degree $\geqslant 5$, we have $k \geqslant 2$, and v_0 is joined to two vertices in $\{u_1, ..., u_k\}$. By Corollary 4.2, s, t_1, x, y, v_0 are the only vertices of G_0 which are not in $G'_{xy} - z_0 = K_5$. Hence $|V(G_0)| = 10$. By (8), G_0 has some vertex joined to all other vertices. That vertex must be $z_1 = u_1$. By Lemma 2.2 applied to $G'_{xy} = K_6$, $m = 4$, $k = 2$, and no vertex distinct from z_1, v_0 is joined to both x and y. In any 5-coloring c of $G_0 - \{v_0, t_1\}$ we must have $c(s) = c(u_2)$. Hence $u_2 \notin \{z_2, z_3, z_4\}$. One of z_2, z_3 is joined to y because y has degree $\geqslant 5$. The other is joined to either x or y. So, if we contract yt_1 into t_1 and v_0x into x, then the only missing edges in the resulting graph are su_2 and edges from $\{z_2, z_3, z_4\}$ to $\{x, t_1\}$ forming a $K_2 \cup P_3$. Hence G is nontoroidal by Proposition 2.3. This proves (16).

By (13), (15), G_0 is induced by $\{v_0\} \cup N(v_0) \cup (G'_{xy} - z_0) \cup \{s\}$. By (8), G'_{xy} has at least three vertices not in $N(v_0)$. If G'_{xy} has four vertices not in $N(v_0)$, then G_0 has a K_5 with only one vertex in $N(v_0)$. As $N(v_0)$ has a K_4, by (16), we obtain a contradiction to Lemma 2.5. Hence G'_{xy} contains precisely three vertices, say w_1, w_2, w_3, not in $N(v_0)$. The vertices of $N(v_0)$ are denoted x, y, t_1, t_2, t_3. By (16), we can assume that $G_0(\{y, t_1, t_2, t_3\}) = K_4$. By (8) we can assume that t_1 has degree 9 in G_0. We can also assume that G'_{xy} contains t_2.

Let $H_1 = G_0(\{v_0, y, t_1, t_2, t_3\})$ and $H_2 = G_0(\{t_1, t_2, w_1, w_2, w_3\})$. Then each of H_1, H_2 is a K_5. Now we consider the facial walk $u_1u_2 \cdots u_qu_1$ of

H_1 whose interior contains w_1, w_2, w_3. By Euler's formula $q \leqslant 8$. Hence no vertex occurs three times in $u_1 u_2 \cdots u_q$. Both t_1 and t_2 must occur twice and they must occur as $\cdots t_1 \cdots t_2 \cdots t_1 \cdots t_2$. Hence the edge $t_1 t_2$ does not occur twice in $u_1 u_2 \cdots u_q$. Those facial walks of $H_1 \cup H_2$, inside which G_0 has at least one (respectively two) vertices, have length at least 5 (respectively 6) by Proposition 4.1. All these conditions can only be met if $q = 8$ and $G_0(H_1 \cup H_2)$ has a facial 6-cycle or two facial 5-cycles whose interior contain x and s. Having 5-colored H_1, the colors of w_1, w_2, w_3 can be chosen such that the coloring can be extended to x, s too. The proof of Theorem 6.1 is complete. ∎

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