Compact and Weakly Compact Homomorphisms between Algebras of Continuous Functions

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In this note we study the relationship between compactness and weak compactness of a continuous homomorphism \( \phi \) from \( C_c(S) \) into \( C_c(T) \) and the associated continuous function \( \varphi: T \to S \), where \( S \) and \( T \) are completely regular Hausdorff spaces.

\[ C_c(S) \to C_c(T) \]

1. INTRODUCTION AND PRELIMINARIES

The purpose of this note is to study compact and weakly compact homomorphisms between algebras of continuous functions. As the main result it is shown that if a continuous homomorphism \( \phi: C_c(S) \to C_c(T) \) is weakly compact, where \( S \) is a completely regular Hausdorff space and \( T \) a connected completely regular Hausdorff \( k_R \)-space, then the associated function \( \varphi: T \to S \) is constant. This result is an extension of a result proved by H. Kamowitz in [5]. Also R. Singh and W. Summers [8] have obtained results in this direction, but only for compact Hausdorff spaces. Further we prove that if \( S \) is an extremally disconnected completely regular Hausdorff space and \( T \) is a completely regular Hausdorff space satisfying the first axiom of countability, then every continuous homomorphism \( \phi: C_c(S) \to C_c(T) \) is compact.

Throughout this note we let \( S \) and \( T \) denote completely regular Hausdorff spaces. The collection of all continuous functions from \( T \) into a locally convex space \( E \) endowed with its compact-open topology is denoted
by $C(\tau, E)$. Further we denote by $C_c^0(\tau)$ the algebra of all continuous scalar-valued functions on $\tau$, endowed with its compact-open topology. Recall that a subset $H$ of $C_c^0(\tau, E)$ is called equicontinuous at $t_0 \in \tau$, if for every neighbourhood $W$ of zero in $E$ there exists a neighbourhood $U$ of $t_0$ in $\tau$ such that $f(t) - f(t_0) \in W$ for all $t \in U$ and $f \in H$. We say that $H$ is 	extit{equicontinuous on} $\tau$, if it is equicontinuous at every point of $\tau$.

In [6] J. Llavona and J. Jaramillo have shown that $A: C_c^0(S) \to C_c^0(T)$ is a continuous homomorphism iff there is a continuous function $\varphi_A: \tau \to S$ such that $A(f) = f \circ \varphi_A$ for all $f \in C(S)$, and if $S$ is a real-compact, then every homomorphism between $C_c^0(S)$ and $C_c^0(T)$ is automatically continuous.

In order to obtain a characterization of compact homomorphisms from $C_c^0(S)$ into $C_c^0(T)$ we will need the following compactness criteria of Arzelà-Ascoli type. This result can be found in [7].

**Proposition 1.** Let $E$ be a quasi-complete locally convex space and $T$ a completely regular Hausdorff $k_r$-space. Then a subset $H$ of $C_c^0(\tau, E)$ is relatively compact iff (i) $H$ is equicontinuous on $\tau$ and (ii) $H(t)$ is relatively compact in $E$ for every $t \in \tau$.

**2. Compact and Weakly Compact Homomorphisms**

Let us now begin to examine the relationship between compactness and weak compactness of $A$ and the associated functions $\varphi_A$.

Given two locally convex spaces $E$ and $F$. In this note we call a continuous linear function $f: E \to F$ compact (respectively weakly compact), if it maps bounded subsets of $E$ into relatively compact (respectively relatively weakly compact) subsets of $F$. Note that for every compact (respectively weakly compact) function $f$ from a quasi-normable locally convex space $E$ into a Banach space $F$ there exists a zero neighbourhood $U$ in $E$ such that $f(U)$ is relatively compact (respectively relatively weakly compact) in $F$.

Further notice that $C_c^0(S)$ is always a quasi-normable locally convex space, when $S$ is a completely regular Hausdorff space.

Using nets, it is easily seen that a subset $H$ of $C(\tau)$ is equicontinuous on $\tau$ if for every $t \in \tau$, every net $t_v \to t$ in $\tau$ implies that $\sup_{f \in H} |f(t_v) - f(t)| \to 0$. This observation is needed in the following result.

**Proposition 2.** Let $S$ and $T$ be completely regular Hausdorff spaces and assume also that $T$ is a $k_r$-space. A continuous homomorphism $A: C_c^0(S) \to C_c^0(T)$ is compact iff for every $t \in T$, every net $t_v \to t$ in $T$ implies that $\sup_{f \in B} |f(\varphi_A(t_v)) - f(\varphi_A(t))| \to 0$ for every bounded set $B$ in $C_c^0(S)$.
Proof. Suppose first that \( A \) is compact. Let \( B \) be an arbitrary bounded subset of \( C_c(S) \) and let \( t \in T \). By assumption \( A(B) \) is relatively compact in \( C_c(T) \). Hence \( A(B) \) is equicontinuous on \( T \), i.e., every net \( t, \to t \) in \( T \) implies that \( \sup_{f \in B} |A(f) t, - A(f) t| = \sup_{f \in B} |f(\varphi_A(t,)) - f(\varphi_A(t))| \to 0 \). Conversely, let \( B \) be an arbitrary bounded set in \( C_c(S) \). By hypothesis \( A(B) \) is equicontinuous on \( T \). Hence \( A \) is compact by Proposition 1, since \( A(B) t \) is relatively compact in \( \mathbb{R} \) for every \( t \in T \) as \( A(B) \) is trivially bounded.

Now we are ready to prove a characterization of compact homomorphisms.

**Proposition 3.** Assume that \( S \) and \( T \) are completely regular Hausdorff spaces and \( T \) is also a \( k_\infty \)-space. A continuous homomorphism \( A: C_c(S) \to C_c(T) \) is compact iff \( \varphi_A \) is locally constant.

Proof. Suppose first that there exists a \( t \in T \) such that \( \varphi_A \) is not constant on any open set in \( T \) containing \( t \). Let now \( I \) be any fixed open neighbourhood base at \( t \) in \( T \). Then the order relation \( U \leq V \) iff \( U \subset V \) directs \( I \). Hence there is a net \( (t_U) \) converging to \( t \) in \( T \) with \( \varphi_A(t_U) \neq \varphi_A(t) \) for each \( U \in I \). Thus for each \( U \in I \) there is a continuous function \( f_U: S \to [0, 1] \) such that \( f_U(\varphi_A(t)) = 0 \) and \( f_U(\varphi_A(t_U)) = 1 \). Now let \( B := \{f_U: U \in I\} \). Then \( B \) is bounded in \( C_c(S) \). But since \( |f_U(\varphi_A(t_U)) - f_U(\varphi_A(t))| = 1 \) for each \( U \in I \), we obtain a contradiction according to Proposition 2. Conversely, suppose that \( \varphi_A \) is locally constant. Consider an arbitrary net \( (t_U) \in T \) converging to \( t \) in \( T \). Then there exists an open set \( U \) in \( T \) containing \( t \) such that \( t_v \in U \) for \( v \geq v_0 \) and \( \varphi_A \) is constant on \( U \). Hence \( \varphi_A(t) = \varphi_A(t_v) \) for \( v \geq v_0 \), and we get that \( A \) is compact.

Our next aim is to find out when every weakly compact homomorphism is compact. In the proof of the following result we use an idea due to S. Warner [9, p. 274].

**Proposition 4.** Let \( S \) and \( T \) be completely regular Hausdorff spaces. If a continuous homomorphism \( A: C_c(S) \to C_c(T) \) is weakly compact, then for each compact subset \( K \subset T \) we have that \( \varphi_A(K) \subset S \) is finite.

Proof. Suppose there exists a compact subset \( K \subset T \) such that \( \varphi_A(K) \) is infinite. Let \( (\varphi_A(x_n)) \) be a sequence of distinct points in the compact set \( \varphi_A(K) \). Then \( (x_n) \) is also a countable collection of distinct points in \( K \). Since \( K \) is compact, the sequence \( (x_n) \) has a cluster point \( x \). We may assume \( \varphi_A(x) \) to be different from each \( \varphi_A(x_n) \). Let \( f_n: S \to [0, 1] \) be continuous with \( f_n(\varphi_A(x)) = 0 \) and \( f_n(\varphi_A(x_k)) = 1 \) for \( k \leq n \). Then \( B := \{f_n: n \in \mathbb{N}\} \) is bounded in \( C_c(S) \). Hence \( A(B) = \{f_n \circ \varphi_A: n \in \mathbb{N}\} \) is
relatively weakly compact in $C_{\text{co}}(T)$. Let now $g$ be a weak cluster point for $(f_n \circ \varphi_A)$ in $C_{\text{co}}(T)$. Then $g \in \{f_n \circ \varphi_A : n \geq m\}$, $m \in \mathbb{N}$, where $\sigma = \sigma(C_{\text{co}}(T), C_{\text{co}}(T)')$. Now for any $\varepsilon > 0$ there exists $n \geq m$ such that $|g(x_m) - f_n(\varphi_A(x_m))| < \varepsilon$, i.e., $|g(x_m) - 1| < \varepsilon$. Thus $g(x_m) = 1$ for each $m \in \mathbb{N}$. In the same way it can be shown that $g(x) = 0$. But $x \in \{x_m : m \in \mathbb{N}\}$, and hence $g(x) \in \{g(x_m) : m \in \mathbb{N}\}$ as $g \in C(T)$. But this is a contradiction.

Before we can formulate our main result, we need one more result.

**Proposition 5.** Let $S$ and $T$ be completely regular Hausdorff spaces and assume also that $T$ is a $k_R$-space. Further let $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ be a continuous homomorphism. If for each compact subset $K \subset T$ we have that $\varphi_A(K) \subset S$ is finite, then $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ is compact.

**Proof.** We first show that $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ is continuous. Here $C_{\text{co}}(S)$ denotes $C(S)$ endowed with the simple topology. Let now $K \subset T$ be an arbitrary compact subset. Then $\varphi_A(K) \subset S$ is finite and

$$\sup_{t \in K} |A(f)(t)| = \sup_{t \in K} |(f \circ \varphi_A)(t)| = \sup_{s \in \varphi_A(K)} |f(s)|,$$

for every $f \in C(S)$. Because $\sigma(C_{\text{co}}(S), C_{\text{co}}(S)')$ is finer than $C_{\text{co}}(S)$, we get that $A$ is continuous from $\sigma(C_{\text{co}}(S), C_{\text{co}}(S)')$ into $C_{\text{co}}(T)$. Since every bounded set in $\sigma(C_{\text{co}}(S), C_{\text{co}}(S)')$ is precompact, it follows that $A$ maps bounded subsets of $C_{\text{co}}(S)$ into precompact subsets of $C_{\text{co}}(T)$. As $C_{\text{co}}(T)$ is complete, the proposition follows.

By collecting together Propositions 3, 4, and 5, we get the following:

**Theorem 6.** Let $S$ and $T$ be completely regular Hausdorff spaces and assume that $T$ also is a $k_R$-space. Further let $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ be a continuous homomorphism. Then the following statements are equivalent:

(i) $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ is compact.

(ii) $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ is weakly compact.

(iii) $\varphi_A$ is locally constant.

(iv) For every compact subset $K \subset T$ we have that $\varphi_A(K) \subset S$ is finite.

**Corollary 7.** If $S$ is a completely regular Hausdorff space, $T$ a connected completely regular Hausdorff $k_R$-space, and $A : C_{\text{co}}(S) \to C_{\text{co}}(T)$ is a weakly compact homomorphism, then $\varphi_A : T \to S$ is constant.

The strength of our theorem on $k_R$-spaces is indicated by observing that there exist a lot of non-discrete completely regular Hausdorff spaces with the property that every compact subset is finite. These spaces are not
Recall that a topological space $T$ is called *extremally disconnected*, if every open set has an open closure. An equivalent condition is that every pair of disjoint open sets in $T$ have disjoint closures (see [3, p. 22]).

**Theorem 8.** Let $S$ be an extremally disconnected completely regular Hausdorff space and let $T$ be a completely regular Hausdorff space satisfying the first axiom of countability. If $\varphi : T \to S$ is a continuous function, then $\varphi$ is locally constant.

**Proof.** Suppose that $\varphi$ is not locally constant. Then there is a $t_0 \in T$ such that for every open set $U$ in $T$ containing $t_0$ there exists a $t \in U$ with $\varphi(t) \neq \varphi(t_0)$. Now let $(U_n)$ be a decreasing countable open neighbourhood base at $t_0$ in $T$. Then there is a sequence $(t_n)$ in $T$ such that $t_n \to t_0$ and $\varphi(t_n) \neq \varphi(t_0)$ for each $n \geq 1$. Because of the continuity of $\varphi$ we can without loss of generality assume that $\varphi(t_n) \neq \varphi(t_m)$ for $n \neq m$. Further we have that $\{\varphi(t_n) : n \geq 1\}$ is compact in $S$. Hence, for every $n \geq 1$ we can choose a closed neighbourhood $V_n$ of $\varphi(t_n)$ such that $(V_n)$ is a pairwise disjoint sequence. Let now $U_1 := \bigcup_{n \text{ odd}} V_n$ and $U_2 := \bigcup_{n \text{ even}} V_n$. Then $U_1$ and $U_2$ are open disjoint sets in $S$. But since $\varphi(t_0) \in U_1 \cap U_2$, we get a contradiction to the assumption of $S$.

We shall now apply Theorem 8 to show that the multiplicative property of a continuous homomorphism $A$ gives a lot more than if we only assume $A$ to be linear and continuous. In [4, p. 168] A. Grothendieck proved the following result: *Let $S$ be an extremally disconnected compact Hausdorff space and let $F$ be a complete separable locally convex space. Then any continuous linear function of $C_0(S)$ into $F$ is weakly compact.*

Note that $C_0(T)$ is separable iff there is a coarser metrisable and separable topology on the completely regular Hausdorff space $T$. Let $\mathcal{F}$ be the class of all complete topological algebras which are topologically isomorphic, in the algebra sense, to a subalgebra of $C_0(T)$ for some completely regular Hausdorff space $T$ satisfying the first axiom of countability. If we now combine Theorem 8 and Proposition 3 and observe that every completely regular Hausdorff space satisfying the first axiom of countability is a $k_\sigma$-space we obtain the following corollary.

**Corollary 9.** Let $S$ be an extremally disconnected completely regular Hausdorff and let $F$ be a topological algebra in $\mathcal{F}$. Then every continuous homomorphism $A : C_0(S) \to F$ is compact.

**Remark.** The following classes of topological algebras are included in $\mathcal{F}$. (See [2]).
— All separable, commutative, and semisimple Banach algebras.
— All complex, complete, separable, and semisimple LMCH algebras which are $Q$-algebras and square algebras.
— All complex, complete, separable, and semisimple LMCH algebras which are barreled and square algebras.
— All separable full Fréchet algebras.

EXAMPLE. Consider an arbitrary homomorphism $A: l^\infty \to F$, $F \in \mathcal{F}$. Since $l^\infty \cong C_0(\beta N)$ we get by Corollary 9 that $A$ is always compact. In particular, every homomorphism $A: l^\infty \to c_0$ is compact.

Finally let us mention that Corollary 7 can be applied to homomorphisms between Fréchet algebras of real-valued functions on a Banach space $E$ which are uniformly weakly continuous when restricted to any bounded subset of $E$ [1].

REFERENCES