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# On a Generalized Matching Problem Arising in Estimating the Eigenvalue Variation of Two Matrices

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It is shown that if $G$ is a graph having vertices $P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n$ and satisfying some conditions, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that there is a path, for $i = 1, 2, \ldots, n$ connecting $P_i$ with $Q_{\sigma(i)}$ having a length at most $\{n/2\}$. This is used to prove a theorem having applications in eigenvalue variation estimation.

For complex $n \times n$-matrices $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $B$ with eigenvalues $\mu_1, \ldots, \mu_n$, it is possible to give bounds for the "spectral-variation" $S_A(B) = \max_i \min_j |\lambda_j - \mu_i|$, depending only on $\|A\|$, $\|B\|$ and $\|A - B\|$. Here $\| \;\|$ denotes the spectral-norm (e.g. [1]). These bounds are also bounds on

$$\delta = \max_{0 \leqslant t \leqslant 1} \max(S_A(tB + (1-t)A), S_B(tB + (1-t)A)).$$

It follows from a continuity argument that each connected component of $\bigcup_{j=1}^{n} \{z : |z - \mu_j| \leqslant \delta\}$ and of $\bigcup_{j=1}^{n} \{z : |z - \lambda_j| \leqslant \delta\}$ contains as many eigenvalues of $A$ as of $B$. One is in fact interested in the "eigenvalue variation"

$$\nu(A, B) = \min_\sigma \max_i |\lambda_i - \mu_{\sigma(i)}|,$$

where $\sigma$ runs through all permutations of $\{1, 2, \ldots, n\}$. It is easy to see that $\nu(A, B) \leqslant (2n-1)\delta$. It was suspected that $2n - 1$ can be replaced by $n$ for $n$ odd and $n - 1$ for $n$ even. Hence the question arose whether the following statement is true.

STATEMENT 1. *Let $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\mu = \{u_1, \mu_2, \ldots, \mu_n\}$ be two sets of not necessarily distinct points in the complex plane. Suppose that for every connected component $D$ of the domain $\bigcup_{i=1}^{n} \{z : |z - \mu_i| \leqslant 1\}$ or of the domain $\bigcup_{i=1}^{n} \{z : |z - \lambda_i| \leqslant 1\}$ the number of elements of $\lambda$ contained in $D$ equals the number of elements of $\mu$ contained in $D$. Then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that for $i = 1, 2, \ldots, n$,*

$$|\lambda_i - \mu_{\sigma(i)}| \leqslant \begin{cases} n & \text{for } n \text{ odd,} \\ n-1 & \text{for } n \text{ even.} \end{cases}$$

Since we shall answer the above question in the affirmative, we will refer to Statement 1 as Theorem 1.

It turns out that a much more general result is true. It will be formulated as Theorem 2 and proved below in graph-theoretical terms.

If $A$ and $B$ are vertices in a connected graph, then we shall use the notation $L(AB)$ for a path with endpoints $A, B$ and $l(AB)$ for its length, i.e. the number of the edges in it or, if the edges are weighted, the sum of the weights of its edges. As usual the distance $d(AB)$ means the length of the shortest path connecting $A, B$.

Denote by $\{m\}$ the least integer not smaller than $m$.

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We define now a class $\Gamma_n$ of graphs. A graph $G$ will belong to $\Gamma_n$ if it has the following structure:

(i) The vertex set of $G$ is the union of two disjoint sets $V_p$ and $V_q$, each containing exactly $n$ elements.

(ii) Let $G_p$ and $G_q$ be the induced subgraphs of $G$ on the sets $V_p$ and $V_q$. Let $B_{pq}$ be the induced bipartite subgraph with cells $V_p$ and $V_q$. Then the following condition holds for each connected component $D$ of $G_p$: the number of vertices in $V_q$ joined by an edge to some vertex in $D$ equals the number of vertices in $D$. A corresponding condition holds for every connected component of $G_q$.

(iii) Edges $B_{pq}$ stemming out from the same vertex of $V_p$ have the other endpoint in the same connected component of $G_q$, and vice versa interchanging $p$ with $q$.

Notice that from (i), (ii), (iii) it follows

(iv) The degree of each vertex in $B_{pq}$ is at least 1.

Actually (i), (ii), (iii) and (i), (ii), (iv) are equivalent.

A path connecting a vertex $P \in V_p$ with a vertex $Q \in V_q$ will be said to be *proper* if it contains exactly one edge of $P_{pq}$, this edge has at least one of the vertices $P, Q$ as endpoints, and

$$l(PQ) < \left\{\frac{n}{2}\right\}. \tag{0}$$

THEOREM 2. *If $G$ is a member of $\Gamma_n$, and $V_p = \{P_1, P_2, \ldots, P_n\}$, $V_q = \{Q_1, \ldots, Q_n\}$, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that for each $i = 1, 2, \ldots, n$ there is a proper path $L(P_iQ_{\sigma(i)})$.*

PROOF. Let $A_i$ be the subset of $V_q$ such that if $Q \in A_i$ then there is a proper path $L(P_iQ)$. The set $A_i$ is non-empty for $i = 1, 2, \ldots, n$ by property (iv). We will prove Theorem 2 by showing that the sets $A_1, A_2, \ldots, A_n$ have a system of distinct representatives. This will be done by verifying Hall's condition [2]. Thus, we shall verify the condition:

$$\left|\bigcup_{j=1}^{k} A_{i_j}\right| \geq k, \qquad k = 1, 2, \ldots, n; \qquad \{i_j\}_{j=1}^{k} \subset \{1, 2, \ldots, n\}. \tag{1}$$

Let $s \geq 1$ be such that $G_q^1, G_q^2, \ldots, G_q^s$ are the connected components of $G_q$ and let $m_1, m_2, \ldots, m_s$ be the cardinalities of the corresponding vertex sets $V_q^1, V_q^2, \ldots, V_q^s$.

Notice that if

$$m_j \leq \left\{\frac{n}{2}\right\} \tag{2}$$

and there is an edge from $P_i$ to a vertex of $G_q^j$, then $|A_i| \geq m_j$, and notice that (2) holds for all but possibly one value of $j$. Choose the notation so that $m_s \geq m_j$ for $j = 1, 2, \ldots, s-1$.

Consider the set $R$ of vertices $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ and the corresponding sets $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$.

*Case 1.* Either $m_s \leq \{n/2\}$ or there is no edge from $P_{i_j}$, $j = 1, 2, \ldots, k$, to $G_q^s$.

We show that in either case (1) holds. Indeed, since each $A_{i_j}$ contains at least one of the components of $G_q^n$, one can group together equal components and get:

$$\left|\bigcup_{j=1}^{k} A_{i_j}\right| \geq \left|\bigcup_{\nu=1}^{h} V_q^{r}\right| = \sum_{\nu=1}^{h} |V_q^{r_\nu}| \textstyle\sum_{\nu=1}^{h} m_{r_\nu}.$$

On the other hand $k$ is less than or equal to the total number of vertices $P$ such that there is an edge from $P$ to one of the sets $V_q^{r_\nu}$, but this number is smaller or equal, by property (ii) of graphs $\Gamma_n$, to $\sum_{\nu=1}^{h} m_{r_\nu}$.

*Case 2.* $m_s \geq \{n/2\}+1$ and there is an edge from the set $R$ to some vertex of $V_q^s$.

In this case clearly (1) holds provided $k \leq \{n/2\}$; indeed, for some $t$, $|A_{i_t}| \geq \{n/2\}$. We claim that (1) holds even if $k > \{n/2\}$.

Suppose the contrary

$$\left|\bigcup_{j=1}^{k} A_{i_j}\right| < k.$$

It follows that there are at least $n-k+1$ elements $Q$ in $V_q$ which are not in $\bigcup_{i=1}^{k} A_{i_j}$. But $k > \{n/2\}$ implies $n-k+1 \leq \{n/2\}$.

Define $B_i$, $i = 1, 2, \ldots, n$, to be the set of elements $P$ of $V_p$ for which there is a proper path $L(Q_iP)$. We have shown that (1) holds for $k \leq \{n/2\}$. Hence, by symmetry, $|\bigcup_{j=1}^{l} B_{i_j}| \geq l$ for $l \leq \{n/2\}$, in particular when $l = n-k+1$. Therefore, for at least one of the $n-k+1$ considered vertices $Q$ there is a proper path $L(QP)$ where $P \in R$, a contradiction, since if $L(QP)$ is proper, then $L(PQ)$ is also proper.

COROLLARY 1. *If $G$ is as in Theorem 2, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that $d(P_iQ_{\sigma(i)}) \leq \{n/2\}$.*

COROLLARY 2. *If $G$ is as in Theorem 2 and if weight 1 is assigned to every edge in $B_{pq}$ and weight 2 to every edge in $G_p$ and $G_q$, then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that*

$$d(P_iQ_{\sigma(i)}) \leq \begin{cases} n & \text{for } n \text{ odd}, \\ n-1 & \text{for } n \text{ even}. \end{cases} \tag{3}$$

We shall omit the proofs.

PROOF OF THEOREM 1. Given the set of points $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\mu = \{\mu_1, \mu_2, \ldots, \mu_n\}$, consider the graph $G$ having vertex set $\lambda \bigcup \mu$. Putting $\lambda = V_p$, $\mu = V_q$ two vertices both in $V_p$ or both in $V_q$ are joined by an edge if the distance between them is at most 2. Two vertices, one in $V_p$ and one in $V_q$, are joined by an edge if the distance between them is at most 1. The graph $G$ is clearly a member of $\Gamma_n$. Assigning weights as in Corollary 2, condition (3) follows and this implies Theorem 1.

It seems to be of interest to formulate a particularization of Theorem 2.

THEOREM 3. *Suppose $G$ is a graph, the vertex set of which consists of the union of two disjoint sets $V_p = \{P_1, P_2, \ldots, P_n\}$ and $V_q = \{Q_1, Q_2, \ldots, Q_n\}$, and the edge set of which satisfies the following two conditions.*

(i) *The induced subgraphs on $V_p$ and $V_q$ are connected.*

(ii) *Each vertex of the bipartite graph induced on $V_p$ and $V_q$ as cells has degree at least* 1.

*Then there is a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that there is a proper path $L(P_iQ_{\sigma(i)})$ for each $i = 1, 2, \ldots, n$.*

REMARK. Theorem 2 is sharp, i.e. for every $n$ there are graphs in $\Gamma_n$ for which it is impossible to choose in the definition of proper paths a shorter length than given in (0).

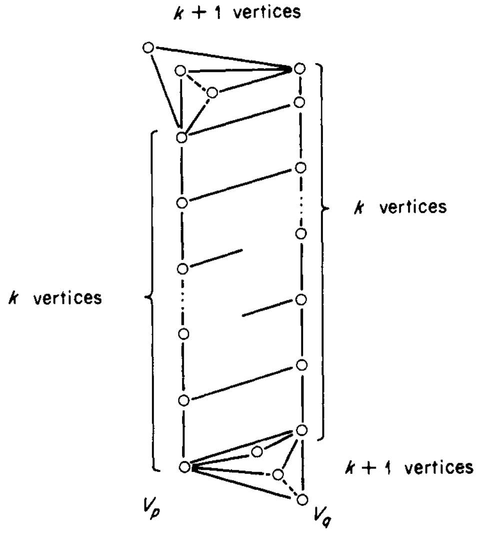


FIG. 1

As an example for odd $n$ consider the graph of Figure 1. This graph is a member of $\Gamma_{2k+1}$. It even satisfies the assumptions of Theorem 3. But clearly for the $k+1$ vertices on top of $V_p$ only $k$ vertices of $V_q$ can be closer than required by condition (0).

This situation can occur in the case of Theorem 1 also, when all the points $\lambda$ and $\mu$ are on the real line. If $k=2$, for instance, let the points of $\lambda$ and of $\mu$ be the points having abscissas

$$\lambda_1=\lambda_2=\lambda_3=0, \quad \lambda_4=2, \quad \lambda_5=4, \quad \mu_1=1, \quad \mu_2=3, \quad \mu_3=\mu_4=\mu_5=5.$$

This shows that the condition in Theorem 1 is also sharp.

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