Discrete Numerical Solution of Strongly Coupled Mixed Diffusion Problems

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Abstract—This paper is concerned with the discrete numerical solution of coupled partial differential mixed problems with non-Dirichlet coupled boundary value conditions. By using a discrete separation of variables method, the proposed numerical solution of the problems is the exact solution of certain coupled partial difference system, appearing from the discretization of the continuous partial differential systems. Existence, stability, and examples are considered. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Coupled partial differential systems with coupled boundary value conditions are frequent in quantum mechanical scattering problems [1,2], chemical physics [2], modelling of coupled thermoelastic-plastic response of clays subjected to nuclear waste heat [3], and coupled diffusion problems [4,5]. The solution of these problems has motivated the study of vector and matrix Sturm-Liouville problems [5].

This paper deals with coupled partial differential systems of the type

\[ u_t(x,t) - A_{uxx}(x,t) = 0, \quad 0 < x < 1, \quad t > 0, \]  
\[ A_1u(0,t) + B_1u_x(0,t) = 0, \quad t > 0, \]  
\[ A_2u(1,t) + B_2u_x(1,t) = 0, \quad t > 0, \]  
\[ u(x,0) = F(x), \quad 0 \leq x \leq 1, \]

where the unknown \( u = (u_1, u_2, \ldots, u_m)^T \), and the function \( F(x) = (f_1, f_2, \ldots, f_m)^T \) are \( m \)-dimensional vectors, \( A_1, B_1, i = 1, 2 \) are complex \( m \times m \) matrices, elements of \( \mathbb{C}^{m \times m} \), and \( A \)

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is a stable positive matrix, i.e., verifies

\[ \text{Re}(z) > 0, \quad \forall z \in \sigma(A). \] (5)

Let us assume that

the block matrix \( A = \begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} \) is invertible

and not all its blocks \( A_1, A_2, B_1, B_2 \) are singular.

Due to physical conditions, the existence of solutions for the proposed problem is sometimes known, and in these cases, the construction of stable discrete numerical solutions is required. Matrix difference schemes have been used in [6] for solving coupled diffusion problems of type (1) with Dirichlet boundary conditions. Standard difference methods for problems without coupling in the boundary conditions, or scalar problems with non-Dirichlet boundary conditions have been considered in [4,7]. Problems (1)–(4) for the case where \( B_1 = 0 \) has been treated in [8], and the case where \( A_1 = B_1 = I \) in [9].

The organization of the paper is as follows. Section 2 deals with the discretization of the problem. The study of the existence of nontrivial stable solutions of the discretized partial boundary value difference system is addressed in Section 3. In Section 4, the construction of numerical solutions of the mixed partial differential is addressed. Stability and examples are considered in Section 5.

Throughout this paper, the set of all eigenvalues of a matrix in \( C^{m \times m} \) is denoted by \( \sigma(D) \), the spectral radius of \( D \), defined by the maximum of the set \( \{ |z| ; z \in \sigma(D) \} \) is denoted by \( \rho(D) \). We recall that \( D \) is said to be convergent if the sequence \( \{ D^n \}_{n \geq 0} \) tends to the zero matrix of \( C^{m \times m} \), and by Theorem 1.3.9 of [10], a matrix \( D \) in \( C^{m \times m} \) is convergent if \( \rho(D) < 1 \). If \( D \) is a Hermitian matrix, then \( \sigma(D) \) is contained in the real line and we denote by \( \lambda_{\text{min}}(D) \), the minimum of \( \sigma(D) \). If \( S \) is a matrix in \( C^{m \times m} \), we denote by \( S^+ \) its Moore-Penrose pseudoinverse. An account of properties, examples, and applications of this concept may be found in [11,12]. In particular, the kernel of \( S \), denoted by \( \ker S \) coincides with \( \text{Im}(I - S^+ S) \), the image of the matrix \( I - S^+ S \). Finally, the vector subspace generated by a vector \( \omega \) is denoted by \( \text{LIN} \{ \omega \} \).

2. ON THE DISCRETIZED PARTIAL DIFFERENCE SYSTEM

Let us divide the domain \([0, 1] \times [0, \infty)\) into equal rectangles of sides \( \Delta x = h \) and \( \Delta t = k \), and introduce coordinates of a typical mesh point \((ih, jk)\) and let us represent \( U(ih, jk) \). Approximating the partial derivatives appearing in (1) by the forward difference approximations

\[ u_t(ih, jk) \approx \frac{U(i, j + 1) - U(i, j)}{k}, \]
\[ u_{xx}(ih, jk) \approx \frac{U(i + 1, j) - 2U(i, j) + U(i - 1, j)}{h^2}, \]

equation (1) takes the form

\[ \frac{1}{k} [U(i, j + 1) - U(i, j)] - \frac{A}{h^2} [U(i, j + 1) - 2U(i, j) + U(i - 1, j)], \]

where \( h = 1/N \), \( 1 \leq i \leq N - 1 \), \( j \geq 0 \). Let \( r = k/h^2 \), \( f(i) = F(i/N) \) and let us write the last equation in the form

\[ rA [U(i + 1, j) + U(i - 1, j)] + (I - 2rA) U(i, j) - U(i + 1, j) = 0. \] (7)

The initial and boundary conditions (2)–(4) take the form

\[ A_1 U(0, j) + NB_1 [U(1, j) - U(0, j)] = 0, \quad j \geq 0, \] (8)
\[ A_2 U(N, j) + NB_2 [U(N, j) - U(N - 1, j)] = 0, \quad j \geq 0, \] (9)
\[ U(i, 0) = f(i), \quad 1 \leq i \leq N - 1. \] (10)

According to [13], it is easy to show that scheme (7) is consistent.
Let us seek solutions of (8)-(10) of the form
\[ U(i,j) = G(j)H(i), \quad G(j) \in \mathbb{C}^{m \times m}, \quad H(i) \in \mathbb{C}^{m}; \]  
then for \( 1 \leq i \leq N - 1, \ j \geq 0 \), one gets
\[ rAG(j)[H(i + 1) - H(i - 1)] + (I - 2rA)G(j)H(i) = G(j + 1)H(i), \]  
\[ A_1G(j)H(0) + NB_1G(j)[H(1) - H(0)] = 0, \]  
\[ A_2G(j)H(N) + NB_2G(j)[H(N) - H(N - 1)] = 0, \]  
where
\[ Nh = 1, \quad r = \frac{k}{h^2}. \]

If \( \rho \) is a real number, (12) can be written in the form
\[ 0 = rAG(j)\left[H(i + 1) + \left(\frac{-2r - \rho}{r}\right)H(i) + H(i - 1)\right] + [(I + \rho A)G(j) - G(j + 1)]H(i). \]

Note that (6) holds true if \( \{H(i)\}, \ \{G(j)\} \) satisfy
\[ G(j + 1) - (I + \rho A)G(j) = 0, \quad j \geq 0, \]  
\[ H(i + 1) + \left(\frac{-2r - \rho}{r}\right)H(i) + H(i - 1) = 0, \quad 1 \leq i \leq N - 1. \]

The solution of (17) satisfying \( G(0) = I \) is given by
\[ G(j) = (I + \rho A)^j, \quad j \geq 0. \]

Since coefficients of vector equation (18) are scalar and if \( \rho \) satisfies
\[ -4r < \rho \leq 0, \]
one gets \( |(2r + \rho/2r)| \leq 1 \). Under the hypothesis
\[ -4r < \rho < 0, \]
the characteristic equation
\[ z^2 - \left(\frac{2r + \rho}{r}\right)z + 1 = 0 \]
has the complex solutions
\[ z_0 = \left\{ \frac{2r + \rho}{2r} + j\sqrt{\left[1 - \left(\frac{2r + \rho}{2r}\right)^2\right]} \right\} = e^{j\theta}, \quad z_1 = e^{-j\theta}, \]
where \( \cos(\theta) = (2r + \rho/2r), \ \theta \in [0, 2\pi] \) and where \( j = \sqrt{-1} \) is the imaginary unity.

The solution set of the vector equation (18) is given by
\[ H(i) = z_0^ic + z_1^id, \quad c, d \in \mathbb{C}^m, \quad 1 \leq i \leq N - 1. \]

For sake of clarity, let us assume that \( A_1 = I \). In this case, condition (13) takes the form
\[ G(j)H(0) + NB_1G(j)[H(1) - H(0)] = 0; \]
by substituting (23) into (24), we obtain that
\[
[I + (z_0 - 1)NB_1]c = -[I + (z_1 - 1)NB_1]d,
\]
where \(d \in \mathbb{C}^m\). By multiplying (23) by \([I + (z_0 - 1)NB_1]\), we obtain
\[
[I + (z_0 - 1)NB_1]H(i) = z^i_0 [I + (z_0 - 1)NB_1]c + z^i_1 [I + (z_0 - 1)NB_1]d
= -z^i_0 [I + (z_1 - 1)NB_1]d + z^i_1 [I + (z_0 - 1)NB_1]d
= \{-2j \sin (i\theta) I + NB_1 [z^i_1 - z^i_0 - (z^i_1 - z^i_0)]\} d
= -2j \{\sin (i\theta) I - NB_1 [\sin (i\theta) - \sin ((i - 1)\theta)]\} d.
\]

Since \([I + (z_0 - 1)NB_1]\) is an invertible matrix, for large enough values of \(N\), it follows that
\[
H(i) = \sin (i\theta) I - 2N \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{2i - 1}{2}\theta\right) B_1, \quad 1 < i < N - 1,
\]
is a solution of (18), for any vector \(d \in \mathbb{C}^m\). Since \(\cos \theta = (2r + \rho/2r)\), it follows that
\[
\rho = -4r \sin^2 \left(\frac{\theta}{2}\right), \quad \theta \in [0, \pi \cup \pi, 2\pi],
\]
because \(\theta = 0\) and \(\theta = \pi\) yield the trivial solution in (26). By using (18) for \(i = 1\), one gets
\[
H(0) = -NB_1 \sin \theta d, \quad d \in \mathbb{C}^m. \quad \text{By substituting (26) and the last equation into (24), one gets}
\]
\[
-N \sin \theta \left[(I + \rho A)^j B_1 - B_1 (I + \rho A)^j\right] d = 0, \quad d \in \mathbb{C}^m, \quad j > 0.
\]

Since \(\theta \in [0, \pi \cup \pi, 2\pi]\), this last equation is equivalent to
\[
\left[(I + \rho A)^j B_1 - B_1 (I + \rho A)^j\right] d = 0, \quad d \in \mathbb{C}^m, \quad j > 0.
\]

Considering (18) for \(i = N - 1\), it is easy to show that
\[
H(N) = (\sin (N\theta) - NB_1 [\sin (N\theta) - \sin ((N - 1)\theta)]) d, \quad d \in \mathbb{C}^m.
\]

By imposing that \(U(i, j)\) satisfies (14), and using (26),(27), one gets
\[
\begin{cases}
A_2 \sin (N\theta) - 2N \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{2N - 1}{2}\theta\right) A_2 B_1 \\
+ 2 \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{2N - 1}{2}\theta\right) B_2 \\
+ 4N^2 B_2 B_1 \sin^2 \left(\frac{\theta}{2}\right) \sin ((N - 1)\theta)
\end{cases}
\]
\[
(I + \rho A)^j d = 0. \quad \text{(28)}
\]

By the Hamilton-Cayley Theorem [14, p. 556], if \(p\) is the degree of the minimal polynomial of \(A\), \((I + \rho A)^j\) can be expressed as a polynomial in \(A\) of degree \(p\), the last equation is equivalent to
\[
\begin{cases}
2N \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{2N - 1}{2}\theta\right) [B_2 - A_2 B_1] \\
+ A_2 \sin (N\theta) + 4N^2 B_2 B_1 \sin^2 \left(\frac{\theta}{2}\right) \sin ((N - 1)\theta)
\end{cases}
\]
\[
A^j d = 0, \quad 0 \leq j < p. \quad \text{(28)}
\]

Since vectors \(d \in \mathbb{C}^m \sim \{0\}\), condition (28) can be written in the form
\[
L(\theta) A^j d = 0, \quad 0 \leq j < p, \quad d \in \mathbb{C}^m \sim \{0\},
\]
\[
\text{(29)}
\]
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\[ L(\theta) = 2N \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{2N - 1}{2} \theta\right) \left[B_2 - A_2 B_1\right] + \frac{1}{N} A_2 \]
\[ + \sin\left((N - 1) \theta\right) \left[A_2 + 4N^2 B_2 B_1 \sin^2\left(\frac{\theta}{2}\right)\right] \]

is singular.

Under the hypothesis,
\[
\begin{bmatrix}
I & B_1 \\
A_2 & B_2
\end{bmatrix}
\]
is invertible,

by the properties of Schur’s complement matrix \[15\], we have that \(B_2 - A_2 B_1\) is invertible, and by Banach’s Lemma, it follows that \((B_2 - A_2 B_1) + (1/N) A_2\) is invertible if \(N\) is large enough. Thus, the values \(0 \in \theta \in [0, \pi \cup \pi, 2\pi]\) such that matrix \(L(\theta)\) is singular, must satisfy \(\sin((N - 1) \theta) \neq 0\), and hence, \(L(\theta)\) is singular if and only if
\[
A_2 + 4N^2 B_2 B_1 \sin^2\left(\frac{\theta}{2}\right) + \frac{2N \sin(\theta/2) \cos((2N - 1/2) \theta)}{\sin((N - 1) \theta)} \left[B_2 - A_2 B_1\right] + \frac{1}{N} A_2
\]
is singular, or equivalently,
\[
(B_2 - A_2 B_1)^{-1} A_2 + 4N^2 (B_2 - A_2 B_1)^{-1} B_2 B_1 \sin^2\left(\frac{\theta}{2}\right)
\]
\[+ \frac{2N \sin(\theta/2) \cos((2N - 1/2) \theta)}{\sin((N - 1) \theta)} \left[I + \frac{1}{N} (B_2 - A_2 B_1)^{-1} A_2\right]
\]
is singular.

Let us introduce the matrices
\[ \tilde{A}_2 = (A_2 B_1 - B_2)^{-1} A_2, \quad \tilde{B}_2 = (A_2 B_1 - B_2)^{-1} B_2. \]

By (32) and the spectral mapping theorem \[14, p. 569\], (32) can be written
\[
N \left[\frac{\sin(N \theta)}{\sin((N - 1) \theta)} - 1\right] \in \sigma \left(\frac{\sin(N \theta)}{\sin((N - 1) \theta)} \tilde{A}_2 + 4N^2 \tilde{B}_2 B_1 \sin^2\left(\frac{\theta}{2}\right)\right),
\]
\[\theta \in [0, \pi \cup \pi, 2\pi].\]

Suppose that matrices \(\tilde{A}_2\) and \(B_1\) have real eigenvalues \(\alpha \in \sigma(\tilde{A}_2)\) and \(\beta \in \sigma(B_1)\) and a common eigenvector \(v \in \mathbb{C}^m\) associated with them,
\[
(B_1 - \beta I) v = (\tilde{A}_2 - \alpha I) v = 0, \quad v \neq 0, \quad (\alpha, \beta) \in \mathbb{R}^2,
\]

then
\[
\left[\frac{\sin(N \theta)}{\sin((N - 1) \theta)} \tilde{A}_2 + 4N^2 \sin^2\left(\frac{\theta}{2}\right) \left(\tilde{A}_2 B_1^2 - B_1\right)\right] v
\]
\[= \left[\frac{\sin(N \theta)}{\sin((N - 1) \theta)} \alpha + 4N^2 \sin^2\left(\frac{\theta}{2}\right) (\alpha^2 - \beta)\right] v,
\]
and for $\theta \in [0, \pi, 2\pi]$, one gets the condition
\[
\frac{\sin (N\theta)}{\sin ((N-1)\theta)} \alpha + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta) \text{ is a real eigenvalue of }
\]
\[
\frac{\sin (N\theta)}{\sin ((N-1)\theta)} \tilde{A}_2 + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \left(\tilde{A}_2 B_1^2 - B_1\right) \text{ and } \nu \text{ is an associated eigenvector of }
\]
\[
\frac{\sin (N\theta)}{\sin ((N-1)\theta)} \alpha + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta)
\]
and
\[
N \left[ \frac{\sin (N\theta)}{\sin ((N-1)\theta)} - 1 \right] = \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \alpha + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta).
\]
Suppose $\alpha \neq N$, then (38) is equivalent to
\[
\frac{\sin (N\theta)}{\sin ((N-1)\theta)} = \frac{1}{(N-\alpha)} N + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta),
\]
or equivalently,
\[
\cot ((N-1)\theta) = \frac{N [1 + 2N\beta (\alpha \beta - 1)] \tan (\theta/2) + \cot \theta}{(N-\alpha)}.
\]
Note that for $0 < k < 2N-3$, there exist $\theta_k \in (k/N - 1)\pi, (k+1/N - 1)\pi$ satisfying (39). Note that by (38), condition (28) takes the form
\[
\left\{ \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \tilde{A}_2 + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \left(\tilde{A}_2 B_1^2 - B_1\right) \right.-
\left[ \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \alpha + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta) I \right]. \right.
\]
\[
A^{ij} \mathbf{d} = 0.
\]
By (27) and the last equation, vectors $H(i)$ is defined by (26) with $\theta \in [0, \pi, \pi, 2\pi]$, where the vectors $\mathbf{d} \in \mathbb{C}^m$ satisfy $T(\theta)\mathbf{d} = 0$, and $T(\theta)$ is the matrix in $\mathbb{C}^{(2p-1)\times m}$ defined by
\[
T(\theta) = \begin{bmatrix}
B_1 A - AB_1 \\
B_1 A^2 - A^2 B_1 \\
\vdots \\
B_1 A^{p-1} - A^{p-1} B_1 \\
S(\theta) A \\
S(\theta) A \\
\vdots \\
S(\theta) A^{p-1}
\end{bmatrix},
\]
where $S(\theta)$ is the matrix in $\mathbb{C}^{m\times m}$ defined by
\[
S(\theta) = \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \tilde{A}_2 + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \left(\tilde{A}_2 B_1^2 - B_1\right)
\]
\[
\frac{\sin (N\theta)}{\sin ((N-1)\theta)} \alpha + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) (\alpha \beta^2 - \beta) I.
\]
Assume that apart from condition (36), vector $\nu$ verifies
\[
\{ A^{ij} \nu; 1 \leq j \leq p-1 \} \subset \ker \left( \tilde{A}_2 - \alpha I \right) \cap \ker (B_1 - \beta I),
\]
then $\nu$ verifies $T(\theta)\nu = 0$ for every value $\theta \in [0, \pi, \pi, 2\pi]$ of equation (41). In fact, note that condition $T(\theta)\nu = 0$ implies
\begin{enumerate}
(i) $(B_1 A^j - A^j B_1)\nu = 0, 1 \leq j \leq p-1$, and
(ii) $S(\theta) A^j \nu = 0, 0 \leq j \leq p-1$.
\end{enumerate}
If (i) holds, then for $1 \leq j \leq p - 1$, one gets

$$0 = (B_1A^j - A^jB_1)v = B_1A^jv - A^jB_1v = (B_1 - \beta I)A^jv - A^j(B_1 - \beta I)v,$$

but for those $v \in \mathbb{C}^m$ such that verifies (36), one gets that $(B_1 - \beta I)v = 0$, thus, taking into account the last expression, it is sufficient that apart from (36) one satisfies that $A^jv \in \text{Ker}(B_1 - \beta I)$. Analogously, if (ii) holds true, then for $0 \leq j \leq p - 1$, one gets

$$0 = \left[ \frac{\sin(N\theta)}{\sin((N - 1)\theta)} \tilde{A}_2 + 4N^2\sin^2\left(\frac{\theta}{2}\right) \left(\tilde{A}_2B_1^2 - B_1\right) \right] A^jv$$

but for those $v \in \mathbb{C}^m$ such that verifies (36), one gets that $(B_1 - \alpha I)v = 0$, thus, taking into account the last expression, it is sufficient that vector $v \in \mathbb{C}^m$ verifies $A^jv \in \text{Ker}(\tilde{A}_2 - \alpha I)$.

Summarizing if conditions (36) and (42) are satisfied, then $T(\theta)v = 0$, $v \neq 0$ guarantee the singularity of matrix $T(\theta)$ and that $v \in \text{Ker}T(\theta)$ for those $\theta$ satisfying (39). By the previous comments, the following set of nontrivial solutions of problems (11)–(13) have been constructed, for $1 \leq k \leq 2N - 3$:

$$U_k(i,j) = (I + \rho A)^j \left\{ \sin(i\theta_k) + 2N\beta \sin\left(\frac{\theta_k}{2}\right) \cos\left(\frac{(2i - 1)\theta_k}{2}\right) \right\} v,$$

$$\rho_k = -4\tau\sin^2\left(\frac{\theta_k}{2}\right), \quad 1 \leq i < N, \quad 0 \leq j < p.$$  \hspace{1cm} (43)

Let us now assume that $\rho = 0$ in (20) which corresponds to the case where $\theta = 0$ in (22),(23). In this case, equation (22) takes the form $t(z - 1)^2 = z^2 - 2z + 1 = 0$, and the solution set of equation (19) is given by

$$H(i) = 1c + id = c + id, \quad c, d \in \mathbb{C}^m.$$  \hspace{1cm} (44)

By imposing to $U(i,j)$ defined by (11) with $G(j) = I$, $j \geq 0$ that satisfies (13) with $A_1 = I$, one gets $H(0) + NB_1[H(1) - H(0)] = 0$. By using (44), it follows that $c = -NB_1d$. Hence, (44) takes the form $H(i) = (iI - NB_1)d, d \in \mathbb{C}^m$. By imposing condition (14) to $U(i,j) = H(i)$, one gets

$$\{A_2 - A_2B_1 + B_2\} d = 0, \quad d \in \mathbb{C}^m \sim \{0\}.$$  \hspace{1cm} (45)

By the definition of $\tilde{A}_2$ and $\tilde{B}_2$ given in (34) and using (36), one gets

$$(A_2B_1 - B_1)^{-1}(A_2 - A_2B_1 + B_2) = \tilde{A}_2 - I,$$

thus, condition (45) is equivalent to

$$\left(\tilde{A}_2 - I\right) d_0 = 0.$$  \hspace{1cm} (46)

Equation (46) has nonzero vector solutions $d_0 \in \mathbb{C}^m$, if and only if $1 \in \sigma(\tilde{A}_2)$, and by Theorem 2.12 of [12], the solution set of (46) is given by

$$d_0 \in \text{Im}\left(I - (\tilde{A}_2 - I)^+\left(\tilde{A}_2 - I\right)\right) = \text{Ker}\left(\tilde{A}_2 - I\right).$$
In this case, we have unbounded solutions as \( N \to \infty \) of (11)-(13) defined by

\[
U(i,j) = (iI - NB_1) d_0, \quad d_0 \in \text{Ker}(\hat{A}_2 - I), \quad 1 \leq i \leq N - 1, \quad j \geq 0.
\]

Consider boundary value problems (11)-(14) with \( B_1 = I \) taking the form

\[
\begin{aligned}
rAG(j) [H(i + 1) - H(i - 1)] + (I - 2rA) G(j) H(i) &= G(j + 1) H(i) \\
A_1 G(j) H(0) + NG(j) [H(1) - H(0)] &= 0, \\
A_2 G(j) H(N) + NB_2 G(j) [H(N) - H(N - 1)] &= 0,
\end{aligned}
\]

\[\text{for } 1 \leq i \leq N - 1, \quad j \geq 0.\]  

(47)

In this case, the function \( H(i) \) will be given by

\[
H(i) = \begin{bmatrix} A_1 \sin(i\theta) - 2N \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{2i - 1}{2} \theta\right) I \end{bmatrix} c, \quad c \in \mathbb{C}^m,
\]

(48)

is a solution of (18). The analogy of condition (28) is

\[
[A_1 G(j) - G(j) A_1] c = 0.
\]

(49)

By using (49), condition (30) takes the form

\[
L(\theta) = 2N \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{2N - 1}{2} \theta\right) \left[ (B_2 A_1 - A_2) + \frac{1}{N} A_2 A_1 \right]
\]

\[+ \sin((N - 1) \theta) \left[ A_2 A_1 + 4N^2 B_2 \sin^2 \left(\frac{\theta}{2}\right) \right] \text{ is singular.} \]

(50)

If we assume that the matrix by blocks

\[
\begin{bmatrix} A_1 & I \\ A_2 & B_2 \end{bmatrix}
\]

is invertible,

(51)

then by properties of Schur’s complement matrix, one gets that \( B_2 A_1 - A_2 \) is invertible, and by Banach’s perturbation lemma, one gets that \( (B_2 A_1 - A_2) + (1/N)A_2 A_1 \) is invertible if \( N \) is large enough. Let us consider

\[
\hat{A}_2 := (B_2 A_1 - A_2)^{-1} A_2, \quad \hat{B}_2 := (B_2 A_1 - A_2)^{-1} B_2,
\]

(52)

and note that \( \hat{A}_2 = \hat{B}_2 A_1 - I \). The analogous condition to (40) in this case is

\[
2N \sin(\theta/2) \cos((2N - 1/2)\theta) = (\alpha \beta^2 - \beta) \frac{\sin(N\theta)}{\sin((N - 1) \theta)} + 4N^2 \alpha \sin^2 \left(\frac{\theta}{2}\right).
\]

(53)

If \( N \neq \alpha \beta^2 - \beta \), then

\[
\frac{\sin(N\theta)}{\sin((N - 1) \theta)} = \frac{N}{N - (\alpha \beta^2 - \beta)} + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \frac{\alpha}{N - (\alpha \beta^2 - \beta)},
\]

where \( \alpha \) and \( \beta \) satisfy

\[
\alpha \in \sigma(\hat{B}_2), \quad \beta \in \sigma(\hat{A}_2), \quad \beta \in \mathbb{R}, \text{ and } v \in \mathbb{C}^m \text{ such that}
\]

\[
(A_1 - \beta I) v = (\hat{B}_2 - \alpha I) v = 0, \quad v \in \mathbb{C}^m \sim \{0\}, \quad (\alpha, \beta) \in \mathbb{R}^2.
\]

(54)
Cases where $A_2 = I$ or $B_2 = I$ can be reduced to the previous cases considering the variable change defined by $i \rightarrow N - i$, $1 \leq i \leq N - 1$, interchanging the roles of $A_2$ and $B_2$ by $A_1$ and $B_1$, respectively. Summarizing, the following general result has been obtained.

**THEOREM 3.1.** Consider the boundary value problems (11)-(13). Let $A$ be the matrix in $\mathbb{C}^{m \times m}$, and let $p \geq 1$ be the degree of the minimal polynomial of $A$, assuming that $A$ is given by (6). Consider the following cases.

**CASE I.** $A_2 = I$.

(i) Assume the conditions

$$\begin{align*}
(B_1 - \beta I)v &= (\hat{A}_2 - \alpha I)v = 0, \quad v \neq 0, \quad (\alpha, \beta) \in \mathbb{R}^2, \\
\{A^j v; 1 \leq j \leq p - 1\} &\subset \ker(\hat{A}_2 - \alpha I) \cap \ker(B_1 - \beta I),
\end{align*}$$

for some vector $v \in \mathbb{C}^m$, where

$$\hat{A}_2 = (B_1 - A_2B_1)^{-1}A_2 \quad \text{and} \quad \alpha \in \sigma(\hat{A}_2), \quad \beta \in \sigma(B_1).$$

Let $T_1(\theta)$ be the matrix in $\mathbb{C}^{(2p-1)m \times m}$ defined by

$$T_1(\theta) = \begin{bmatrix}
B_1A - AB_1 \\
B_1A^2 - A^2B_1 \\
\vdots \\
B_1A^{p-1} - A^{p-1}B_1 \\
\vdots \\
S_1(\theta)A \\
S_1(\theta)A^{p-1}
\end{bmatrix},$$

then $T_1(\theta)v = 0$ where $S_1(\theta)$ is given by

$$S_1(\theta) = \frac{\sin(N\theta)}{\sin((N - 1)\theta)} \hat{A}_2 + 4N^2 \sin^2\left(\frac{\theta}{2}\right)\left(\hat{A}_2B_1^2 - B_1\right) - \left[\frac{\sin(N\theta)}{\sin((N - 1)\theta)}\alpha + 4N^2 \sin^2\left(\frac{\theta}{2}\right)\left(\alpha\beta^2 - \beta\right)\right]I.$$

Let $\theta_k$ be a solution of equation

$$\cot((N - 1)\theta_k) = \frac{N[1 + 2N\beta(\alpha\beta - 1)]\tan(\theta_k/2) + \cot\theta_k}{(N - \alpha)},$$

in $J_k = [(k - 1)/N - 1, (k/N - 1)\pi]$, and $\rho_k = -4\pi^2(\theta_k/2)$, for $1 \leq k \leq 2N - 3$. Then a set of nontrivial solutions of the boundary value problems (11)-(13) is given by

$$U_k(i,j) = (I + \rho_k A)^j \left\{\sin(\theta_k) + 2N\beta\sin\left(\frac{\theta_k}{2}\right)\cos\left(\frac{2i - 1}{2}\theta_k\right)\right\}v,$$

$$1 \leq i \leq N - 1, \quad j \geq 0.$$

(ii) If $1$ is an eigenvalue of $\hat{A}_2$, then there exist nontrivial solutions of problems (11)-(13), given by

$$U(i,j) = (iI - NB_1)d_0, \quad d_0 \in \ker(\hat{A}_2 - I), \quad 1 \leq i \leq N - 1, \quad j \geq 0,$$

which remain unbounded as $N \to \infty$. 

CASE II. $A_2 = I$.

(i) Assume the conditions

\[
\begin{aligned}
(B_1 - \beta I) v = \left( A_1 - \alpha I \right) v &= 0, \quad \alpha \neq 0, \quad (\alpha, \beta) \in \mathbb{R}^2, \\
\{ A^j v; 1 \leq j \leq p - 1 \} \subset \text{ker} \left( A_1 - \alpha I \right) \cap \text{ker} \left( B_1 - \beta I \right),
\end{aligned}
\]

where

\[
\hat{A}_2 = \left( A_1 - A_2 B_1 \right)^{-1} A_1 \quad \text{and} \quad \alpha \in \sigma \left( \hat{A}_1 \right), \quad \beta \in \sigma \left( B_2 \right).
\]

Let $T_2(\theta)$ be the matrix in $\mathbb{C}^{(2p-1) \times m}$, defined by

\[
T_2 (\theta) = \begin{bmatrix}
B_2 A - A B_2 \\
B_2 A^2 - A^2 B_2 \\
\vdots \\
B_2 A^{p-1} - A^{p-1} B_2 \\
S_2 (\theta) A \\
\vdots \\
S_2 (\theta) A^{p-1}
\end{bmatrix},
\]

then $T_2 (\theta) v = 0$, where $S_2 (\theta)$ is given by

\[
S_2 (\theta) = \frac{\sin \left( N \theta \right)}{\sin \left( \left( N - 1 \right) \theta \right)} \hat{A}_1 + 4 N^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \hat{A}_1 B_2 - B_2 \right) - \frac{\alpha \beta^2 - \beta}{\sin \left( \left( N - 1 \right) \theta \right)} 4 N^2 \sin^2 \left( \frac{\theta}{2} \right) I.
\]

Let $\theta_k$ be a solution of equation

\[
\cot \left( \left( N - 1 \right) \theta_k \right) = \frac{N \left[ 1 + 2 N \beta \left( \alpha \beta - 1 \right) \right] \tan \left( \theta_k / 2 \right) + \cot \theta_k}{N - \alpha},
\]

in $I_k = \left( (k-1)/N - 1 \right) \pi, \left( k/N - 1 \right) \pi \right]$, and $\rho_k = -4 \pi \sin^2 (\theta_k / 2)$, for $1 \leq k \leq 2N - 3$. Then a set of nontrivial solutions of the boundary value problems (11)–(13), is given by

\[
U_k (i, j) = (I + \rho_k A)^j \left\{ \sin (i \theta_k) + 2 N \beta \sin \left( \frac{\theta_k}{2} \right) \cos \left( \frac{2 i - 1}{2} \theta_k \right) \right\} v, \quad 1 \leq i \leq N - 1, \quad j \geq 0.
\]

(ii) If $1$ is an eigenvalue of $\hat{A}_1$, then there exist nontrivial solutions of problems (11)–(13) given by

\[
U (i, j) = (i I - N B_1) d_0, \quad d_0 \in \text{Ker} \left( \hat{A}_1 - I \right), \quad 1 \leq i \leq N - 1, \quad j \geq 0,
\]

which remain unbounded as $N \to \infty$.

CASE III. $B_1 = I$.

(i) Assume the conditions

\[
\begin{aligned}
(\hat{B}_2 - \beta I) v &= (A_1 - \alpha I) v = 0, \quad \alpha \neq 0, \quad (\alpha, \beta) \in \mathbb{R}^2, \\
\{ A^j v; 1 \leq j \leq p - 1 \} \subset \text{ker} \left( A_1 - \alpha I \right) \cap \text{ker} \left( \hat{B}_2 - \beta I \right),
\end{aligned}
\]

(57)
Mixed Diffusion Problems

for some vector $v \in \mathbb{C}^m$, where

$$\tilde{B}_2 = (A_2 - B_2 A_1)^{-1} B_2 \quad \text{and} \quad \alpha \in \sigma(A_1), \quad \beta \in \sigma(\tilde{B}_2).$$

Let $T_3(\theta)$ be the matrix in $\mathbb{C}^{(2p-1)mxm}$, defined by

$$T_3(\theta) = \begin{bmatrix}
A_1 A - A A_1 \\
A_1 A^2 - A^2 A_1 \\
\vdots \\
A_1 A^{p-1} - A^{p-1} A_1 \\
S_3(\theta) \\
S_3(\theta) A \\
\vdots \\
S_3(\theta) A^{p-1}
\end{bmatrix},$$

then $T_3(\theta)v = 0$, where $S_3(\theta)$ is given by

$$S_3(\theta) = \frac{\sin(N\theta)}{\sin((N-1)\theta)} \left(A_1 - \tilde{B}_2 A_2^2\right) + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \tilde{B}_2$$

$$- \left[\frac{\sin(N\theta)}{\sin((N-1)\theta)} \left(\alpha - \alpha^2 \beta\right) + 4N^2 \sin^2 \left(\frac{\theta}{2}\right) \beta\right] I.$$

Let $\theta_k$ be a solution of equation

$$\cot((N-1)\theta_k) = \frac{N[1 + 2N\beta] \cot(\theta_k / 2) + \alpha (1 - \alpha \beta) \cot \theta_k}{[N + \alpha (\alpha \beta - \alpha)]},$$

in $I_k = [(k - 1/N - 1)\pi, (k/N - 1)\pi]$, and $\rho_k = -4\rho \sin^2(\theta_k / 2)$, for $1 \leq k \leq 2N - 3$. Then a set of nontrivial solutions of the boundary value problems (11)-(13) is given by

$$U_k(i, j) = (I + \rho_k A)^j \left\{\alpha \sin(i \theta_k) + 2N \sin \left(\frac{\theta_k}{2}\right) \cos \left(\frac{(2i - 1)}{2}\theta_k\right)\right\} v,$$

$$1 \leq i \leq N - 1, \quad j \geq 0.$$
Let $T_4(\theta)$ be the matrix in $\mathbb{C}^{(2p-1)m \times m}$, defined by

$$T_4(\theta) = \begin{bmatrix}
A_2A - AA_2 \\
A_2A^2 - A^2A_2 \\
\vdots \\
A_2A^{p-1} - A^{p-1}A_2 \\
S_4(\theta) \\
\vdots \\
S_4(\theta)A \\
\end{bmatrix},$$

then $T_4(\theta)v = 0$, where $S_4(\theta)$ is given by

$$S_4(\theta) = \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \left( A_1 - \hat{B}_2A_1^2 \right) + 4N^2\sin^2\left(\frac{\theta}{2}\right)\hat{B}_2 - \left[ \frac{\sin (N\theta)}{\sin ((N-1)\theta)} \left( \alpha - \alpha^2\beta + 4N^2\sin^2\left(\frac{\theta}{2}\right)\beta \right) \right]I.$$

Let $\theta_k$ be a solution of equation

$$\cot ((N-1)\theta_k) = \frac{N \left[ 1 + 2N\beta \cot (\theta_k/2) + \alpha (1 - \alpha/2) \cot \theta_k \right]}{[N + \alpha (\alpha\beta - \alpha)]},$$

in $I_k = \left[ (k-1/N-1)\pi, (k/N-1)\pi \right]$, and $\rho_k = -4r \sin^2(\theta_k/2)$, for $1 \leq k \leq 2N-3$. Then a set of nontrivial solutions of the boundary value problems (11)–(13) is given by

$$U_k(i,j) = (I + \rho_kA)^j \left\{ a \sin (i\theta_k) + 2N \sin \left( \frac{\theta_k}{2} \right) \cos \left( \frac{2i-1}{2} \theta_k \right) \right\} v,$$

$1 \leq i \leq N-1, \quad j \geq 0.$

(ii) If 1 is an eigenvalue of $A_2 + \hat{B}_1A_2^2$, then there exist nontrivial solutions of problems (11)–(13) given by

$$U(i,j) = \frac{1}{N} (NI - iA_2) d_0, \quad d_0 \in \text{Ker} \left( A_2 + \hat{B}_1A_2^2 - I \right), \quad 1 \leq i \leq N-1, \quad j \geq 0,$$

which remain unbounded as $N \to \infty$.

4. CONSTRUCTION OF NUMERICAL SOLUTION OF THE MIXED PROBLEM

Theorem 3.1, (i) of Section 3 provides nonzero solutions of boundary value problems (11)–(13) of the form

$$U_k(i,j) = G_k(j)H_k(i), \quad G_k \in \mathbb{C}^{m \times m}, \quad H_k \in \mathbb{C}^m, \quad 1 \leq k \leq N-1,$$

where $G_k(j) = (I - 4r A \sin^2(\theta_k/2))^j$, $0 \leq j \leq p-1$, and for $\beta \in \sigma(B_1)$,

$$H_k(i) = \left\{ \left( 1 + \frac{N\beta \rho_k}{2r} \right) \sin (i\theta_k) - N\beta \sin \theta_k \cos (i\theta_k) \right\} d_k,$$

$1 \leq i \leq N-1, \quad d_k \in \mathbb{C}^m \sim \{0\},$ (59)

moreover, $\theta_k$ are real numbers in $\left[ (k-1/N-1)\pi, (k/N-1)\pi \right], \ k = 1, 2, \ldots, N-1,$ and the vectors

$$d_k = \left( I - T(\theta_k)^+ T(\theta_k) \right) s_k, \quad s_k \in \mathbb{C}^m \sim \{0\}, \quad 1 \leq k \leq N-1,$$

where $T(\theta_k)^+ T(\theta_k)$ is the Moore-Penrose pseudoinverse of $T(\theta_k)$.
and $T(\theta_k)$ is the matrix in $\mathbb{C}^{(2p-1)\times m}$, defined by (40). Superposition suggests to seek a solution of problems (11)–(13) of the form

$$U(i, j) = \sum_{k=1}^{N-1} (I + \rho_k A)^j \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin(i \theta_k) - N \beta \sin \theta_k \cos(i \theta_k) \right\} d_k,$$

$$\rho_k = -4r \sin^2 \left( \frac{\theta_k}{2} \right), \quad 1 \leq k \leq N - 1.$$  

$\{U_k(i, j)\}_{k=1}^{N-1}$ defined by (60) that satisfies the initial condition $U(i, 0) = f(i)$, $1 \leq i \leq N$, one gets

$$f(i) = \sum_{k=1}^{N-1} \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin(i \theta_k) - N \beta \sin \theta_k \cos(i \theta_k) \right\} d_k, \quad 1 \leq i \leq N - 1.$$  

Note that according to [16, Chapter 11],

$$H(i - 1) - \left( \frac{2r + \rho}{2} \right) H(i) + H(i + 1) = 0, \quad 1 \leq i \leq N - 1,$$

$$H(0) + N \beta [H(1) - H(0)] = 0,$$

$$\alpha H(N) + N (\alpha - 1) [H(N) - H(N - 1)] = 0$$

is a discrete Sturm-Liouville problem. To guarantee well conditioning of the problem, we assume that the function $f(i)$ appearing in (4) satisfies

$$f_q(0) + N \beta [f_q(1) - f_q(0)] = 0, \quad \text{for } q = 1, 2, \ldots, m,$$

$$\alpha f_q(N) + N (\alpha - 1) [f_q(N) - f_q(N - 1)] = 0,$$

where $f_q$ is $q$th component of $f$. Then by [16, p. 675], one gets

$$f_q(i) = \sum_{k=1}^{N-1} \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin(i \theta_k) - N \beta \sin \theta_k \cos(i \theta_k) \right\} d_{k,q}, \quad 1 \leq i \leq N - 1,$$

where

$$d_{k,q} = \frac{\sum_{i=1}^{N-1} \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin(i \theta_k) - N \beta \sin \theta_k \cos(i \theta_k) \right\} f_q(i)}{\sum_{i=1}^{N-1} \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin(i \theta_k) - N \beta \sin \theta_k \cos(i \theta_k) \right\}^2}, \quad 1 \leq k \leq N - 1.$$  

Note that if we define vector $d_k$ appearing in (60) by

$$d_k = [d_{k,1}, d_{k,2}, \ldots, d_{k,m}]^T,$$

then $\{U_k(i, j)\}$ defined by (60) satisfies $U(i, 0) = f(i)$, $1 \leq i \leq N$.

For the case $\rho = 0$, $H(i)$ is defined by $H(i) = (i - NB_1)d_0$, $d_0 \in \ker(A_1 - I)$, $1 \leq i \leq N - 1$ if the function $f(i)$ that appears in (4) verifies (61) with $\alpha = 1$, and if apart from $d_{k,q}$ defined by (62), we consider for $q = 1, 2, \ldots, m$,

$$d_{0,q} = \frac{\sum_{i=1}^{N-1} (i - N \beta) f_q(i)}{\sum_{i=1}^{N-1} (i - N \beta)^2}, \quad d_0 = \begin{bmatrix} d_{0,1} \\ d_{0,2} \\ \vdots \\ d_{0,m} \end{bmatrix},$$

(63)
then \( \{U_k(i,j)\} \) verifying \( U(i,0) = f(i) \) becomes defined by

\[
U(i,j) = (iI - NB_1)d_j, \quad 1 \leq i \leq N - 1, \quad j \geq 0.
\]

(64)

Note that where conditions (1)-(3) hold true, the vector \( d_k \in \mathbb{C}^m \) will be verified by the conditions of Theorem 3.1. By definition of the vector \( d_k \), conditions of Theorem 3.1 are satisfied if

\[
T_1(\theta) f(i) = 0 \quad \text{and} \quad (B_1 - \beta I) f(i) = 0, \quad 1 \leq i \leq N - 1.
\]

(65)

Note that by the definition of \( T_1(\theta) \), condition (65) is satisfied if

\[
\left( \tilde{A}_2 - \alpha I \right) A^j f(i) = 0 \quad \text{and} \quad (B_1 - \beta I) A^j f(i) = 0, \quad 1 \leq i \leq N - 1, \quad 0 \leq j < p.
\]

(66)

In fact, by (65), one gets

\[
S_1(\theta) A^j f(i) = \left\{ \begin{array}{c}
\sin \left( \frac{N\theta}{2} \right) \tilde{A}_2 + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \tilde{A}_2 B_1^2 - B_1 \right) \\
\sin \left( \frac{(N - 1)\theta}{2} \right) \alpha + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \alpha \beta^2 - \beta \right) I
\end{array} \right\} A^j f(i)
\]

\[
= \sin \left( \frac{(N - 1)\theta}{2} \right) \left( \tilde{A}_2 - \alpha I \right) A^j f(i) + 4N^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \tilde{A}_2 - \alpha I \right) \tilde{B}_1^2 A^j f(i)
\]

= \[4N^2 \sin^2 \left( \frac{\theta}{2} \right) \left( \tilde{A}_2 - \alpha I \right) \tilde{B}_1^2 A^j f(i)
\]

(67)
Condition (67) can be written in the form
\[ f(i) \in \text{Im} T_1(\alpha, \beta) \quad \text{and} \quad \left[ I - T_1(\alpha, \beta) T_1(\alpha, \beta)^+ \right] AT_1(\alpha, \beta) = 0, \quad 1 \leq i \leq N - 1, \quad (68) \]

\[ T_1(\alpha, \beta) = \left( I - M_\omega^+ M_\alpha \right) \left\{ I - \left[ N_\beta \left( I - M_\omega^+ M_\alpha \right) \right]^+ \left[ N_\beta \left( I - M_\omega^+ M_\alpha \right) \right] \right\}, \quad (69) \]

\[ M_\alpha = \tilde{A}_2 - \alpha I, \quad N_\beta = B_1 - \beta I. \]

Note that condition (68) means that \( f(i) \) belongs to \( \text{Im} T_1(\alpha, \beta) \) and that \( \text{Im} T_1(\alpha, \beta) \) is invariant subspace of matrix \( A \). Thus, the following general result has been established.

**Theorem 4.1.** Consider the initial and boundary value problems (1)-(4) and let \( A \) be a positive stable matrix in \( \mathbb{C}^{m \times m} \), and assume that \( A \) is given by (6), with the previous notation and under the hypotheses of Theorem 4.1. Consider the following cases.

**Case I.** \( A_1 = I \).

(i) Assume the conditions
\[ \left( \tilde{A}_2 - \alpha I \right) f(i) = 0, \quad \left( B_1 - \beta I \right) f(i) = 0, \quad 1 \leq i \leq N - 1, \]

\[ \ker \left( \tilde{A}_2 - \alpha I \right) \cap \ker \left( B_1 - \beta I \right) \text{ is an invariant subspace of } A, \]

where
\[ \tilde{A}_2 = (B_1 - A_2B_1)^{-1} A_2 \quad \text{and} \quad \alpha \in \sigma \left( \tilde{A}_2 \right), \quad \beta \in \sigma \left( B_1 \right). \]

Let \( T_1(\alpha, \beta) \) given by (69) and \( f(i) = F(i/N) \) satisfy \( f(i) \in \text{Im} T_1(\alpha, \beta), \quad 1 \leq i \leq N - 1, \)

and \( \left[ I - T_1(\alpha, \beta) T_1(\alpha, \beta)^+ \right] AT_1(\alpha, \beta) = 0 \). Then problems (1)-(4) admit a solution defined by
\[ U(i, j) = \sum_{k=1}^{N-1} \left( I + \rho_k A \right)^j \left\{ \left( 1 + \frac{N \beta \rho_k}{2r} \right) \sin \left( i \theta_k \right) - N \beta \sin \theta_k \cos \left( i \theta_k \right) \right\} \rho_k, \quad (70) \]

\[ \rho_k = -4r \sin^2 \left( \frac{\theta_k}{2} \right), \quad 1 \leq k \leq N - 1, \quad j \geq 0, \quad 1 \leq i \leq N - 1, \]

\[ d_k = \begin{bmatrix} d_{k,1} \\ d_{k,2} \\ \vdots \\ d_{k,m} \end{bmatrix}, \quad d_{k,q} = \frac{\sum_{i=1}^{N-1} \left[ (1 + (N \beta \rho/2r)) \sin \left( i \theta_k \right) - N \beta \sin \theta_k \cos \left( i \theta_k \right) \right] f_q(i)}{\sum_{i=1}^{N-1} \left[ (1 + (N \beta \rho/2r)) \sin \left( i \theta_k \right) - N \beta \sin \theta_k \cos \left( i \theta_k \right) \right]^2}. \quad (71) \]

(ii) If \( 1 \) is an eigenvalue of \( \tilde{A}_2 \), then there exist nontrivial solutions of problems (11)-(13) given by
\[ U(i, j) = (i - NB_1) d_0, \quad d_0 \in \text{Ker} \left( \tilde{A}_2 - I \right), \quad 1 \leq i \leq N - 1, \quad j \geq 0, \]

where
\[ d_{0,q} = \frac{\sum_{i=1}^{N-1} (i - N \beta) f_q(i)}{\sum_{i=1}^{N-1} (i - N \beta)^2}, \]

these solutions remain unbounded as \( N \to \infty \).
CASE II. $A_2 = I$.

(i) Assume the conditions

$$(B_1 - \beta I) f(i) = (A_1 - \alpha I) f(i) = 0, \quad 1 \leq i \leq N - 1,$$

where

$$\ker (A_1 - \alpha I) \cap \ker (B_1 - \beta I)$$

is an invariant subspace of $A$,

where

$$\tilde{A}_2 = (A_1 - A_2 B_1)^{-1} A_1 \quad \text{and} \quad \alpha \in \sigma (\tilde{A}_1), \quad \beta \in \sigma (B_2).$$

Let $T_2(a, \beta) = (I - M_2^+ M_0) [I - [N_\beta (I - M_0^+ M_0)]^+ [N_\beta (I - M_0^+ M_0)]]$, where $M_\alpha = \tilde{A}_1 - \alpha I$, $N_\beta = B_2 - \beta I$. Assume that $f(i)$ verifies $f(i) \in \text{Im} T_2(a, \beta), 1 \leq i \leq N - 1$, and

$$[I - T_2(a, \beta) T_2(a, \beta)^+] A T_2(a, \beta) = 0.$$

Then problems (1)–(4) admit a solution defined by (70) and $d_k$ is defined by (71) with $eta \in \sigma (B_2)$.

(ii) If 1 is an eigenvalue of $\tilde{A}_1$, then there exist nontrivial unbounded solutions of problems (11)–(13) given by

$$U(i, j) = (i I - N B_2) d_0, \quad d_0 \in \ker (\tilde{A}_1 - I), \quad 1 \leq i \leq N - 1, \quad j \geq 0,$$

where $d_0$ is as in Case (i), with $\beta \in \sigma (B_2)$.

CASE III. $B_1 = I$.

(i) Assume the conditions

$$(\tilde{B}_2 - \beta I) f(i) = (A_1 - \alpha I) f(i) = 0, \quad 1 \leq i \leq N - 1,$$

where

$$\ker (A_1 - \alpha I) \cap \ker (\tilde{B}_2 - \beta I)$$

is an invariant subspace of $A$,

where

$$\tilde{B}_2 = (A_2 - B_2 A_1)^{-1} B_2 \quad \text{and} \quad \alpha \in \sigma (A_1), \quad \beta \in \sigma (\tilde{B}_2).$$

Let $T_3(a, \beta) = (I - M_3^+ M_0) [I - [N_\alpha (I - M_0^+ M_0)]^+ [N_\alpha (I - M_0^+ M_0)]]$, where $M_\beta = \tilde{B}_2 - \beta I$, $N_\alpha = A_1 - \alpha I$. Assume that

$$f(i) \in \text{Im} T_3(a, \beta), \quad 1 \leq i \leq N - 1, \quad \text{and} \quad [I - T_3(a, \beta) T_3(a, \beta)^+] A T_3(a, \beta) = 0.$$

Then problems (1)–(4) admit a solution defined by

$$U(i, j) = \sum_{k=1}^{N-1} (I + \rho_k A)^j \left\{ \frac{\alpha \sin (i \theta_k) + 2 N \sin \frac{\theta_k}{2} \cos \left( \frac{2i - 1}{2} \theta_k \right)}{\rho_k} \right\} d_k,$$

$$\rho_k = -4 r \sin^2 \left( \frac{\theta_k}{2} \right), \quad 1 \leq k \leq N - 1.$$

$$d_k = \begin{bmatrix} d_{k,1} \\ d_{k,2} \\ \vdots \\ d_{k,m} \end{bmatrix}, \quad d_{k,1} = \sum_{i=1}^{N-1} \left\{ \frac{\alpha \sin (i \theta_k) + 2 N \sin \frac{\theta_k}{2} \cos ((2i - 1) \theta_k/2)}{\rho_k} \right\} f_q(i),$$

$$1 \leq k \leq N - 1.$$
(ii) If 1 is an eigenvalue of $A_1 + \widehat{B}_2 A_2^2$, then there exist nontrivial solutions of problems (11)–(13) given by

$$U(i,j) = \frac{1}{N} (N I - i A_1) d_0, \quad d_0 \in \text{Ker} \left( A_1 + \widehat{B}_2 A_2^2 - I \right), \quad 1 \leq i \leq N - 1, \quad j \geq 0,$$

where

$$d_0 = \begin{bmatrix} d_{0,1} \\ d_{0,2} \\ \vdots \\ d_{0,m} \end{bmatrix}, \quad d_{0,q} = \frac{N \sum_{i=1}^{N-1} \left( N - \alpha i \right) f_q(i)}{\sum_{i=1}^{N-1} \left( N - \alpha i \right)^2},$$

these solutions remain unbounded as $N \to \infty$.

**Case IV.** $B_2 = I$.

(i) Assume the conditions

$$\left( \widehat{B}_1 - \beta I \right) f(i) = (A_2 - \alpha I) f(i) = 0, \quad 1 \leq i \leq N - 1,$$

$\text{Ker} \left( A_2 - \alpha I \right) \cap \text{Ker} \left( \widehat{B}_1 - \beta I \right)$ is an invariant subspace of $A_1$.

where

$$\widehat{B}_1 = (A_1 - B_1 A_1)^{-1} B_1 \quad \text{and} \quad \alpha \in \sigma (A_1), \quad \beta \in \sigma \left( \widehat{B}_2 \right).$$

Let

$$T_4(\alpha, \beta) = \left( I - M_{\beta}^+ M_{\beta} \right) \left\{ I - \left[ N_{\alpha} \left( I - M_{\beta}^+ M_{\beta} \right) \right]^+ \left[ N_{\alpha} \left( I - M_{\beta}^+ M_{\beta} \right) \right] \right\},$$

where $M_{\beta} = \widehat{B}_1 - \beta I, \ N_{\alpha} = A_2 - \alpha I$. Assume that $f(i) \in \text{Im} T_4(\alpha, \beta), 1 \leq i \leq N - 1$, and

$$\left[ I - T_4(\alpha, \beta) T_4(\alpha, \beta)^+ \right] A T_4(\alpha, \beta) = 0.$$

Then problems (1)–(4) admit a solution defined by (72) where $d_k$ is defined by (73), with $\beta \in \sigma (\widehat{B}_2)$.

(ii) If 1 is an eigenvalue of $A_2 + \widehat{B}_1 A_2^2$, then there exist nontrivial unbounded solutions of problems (11)–(13) given by

$$U(i,j) = \frac{1}{N} (N I - i A_2) d_0, \quad d_0 \in \text{Ker} \left( A_2 + \widehat{B}_1 A_2^2 - I \right), \quad 1 \leq i \leq N - 1, \quad j \geq 0,$$

where $d_0$ is as in Case (iii), with $\alpha \in \sigma \left( \alpha_1 \right)$.

**Remark.** It is important to point out that although in Section 3, they appear more that $N - 1$ eigenvalues of the associated discrete Sturm-Liouville problem, by [16, p. 675], it is sufficient to consider $N - 1$ orthogonal eigenfunctions. This is the reason why we only consider $N - 1$ eigenvalues and eigenvectors in Theorem 4.1.

**5. Stability and Examples**

In this section, we are concerned with the study of the stability of the solution of problems (7)–(10), that is based on the following result (see [8]).
**THEOREM 5.1.** Let $\theta \in [0, 2\pi[$, $r > 0$ and $A \in \mathbb{C}^{n \times m}$. If $A$ satisfies (5), $A_1 = (A + A^H/2)$ is the real part of $A$, and

$$r < \frac{\lambda_{\min}(A_1)}{2|\rho(A)\sin(\theta/2)|^2}, \quad (74)$$

then the matrix $I - 4rA\sin^2(\theta/2)$ is convergent.

Taking into account expression (60) for the solution of problems (7)-(10), it is sufficient that apart from the hypothesis of Theorem 5.1, one satisfies that matrices $I - 4rA\sin^2(\theta_k/2)$, for each $k = 1, 2, \ldots, N - 1$ are convergent, because the Fourier vector coefficients $d_k$ defined by (61) are bounded if $\{f(i)\}_{i \geq 1}$ is bounded. Note that by Theorem 5.1, a sufficient condition for the

$$r < \frac{\lambda_{\min}(A_1)}{2(\rho(A))^2}. \quad (75)$$

**THEOREM 5.2.** With the hypothesis and with the notation of Theorem 5.1, the solution $\{U(i, j)\}$ of problems (7)-(10) is stable if $A$ satisfies condition (74), $\{F(i)\}_{i \geq 1}$ is bounded and $r$ satisfies (75).

**EXAMPLE 5.1.** Consider problems (1)-(4) where

$$A_1 = I \in \mathbb{C}^{3 \times 3}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and $f(i) = [f_1(i) \ 0 \ 0]^T$. Note that hypothesis (5) is satisfied because $\sigma(A) = \{1, 2\}$. It is easy to check $(\hat{A}_2 - \alpha I)v = (B_1 - \beta I)v = 0$ holds, for $v = [1 \ 0 \ 0]^T$, and $\alpha \in \sigma(\hat{A}_2) = 1 = \beta \in \sigma(B_1)$, where

$$\hat{A}_2 = (A_2B_1 - B_2)^{-1}A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

By using the notation of (71), and straightforward calculations, one gets

$$M_\alpha = \left(\hat{A}_2 - \alpha I\right) \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_\alpha^+ = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

$$(I - M_\alpha^+M_\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_\beta = (B_1 - \beta I) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that,

$$N_\beta(I - M_\alpha^+M_\alpha) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [N_\beta(I - M_\alpha^+M_\alpha)]^+.$$

Using (71), one gets

$$T(\alpha, \beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left(I - T(\alpha, \beta) [T(\alpha, \beta)]^+ \right)AT(\alpha, \beta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
Also,

\[
\begin{pmatrix}
\hat{A}_2 - \alpha I
\end{pmatrix} f(i) = \begin{pmatrix}
0 & 1 & 2 \\
0 & -1 & -1 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
f_1(i) \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = (B_1 - \beta I) f(i)
\]

and

\[
f(i) \in \ker \left( \hat{A}_2 - \alpha I \right) \cap \ker (B_1 - \beta I) = \text{LIN} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.
\]

By Theorem 5.1, the vector sequence

\[
U(i,j) = \sum_{k=1}^{N-1} (I + \rho_k A)^j \left\{ \alpha \sin (i \theta_k) + 2N \sin \frac{\theta_k}{2} \cos \left( \frac{(2i - 1) \theta_k}{2} \right) \right\} d_k,
\]

\[
\rho_k = -4 \pi \sin^2 \left( \frac{\theta_k}{2} \right), \quad 1 \leq k \leq N - 1,
\]

where \( \theta_k \in [(k - 1/N - 1)\pi, (k/N - 1)\pi] \), and \( d_k \) defined by (72), provides solutions of problems (7)-(10). In this case, we have

\[
A_1 = \frac{A + A^H}{2} = \begin{pmatrix}
1 & 0 & 1/2 \\
0 & 2 & 1/2 \\
1/2 & 1/2 & 1
\end{pmatrix}, \quad \lambda_{\min} (A_1) = 0.414956.
\]

By Theorem 5.2, since \( \rho(A) = 2 \), the solution is stable for \( r < (0.414956/2|\rho(A)|^2) = 0.0518695 \).

REFERENCES