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A Criterion for the Boundedness from Below with a Class of Symmetric Operators and Its Applications

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A class of symmetric operators H acting in $L^2(M, \mu)$ with an abstract measure space $\langle M, \mu \rangle$ is considered and a criterion for the boundedness from below of H is presented. The criterion gives also a method to estimate the lower bound of H. As a special case of the criterion, an affirmative answer is given to the question: "If an eigenfunction of a self-adjoint operator in $L^2(M, \mu)$ is strictly positive or "nodeless," then is the corresponding eigenvalue the lowest point of the spectrum?" The class includes Schrödinger operators on $\mathbb{R}^n$ with $M = \mathbb{R}^n$ and μ being the Lebesgue measure. It is illustrated how the method may give a simple proof of the boundedness from below for some Schrödinger operators. Further, it is shown (1) that, in a special case, there exists an operator in $L^2(M, \mu)$ essentially isospectral to H and (2) that the existence of a "nodeless" eigenfunction of H implies the existence of a supersymmetric quantum mechanics in which $H - \lambda$ is unitarily equivalent to the restriction of the supersymmetric Hamiltonian to a subspace, where λ is the lower bound of H.

I. INTRODUCTION AND THE MAIN THEOREMS

Let $\langle M, \mu \rangle$ be a measure space (the Borel field is suppressed). Let L_j, $j = 1, ..., n$, be densely defined closed linear operators in $L^2(M, \mu)$ and V be a real-valued measurable function on M. In this paper, we are concerned wth a symmetric operator of the form

$$H = \sum_{j=1}^{n} L_j^* L_j + V \tag{1.1}$$

and present a criterion for H to be bounded from below.

A novelty of our criterion is in that (1) it is based on a "pointwise variational condition" (see (1.3)) rather than technical assumptions for V as is usually done and (2) it gives also a method to estimate the lower bound

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of H. In applications to some Schrödinger operators on $\mathbb{R}^n$, the criterion yields a simple proof of their boundedness from below. In particular, in the case of the Hamiltonian of the hydrogen-like atom, we can obtain easily the lower bound without any detailed spectral analysis.

We now proceed to state the main theorems. We shall denote by $D(A)$ the operator domain of operator A. A basic assumption is the following:

(A) There exists a dense subspace D in $L^2(M, \mu)$ such that $D \subset D(L_j^* L_j) \cap D(V)$, $j = 1, ..., n$.

With this assumption, H is a densely defined symmetric operator on D. We shall denote by $(\, , \,)$ (resp, $\| \cdot \|$) the inner product (resp. norm) of $L^2(M, \mu)$ and by $\langle f \rangle$ the expectation value of the measurable function f on M with respect to μ:

$$\langle f \rangle = \int_M d\mu(m) f(m).$$

The main results are as follows:

THEOREM 1.1. *Suppose that there exist measurable functions Ω_j on M, $j = 1, ..., n$, and a real-valued measurable function Ψ on M with the following properties*:

(a) $D \subset D(\Omega_j)$

(b) *For all f in D, the inequality*

$$\langle |f|^2 \Psi \rangle \leqslant \sum_{j=1}^{n} \{2 \operatorname{Re}(L_j f, \Omega_j f) - \|\Omega_j f\|^2\} \tag{1.2}$$

holds.

(c) *The pointwise inequality*

$$\Psi(m) + V(m) \geqslant \lambda, \qquad a.e. \quad m \in M, \tag{1.3}$$

holds with a real constant λ.

Then, H is bounded from below on D with $H \geqslant \lambda$.

Remark. In examples given later (Section III), equality holds in (1.2) with a suitable choice of $\{\Omega_j, \Psi\}$.

We denote by H_0 (resp. $\bar{H}$) the closure of $(\sum_{j=1}^n L_j^* L_j) \upharpoonright \bigcap_{j=1}^n D(L_j^* L_j)$(resp. $H \upharpoonright D$).

THEOREM 1.2. *Suppose that there exists a real-valued measurable function Ω on M with the following properties*:

(a) *$\Omega \in D(H_0) \cap D(V) \cap D(\bar{H})$ and*

$$\bar{H}\Omega = (H_0 + V)\,\Omega. \tag{1.4}$$

(b) *$\Omega(m) \neq 0$ for a.e. $m \in M$ and*

$$\bar{H}\Omega = \lambda\Omega \tag{1.5}$$

with a real constant λ.

(c) *There exist measurable functions Ω_j on M, $j = 1, ..., n$, such that $D \subset D(\Omega_j)$ and (1.2) holds with $\Psi = \Omega^{-1}(H_0\Omega)$.*

Then, $\bar{H}$ is bounded from below on $D(\bar{H})$ and λ is the lower bound of $\bar{H}$.

Theorem 1.2 has a theoretical interest in the following sense: Let A be a self-adjoint operator in $L^2(M, \mu)$ and bounded from below. It is well known that, under suitable conditions, an eigenfunction of A corresponding to the lowest point of the spectrum can be taken to be strictly positive (e.g., [11, Section XIII.12 and references in the Notes there]). It seems, however, that no answers have been given to the question concerning the converse statement: If an eigenfunction of A is strictly positive, then is the corresponding eigenvalue the lowest point of the spectrum? Theorem 1.2 gives an affirmative answer to this question.

In Section II, we give the proof of Theorems 1.1 and 1.2. In Section III, we discuss some examples of partial differential operators of second order. In Section IV, considering some concrete Schrödinger operators, we show how our method works to give a simple proof of their boundedness from below. In the last section, we discuss some aspects related to Theorems 1.1 and 1.2: We shall first prove the existence of an operator essentially isospectral [7] to H in the case $n = 1$. Then we construct a supersymmetric quantum mechanics [1, 8] in the case of general n in which $H - \lambda$ is unitarily equivalent to the restriction of the supersymmetric Hamiltonian to a subspace. This gives a general solution to the problem of supersymmetric embedding of non-supersymmetric quantum mechanics (e.g., [5, 2, 3]).

II. Proof of the Main Theorems

2.1. *Proof of Theorem* 1.1

Let

$$A_j = L_j - \Omega_j, \qquad j = 1, ..., n. \tag{2.1}$$

Then, by the assumption, A_j is defined on D. Using (1.2), we have

$$\sum_{j=1}^{n} \|A_j f\|^2 \leqslant \sum_{j=1}^{n} \|L_j f\|^2 - \langle |f|^2\, \Psi\rangle$$

for all f in D. On the other hand, (1.3) gives

$$-\langle |f|^2 \Psi \rangle \leqslant \langle (f, (V-\lambda)f), \qquad f \in D.$$

Hence we get

$$\sum_{j=1}^{n} \|A_j f\|^2 \leqslant \sum_{j=1}^{n} \|L_j f\|^2 + (f, (V-\lambda)f)$$
$$= (f, (H-\lambda)f).$$

The left hand side is non-negative. Therefore we obtain for all f in D

$$(f, (H-\lambda)f) \geqslant 0,$$

which is the desired result.

2.2. *Proof of Theorem* 1.2

Let $\Psi = \Omega^{-1}(H_0 \Omega)$. Then, by (1.4) and (1.5), we have $\Psi = \lambda - V$. Hence Ψ is real and (1.3) is trivially satisfied with equality. Therefore, we can apply Theorem 1.1 and conclude that $\bar{H}$ is bounded from below with $\bar{H} \geqslant \lambda$. On the other hand, we have (1.5). Therefore λ is actually the lower bound of $\bar{H}$.

III. Examples

In the following examples, we take the measure space $\langle M, \mu \rangle$ to be $\langle \mathbb{R}^n, \mu = \text{the Lebesgue measure} \rangle$.

Example 1. Let $a(x)$ be an $r \times r$ real symmetric matrix-valued function on $\mathbb{R}^n$. We assume for simplicity that each matrix element $a_{ij}(i, j = 1, ..., n)$ of a is in $C^1(\mathbb{R}^n)$ and that there exists an $r \times r$ matrix-valued function $b(x)$ on $\mathbb{R}^n$ such that each matrix element $b_{ij}(i, j = 1, ..., n)$ of b is in $C^1(\mathbb{R}^n)$ and

$$b^*(x)\, b(x) = a(x), \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n. \tag{3.1}$$

Obviously the operator

$$L_j = \sum_{k=1}^{n} b_{jk}(x)\, \partial_k, \qquad j = 1, ..., n, \tag{3.2}$$

is densely defined on $C_0^\infty(\mathbb{R}^n)$ and closable in $L^2(\mathbb{R}^n)$. We denote the closure of $L_j \restriction C_0^\infty(\mathbb{R}^n)$ by the same symbol.

Let $S = \{a_1, ..., a_N\}$ with fixed points $a_1, ..., a_N$ in $\mathbb{R}^n$ or $S = \varnothing$ and put

$$D = C_0^\infty(\mathbb{R}^n \setminus S). \tag{3.3}$$

Let V be a real valued measurable function on $\mathbb{R}^n$ such that $D \subset D(V)$. It is obvious that $D \subset D(L_j^* L_j)$, $j = 1, ..., n$, and, in the present case, the operator H defined by (1.1) takes the form

$$H = H_0 + V \tag{3.4}$$

on D with

$$H_0 = -\sum_{j,\,k=1}^{n} \partial_j a_{jk} \partial_k. \tag{3.5}$$

If we take $a_{jk}(x) \equiv \delta_{jk}$, j, $k = 1, ..., n$, then H turns out to be an n-dimensional Schrödinger operator.

LEMMA 3.1. *Let Ω be in $C^2(\mathbb{R}^n \setminus S)$ and be strictly positive. Let*

$$\Omega_j(x) = \Omega(x)^{-1} (L_j \Omega)(x), \qquad x \in \mathbb{R}^n \setminus S, \tag{3.6}$$

and

$$\Psi(x) = \Omega(x)^{-1} (H_0 \Omega)(x), \qquad x \in \mathbb{R}^n \setminus S. \tag{3.7}$$

Then, $D \subset D(\Omega_j) \cap D(\Psi)$ and (1.2) *holds with equality.*

Proof. It is obvious from the definition of D, Ω_j, and Ψ that $D \subset D(\Omega_j) \cap D(\Psi)$. To prove the second half, we need only to apply integration by parts to the quantity $\langle |f|^2 \Psi \rangle$, which gives (1.2) with equality as an identity. ▮

Lemma 3.1 shows that, in the present example, property (b) in the assumption of Theorem 1.1 is satisfied under suitable regularities for Ω.

COROLLARY 3.2. *Let H be given by* (3.4) *and Ω be as in Lemma* 3.1. *Suppose that*

$$(H\Omega)(x) \geqslant \lambda \Omega(x), \qquad x \in \mathbb{R}^n \setminus S,$$

with a real constant λ. Then, H is bounded from below on D with $H \geqslant \lambda$.

Proof. By Lemma 3.1 and the assumption, we can apply Theorem 1.1 with Ω_j and Ψ given by (3.6) and (3.7), respectively. Thus the desired result follows. ▮

LEMMA 3.3. *Consider the case* $S=\varnothing$ *so that* $D=C_0^\infty(\mathbb{R}^n)$. *Let* Ω *be a strictly positive function such that* $\Omega\in H^2(\mathbb{R}^n)$ *and* $\Omega^{-1}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. *Let* Ω_j *and* Ψ *be as in Lemma* 3.1 (*but, in this case, the derivatives are taken in the generalized sense*). *Then,* $D\subset D(\Omega_j)$, $j=1,...,n$, *and* (1.2) *holds with equality.*

Proof. Let f be in D and $K=\operatorname{supp} f$. Since $\Omega^{-1}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, *we have*

$$\Omega(x)^{-1}\,|f(x)|\leqslant C_{K,f},\qquad x\in K$$

with a constant $C_{K,f}>0$, from which it follows that $\Omega_j f$ is in $L^2(\mathbb{R}^n)$ for $j=1,...,n$. Hence we get $D\subset D(\Omega_j)$.

Since $C^2(\mathbb{R}^n)\cap H^2(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$ and $\Omega>0$ with $\Omega^{-1}\in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, there exists a sequence $\{u_N\}_{N=1}^\infty\subset C^2(\mathbb{R}^n)\cap H^2(\mathbb{R}^n)$ such that (i) $u_N\to\Omega$ in $H^2(\mathbb{R}^n)$ as $N\to\infty$; (ii) $u_N>0$; (iii) u_N^{-1} is locally bounded uniformly in N. By Lemma 3.1, we have

$$(f,u_N^{-1}(H_0u_N)f)=\sum_{j=1}^n\{2\operatorname{Re}(L_jf,u_N^{-1}(L_ju_N)f)-\|u_N^{-1}(L_ju_N)f\|^2\}$$

for all $f\in C_0^\infty(\mathbb{R}^n)$. Then, it is easy to see that $(f,u_N^{-1}(H_0u_N)f)\to(f,\Omega^{-1}(H_0\Omega)f)$, $(L_jf,u_N^{-1}(L_ju_N)f)\to(L_jf,\Omega^{-1}(L_j\Omega)f)$, and $\|u_N^{-1}(L_ju_N)f\|\to\|\Omega^{-1}(L_j\Omega)f\|$. Therefore (1.2) holds with equality. ∎

In the same way as in the proof of Corollary 3.2, we get

COROLLARY 3.4. *Let* H *be given by* (3.4) *and* Ω *be as in Lemma* 3.3. *Suppose that*

$$(H\Omega)(x)\geqslant\lambda\Omega(x),\qquad a.e.\quad x\in\mathbb{R}^n$$

with a real constant λ. *Then* H *is bounded from below on* $C_0^\infty(\mathbb{R}^n)$ *with* $H\geqslant\lambda$.

A simple example of V is given as follows. Let U be a real-valued function on $\mathbb{R}^n$ with $U\in C^2(\mathbb{R}^n)$ and put

$$V=\sum_{j,k=1}^n\{a_{jk}(\partial_jU\partial_kU-\partial_j\partial_kU)-(\partial_ja_{jk})\,\partial_kU\}.$$

Let

$$\Omega=e^{-U}.$$

Then, we have

$$(H_0+V)\,\Omega=0$$

and the assumption in Corollary 3.2 is satisfied with $D = C_0^\infty(\mathbb{R}^n)$. Thus, we get $H_0 + V \geqslant 0$ on $C_0^\infty(\mathbb{R}^n)$.

EXAMPLE 2. *Let* $B_j(x)$, $j = 1, ..., n$, be real-valued functions on $\mathbb{R}^n$ with $B_j \in C^1(\mathbb{R}^n)$. Let L_j be the closure of $(\partial_j + B_j) \upharpoonright C_0^\infty(\mathbb{R}^n)$. Let D be as in (3.3) and V be as in Example 1. Then, in the present case, the operator H defined by (1.1) takes the form

$$H = -\sum_{j=1}^{n} (\partial_j - B_j)(\partial_j + B_j) + V$$

on D. In the same way as in Example 1, we can show that Lemma 3.1, Corollary 3.2, Lemma 3.3, and Corollary 3.4 hold for the present case as well.

IV. APPLICATIONS TO SCHRÖDINGER OPERATORS

In this section we illustrate how our method may give a simple proof for the boundedness from below of some Schrödinger operators. We consider only two examples here.

4.1. *The Hamiltonian of a Hydrogen-Like Atom*

Let $n \geqslant 3$ and Δ be the n-dimensional Laplacian. Let

$$H = -\Delta - \frac{Z}{r}, \qquad r = |x|, \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{4.1}$$

with a constant $Z > 0$. In the case $n = 3$, H is the Hamiltonian of a hydrogen-like atom with a suitable system of units. Taking

$$L_j = \partial_j,$$

the generalized derivative with respect to x_j, and

$$V(x) = -\frac{Z}{r},$$

we see that H is a special case of Example 1 in Section III.

Let $t \geqslant Z/(n-1)$ and

$$\Omega(x) = e^{-tr}. \tag{4.2}$$

Then, it is not so difficult to see that Ω satisfies the assumption in Lemma 3.3 and $\Omega \in D(V)$ with $(H\Omega)(x) \geqslant -t^2\Omega(x)$, $x \in \mathbb{R}^n \setminus \{0\}$. Since

$t \geqslant Z/(n-1)$ is arbitrary, we conclude from Corollary 3.4 that H is bounded from below on $C_0^\infty(\mathbb{R}^n)$ with

$$H \geqslant -\frac{Z^2}{(n-1)^2}.$$

If we take $t = Z/(n-1)$ in (4.2), then we have

$$H\Omega = -\frac{Z^2}{(n-1)^2}\,\Omega.$$

It is known that, for $n \geqslant 3$, $V = -Z/r$ is intifinitesimally small with respect to Δ (e.g., [10, Theorems X.15 and X.20]) and hence H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Then, one sees that $\Omega \in D(\bar{H})$ and $\bar{H}\Omega = (-\Delta + V)\,\Omega$, where $\bar{H}$ is the closure of $H \upharpoonright C_0^\infty(\mathbb{R}^n)$. Thus, by Theorem 1.2, we conclude that $-Z^2/(n-1)^2$ is the infimum of the spectrum of $\bar{H}$.

4.2. *A General Case*

Let $n \geqslant 3$ and V be a real-valued function on $\mathbb{R}^n$. Suppose that $D \equiv C_0^\infty(\mathbb{R}^n \setminus \{0\}) \subset D(V)$ and

$$V(x) \geqslant -\frac{Z}{r^\alpha}, \qquad x \in \mathbb{R}^n \setminus \{0\}$$

with constants $Z > 0$ and $0 < \alpha \leqslant 2$. Let

$$H = -\Delta + V \tag{4.3}$$

on $D = C_0^\infty(\mathbb{R}^n \setminus \{0\})$. We shall prove

PROPOSITION 4.1. (a) *Let $0 < \alpha < 2$. Then, H is bounded from below on D with*

$$H \geqslant \sup_{\substack{0 < \lambda < n-2 \\ \mu \in \mathbb{R}}} \inf_{t > 0} E^{(\alpha)}_{\lambda,\mu}(t), \tag{4.4}$$

where

$$E^{(\alpha)}_{\lambda,\mu}(t) = \lambda(n-2-\lambda)\,t^2 - Zt^\alpha + \mu(n-1-2\lambda)\,t - \mu^2. \tag{4.5}$$

(b) *Let $\alpha = 2$ and $Z < (n-2)^2/4$. Then, H is bounded from below on D with*

$$H \geqslant \sup_{\substack{\lambda_- < \lambda < \lambda_+ \\ \mu \in \mathbb{R}}} \inf_{t > 0} E^{(2)}_{\lambda,\mu}(t), \tag{4.6}$$

where

$$\lambda_{\pm} = \frac{n-2 \pm \sqrt{(n-2)^2 - 4Z}}{2}. \tag{4.7}$$

(c) *Let* $\alpha = 2$ and $Z = (n-2)^2/4$. *Then, H is bounded from below on D with*

$$H \geqslant 0. \tag{4.8}$$

Proof. Let

$$H_\alpha = -\Delta - \frac{Z}{r^\alpha}.$$

Then, we have $H \geqslant H_\alpha$ on D and hence we need only to consider H_α. Let $\lambda > 0$, $\mu \in \mathbb{R}$, and

$$\Omega_{\lambda,\mu}(x) = r^{-\lambda} e^{-\mu r}, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

Then we have

$$\Omega_{\lambda,\mu}(x)^{-1}\,(H_\alpha \Omega_{\lambda,\mu})(x) = E^{(\alpha)}_{\lambda,\mu}(r^{-1}), \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

We first consider the case $0 < \alpha < 2$. Let $0 < \lambda < n-2$. Then, $E^{(\alpha)}_{\lambda,\mu}(r^{-1})$ is bounded from below in $r > 0$ and we have

$$\Omega_{\lambda,\mu}(x)^{-1}\,(H_\alpha \Omega_{\lambda,\mu})(x) \geqslant E^{(\alpha)}_{\lambda,\mu}$$

with

$$E^{(\alpha)}_{\lambda,\mu} = \inf_{t>0} E^{(\alpha)}_{\lambda,\mu}(t).$$

Therefore, by Corollary 3.2, we get $H_\alpha \geqslant E^{(\alpha)}_{\lambda,\mu}$. Since $\lambda \in (0, n-2)$ and $\mu \in \mathbb{R}$ are arbitrary, we obtain

$$H_\alpha \geqslant \sup_{\substack{0<\lambda<n-2 \\ \mu \in \mathbb{R}}} E^{(\alpha)}_{\lambda,\mu}.$$

Thus part (a) follows.

We next consider the case $\alpha = 2$ and $Z < (n-2)^2/4$. We have $\lambda(n-2-\lambda) - Z > 0$ if and only if $\lambda_- < \lambda < \lambda_+$. In that case, $E^{(2)}_{\lambda,\mu}(r^{-1})$ is bounded from below in $r > 0$. Then, in the same way as in the preceding case, we get

$$H_2 \geqslant \sup_{\substack{\lambda_- < \lambda < \lambda_+ \\ \mu \in \mathbb{R}}} \inf_{t>0} E^{(2)}_{\lambda,\mu}(t).$$

Thus part (b) follows.

Finally we consider the case $\alpha = 2$ and $Z = (n-2)^2/4$. In this case, the coefficient $c \equiv \lambda(n-2-\lambda) - Z$ of r^{-2} in $E^{(2)}_{\lambda,\mu}(r^{-1})$ is non-positive and hence $E^{(2)}_{\lambda,\mu}(r^{-1})$ may be bounded from below in $r > 0$ only if $c = 0$. Therefore, let $c = 0$, which implies that $\lambda = \lambda_0 \equiv (n-2)/2$. Then, $E^{(2)}_{\lambda_0,\mu}(r^{-1})$ is bounded from below in r if and only if $\mu \geqslant 0$ and, in that case, we have

$$\inf_{r>0} E^{(2)}_{\lambda_0,\mu}(r^{-1}) \geqslant -\mu^2.$$

Since $\mu \geqslant 0$ is arbitrary, we get $H_2 \geqslant 0$. Thus part (c) follows. ∎

Remark. For the problem of the essential self-adjointness of H_α on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ or $C_0^\infty(\mathbb{R}^n)$, see, e.g., [10, Section X.2 and references in the "Notes"].

V. Some Related Aspects

In this section we discuss some aspects related to Theorems 1.1 and 1.2. The notation follows that in Section I.

5.1. *Existence of an Operator Essentially Isospectral to H*

In this sub-section, we restrict ourselves to the case $n = 1$ and we write as $L_1 = L$.

Proposition 5,1, *Let Ω be as in Theorem 1.2. Suppose that $D \subset D(L^*)$ and H is essentially self-adjoint on D. Let $\bar{H} = \overline{H \restriction D}$ and*

$$\hat{H} = \bar{H} - \lambda.$$

Then, the operator

$$A = L - \Omega_1$$

defined on D is closable and we have

$$\sigma(\hat{H}) \setminus \{0\} = \sigma(\bar{A}A^*) \setminus \{0\},$$

$$\sigma_p(\hat{H}) \setminus \{0\} = \sigma_p(\bar{A}A^*) \setminus \{0\},$$

where $\sigma(\cdot)$ (resp. $\sigma_p(\cdot)$) denotes the (resp. point) spectrum and $\bar{A}$ is the closure of $A \restriction D$. Further, the multiplicity of each corresponding non-zero eigenvalue of $\hat{H}$ and $\bar{A}A^*$ coincides.

Remark. By standard theorems in functional analysis, we have $(\bar{A})^* = A^*$ and $\bar{A}A^* = (A^*)^* A^*$ is non-negative self-adjoint on its natural domain.

Proof. Since $D \subset D(L^*)$ by assumption, we have

$$A^* = L^* - \bar{\Omega}_1$$

on D and hence A^* is densely defined. Therefore, $A \restriction D$ is closable. By the proof of Theorem 1.1 and the condition $H\Omega = \lambda\Omega$, we have

$$H = A^*A + \lambda$$

on D. Since D is a core for H, it follows that $D(\bar{H}) = D(A^*\bar{A})$ and

$$\hat{H} = A^*\bar{A}.$$

Then, Deift's theorem [7] gives the desired result. ▮

Remark. Proposition 5.1 shows that the operator $\bar{A}^*A$, which is given by

$$\bar{A}A^*f = AA^*f - A\bar{\Omega}_1 f - \Omega_1 A^*f + |\Omega_1|^2 f$$

for $f \in D$ such that $A^*f \in D(A) \cap D(\Omega_1)$ and $\bar{\Omega}_1 f \in D(\Omega_1) \cap D(A)$, is essentially isospectral [7] to $\hat{H}$. In the case of Examples 1 and 2 in Section III, $\bar{A}A^*$ is also a Schrödinger operator.

5.2. *Connection with a Supersymmetric Quantum Mechanics* (*SSQM*)

We first recall an axiomatic definition of an SSQM [1, 8, 3]. We say that a quadruple $\{\mathscr{H}, \{Q_r\}_{r=1}^N, H_{SS}, N_F\}$ consisting of a Hilbert space $\mathscr{H}$, a set of self-adjoint operators $\{Q_r\}_{r=1}^N$, self-adjoint operators H and N_F in $\mathscr{H}$ is called an SSQM if it has the following properties:

(a) $\mathscr{H}$ is decomposed into two mutually orthogonal closed subspaces $\mathscr{H}_\pm$:

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$$

and, for all $\Psi_\pm \in \mathscr{H}_\pm$,

$$N_F \Psi_\pm = \pm \Psi_\pm.$$

(b)

$$H_{SS} = Q_1^2 = Q_2^2 = \cdots = Q_N^2. \tag{5.1}$$

(c) For every $r = 1, ..., N$, N_F maps $D(Q_r)$ into itself and the anticommutation relation

$$N_F Q_r + Q_r N_F = 0$$

holds on $D(Q_r)$.

(d) For $p \neq r$, Q_p anti-commutes with Q_r in the sense of sesquilinear form on $D(Q_r) \cap D(Q_p)$:

$$(Q_p \Psi, Q_r \Phi) + (Q_r \Psi, Q_p \Phi) = 0 \qquad p \neq r, \qquad \Psi, \Phi \in D(Q_r) \cap D(Q_p). \quad (5.2)$$

Remark. Property (a) is equivalent to that the spectrum of N_F is purely discrete with two eigenvalues ± 1 and $\mathscr{H}_+$ (resp. $\mathscr{H}_-$) is the eigenspace of N_F corresponding to the eigenvalue 1 (resp. -1). Property (b) implies that $H_{SS} \geqslant 0$ and $D(H_{SS}^{1/2}) = D(|Q_r|) = D(Q_r)$, $r = 1, ..., N$. Hence (5.2) holds as an equation for Ψ and Φ in $D(H_{SS}^{1/2})$. In the physics literature, the operators Q_r (resp. H_{SS}, N_F) are called the supercharges (resp. the supersymmetric Hamiltonian, the Fermion number operator). Equations (5.1) and (5.2) are regarded as a representation of the super-Lie algebra $S(N)$ with N supercharges [6].

We shall show that, under the assumption of Theorem 1.2 together with some additional conditions, there exists an SSQM such that $H - \lambda$ is unitarily equivalent to the restriction of the supersymmetric Hamiltonian to a subspace in $\mathscr{H}$. This gives a general solution to the problem of supersymmetric embedding of non-supersymmetric quantum Hamiltonians (e.g., [5, 2, 3]).

To introduce fermionic degrees of freedom, let $A^p(\mathbb{C}^n)$ be the p-fold anti-symmetric tensor product of $\mathbb{C}^n$ with $A^0(\mathbb{C}^n) \equiv \mathbb{C}$ and

$$\mathscr{F}_F(\mathbb{C}^n) = \bigoplus_{p=0}^{n} A^p(\mathbb{C}^n) \quad (5.3)$$

be the Fermion Fock space over $\mathbb{C}^n$ (e.g., [9, Section II.4, 4]). Let ψ_j, $j = 1, ..., n$, be the annihilation operators on $\mathscr{F}_F(\mathbb{C}^n)$, so that ψ_j: $A^p(\mathbb{C}^n) \to A^{p-1}(\mathbb{C}^n)$, $p \geqslant 0$ with $A^{-1}(\mathbb{C}^n) \equiv \{0\}$ and the aniti-commutation relations

$$\{\psi_j, \psi_k^*\} = \delta_{jk}, \quad (5.4)$$

$$\{\psi_j, \psi_k\} = 0 = \{\psi_j^*, \psi_k^*\}, \qquad j, k = 1, ..., n, \quad (5.5)$$

hold, where $\{A, B\} \equiv AB + BA$.

Let $\mathscr{H}$ be the Hilbert space given by

$$\mathscr{H} = L^2(M, \mu) \otimes \mathscr{F}_F(\mathbb{C}^n). \quad (5.6)$$

We can write as

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- \quad (5.7)$$

with

$$\mathscr{H}_+ = \bigoplus_{p:\text{even}} L^2(M, \mu) \otimes A^p(\mathbb{C}^n), \tag{5.8}$$

$$\mathscr{H}_- = \bigoplus_{p:\text{odd}} L^2(M, \mu) \otimes A^p(\mathbb{C}^n). \tag{5.9}$$

Let $P_\pm$ be the orthogonal projection onto $\mathscr{H}_\pm$ and put

$$N_F = P_+ - P_-. \tag{5.10}$$

THEOREM 5.2. *Let Ω, Ω_j, and λ be as in Theorem 1.2. Suppose that $D \subset D(L_j^*)$, $j = 1, ..., n$, and, for all f and g in D and every j, $k = 1, ..., n$,*

$$(L_j^* f, L_k g) - (L_k^* f, L_j g) - (L_j^* f, \Omega_k g) + (\bar{\Omega}_k f, L_j g) \\ - (\bar{\Omega}_j f, L_k g) + (L_k^* f, \Omega_j g) = 0. \tag{5.11}$$

Let

$$D_F = D \otimes \mathscr{F}_F(\mathbb{C}^n) \tag{5.12}$$

and define the operators

$$Q_1 = \sum_{j=1}^{n} (A_j \otimes \psi_j^* + A_j^* \otimes \psi_j), \tag{5.13}$$

$$Q_2 = i \sum_{j=1}^{n} (A_j \otimes \psi_j^* - A_j^* \otimes \psi_j), \tag{5.14}$$

on D_F, where A_j is given by (2.1). *Then, the following hold*:

(a) *Q_1 and Q_2 are symmetric on D_F.*

(b) *For all Ψ and Φ in D_F,*

$$(Q_1 \Psi, Q_2 \Phi) + (Q_2 \Psi, Q_1 \Phi) = 0, \tag{5.15}$$

$$(Q_1 \Psi, Q_1 \Phi) = (Q_2 \Psi, Q_2 \Phi). \tag{5.16}$$

$$(N_F \Psi, Q_j \Phi) + (Q_j \Psi, N_F \Phi) = 0, \qquad j = 1, 2. \tag{5.17}$$

Further, if Q_1 and Q_2 are essentially self-adjoint on D_F, then the following hold:

(c) *$D(\bar{Q}_1) = D(\bar{Q}_2)$ and*

$$\bar{Q}_1^2 = \bar{Q}_2^2, \tag{5.18}$$

where $\bar{Q}_j$ denotes the closure of $Q_j \upharpoonright D_F$.

(d) *The quadruple* $\{\mathscr{H}, \{\bar{Q}_1, \bar{Q}_2\}, H_{SS}, N_F\}$ *with*

$$H_{SS} = \bar{Q}_1^2 \tag{5.19}$$

is an SSQM.

(f) *If* $D \otimes A^0(\mathbb{C}^n) \subset D(\bar{Q}_1^2)$, *then* $H_{SS}: D \otimes A^0(\mathbb{C}^n) \to L^2(M, \mu) \otimes A^0(\mathbb{C}^n)$ *and*

$$(H - \lambda) \otimes I = H_{SS} \tag{5.20}$$

on $D \otimes A^0(\mathbb{C}^n)$. *In particular, if* H *is essentially self-adjoint on* D, *then* $H_{SS} \restriction D \otimes A^0(\mathbb{C}^n)$ *is essentially self-adjoint in* $L_2(M, \mu) \otimes A^0(\mathbb{C}^n)$.

Remarks. (1) Condition (5.11) is just that A_j commutes with A_k in the sense of sesquilinear form on $D \times D$. In the examples in Section III, this condition is satisfied under suitable regularities for Ω.

(2) Part (f) shows that $H - \lambda$ is unitarily equivalent to the restriction of the supersymmetric Hamiltonian to $L^2(M, \mu) \otimes A^0(\mathbb{C}^n)$. In this sense, $H - \lambda$ is supersymmetrically embeddable.

Proof. Part (a) is obvious from the definition of Q_1 and Q_2.

Using (5.5) and (5.11), we have

$$(A_j \otimes \psi_j^* \Psi, A_k^* \otimes \psi_k \Phi) = -(A_k \otimes \psi_k^* \Psi, A_j^* \otimes \psi_j \Phi), \qquad j, k = 1, ..., n.$$

Hence we get

$$(Q\Psi, Q^*\Phi) = 0 \tag{5.21}$$

with

$$Q = \sum_{j=1}^{n} A_j \otimes \psi_j^*. \tag{5.22}$$

Note that Q_j is written as

$$Q_1 = Q + Q^*, \qquad Q_2 = i(Q - Q^*), \tag{5.23}$$

on D_F. By (5.21) and (5.23), we get (5.15) and (5.16) with

$$(Q_j \Psi, Q_j \Phi) = (Q\Psi, Q\Phi) + (Q^*\Psi, Q^*\Phi), \qquad j = 1, 2. \tag{5.24}$$

Equation (5.17) follows from the definition of N_F and the fact that $Q_j: D_F \cap \mathscr{H}_\pm \to \mathscr{H}_\mp$.

Let Q_1 and Q_2 be essentially self-adjoint on D_F. Let Ψ be in $D(\bar{Q}_1)$. Then, there exists a sequence $\{\Psi_n\} \subset D_F$ such that $\Psi_n \to \Psi$ and $Q_1 \Psi_n \to \bar{Q}_1 \Psi$. Equation (5.16) implies that $\{Q_2 \Psi_n\}$ is a Cauchy sequence.

Hence Ψ is in $D(\bar{Q}_2)$ and $Q_2\Psi_n \to \bar{Q}_2\Psi$. Therefore we get $D(\bar{Q}_1) \subset D(\bar{Q}_2)$. Similarly we have $D(\bar{Q}_2) \subset D(\bar{Q}_1)$. Thus we obtain $D(\bar{Q}_1) = D(\bar{Q}_2)$. This argument shows also that (5.16) can be extended to all Ψ and Φ in $D(\bar{Q}_1) = D(\bar{Q}_2)$. Therefore (5.18) follows. Thus part (c) is proved.

To prove (d), we note that Equations (5.15) and (5.17) also can be extended to all Ψ and Φ in $D(\bar{Q}_1) = D(\bar{Q}_2)$. In particular, the extension of (5.17) implies that, if Φ is in $D(\bar{Q}_j)$, then $N_F\Phi$ is in $D(\bar{Q}_j)$ and

$$\{N_F, \bar{Q}_j\} = 0$$

on $D(\bar{Q}_j)$. These results together with (c) mean that $\{\mathscr{H}, \{\bar{Q}_1, \bar{Q}_2\}, H_{SS}, N_F\}$ is an SSQM with H_{SS} given by (5.19).

Finally we prove part (f). From the proof of Theorem 1.1, we have in the present case (use (1.4) and (1.5))

$$\sum_{k=1}^{n} \|(A_k \otimes I)\,\Psi\|^2 = (\Psi, (H-\lambda)\otimes I\Psi), \qquad \Psi \in D \otimes A^0(\mathbb{C}^n). \quad (5.25)$$

On the other hand, using (5.4) and the fact that $\psi_j \colon A^0(\mathbb{C}^n) \to \{0\}$, we have

$$(Q_j\Psi, Q_j\Psi) = \sum_{k=1}^{n} \|(A_k \otimes I)\,\Psi\|^2, \qquad \Psi \in D \otimes A^0(\mathbb{C}^n). \qquad (5.26)$$

Since we have $D \otimes A^0(\mathbb{C}^n) \subset D(\bar{Q}_1^2)$ by assumption, (5.25) and (5.26) give the desired result. ∎

Remark. Let $\mathscr{K}$ be a Hilbert space and T_j, $j = 1, ..., n$, be closed linear operators in $\mathscr{K}$. Suppose that there exists a dense subspace D_0 in $\mathscr{K}$ with $D_0 \subset D(T_j) \cap D(T_j^*)$, $j = 1, ..., n$, and, for every $j, k = 1, ..., n$, T_j commutes with T_k in the sense of sesquilinear form on $D_0 \times D_0$. Let

$$\mathscr{H} = \mathscr{K} \otimes \mathscr{F}_F(\mathbb{C}^n)$$

and Q_j be defined by (5.13) and (5.14) with T_j in place of A_j. Then, in quite the same way as in the proof of Theorem 5.2, one can construct an SSQM(cf. [6]).

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