Generalized uncertainty principle and black hole entropy

Zhao Ren a,b,∗, Zhang Sheng-Li a

a Department of Applied Physics, Xi’an Jiaotong University, Xi’an 710049, PR China
b Department of Physics, Shanxi Datong University, Datong 037009, PR China

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Abstract
Recently, there has been much attention devoted to resolving the quantum corrections to the Bekenstein–Hawking black hole entropy. In particular, many researchers have expressed a vested interest in the coefficient of the logarithmic term of the black hole entropy correction term. In this Letter, we calculate the correction value of the black hole entropy by utilizing the generalized uncertainty principle and obtain the correction terms of entropy, temperature and energy caused by the generalized uncertainty principle. We calculate Cardy–Verlinde formula after considering the correction. In our calculation, we only think that the Bekenstein–Hawking area theorem is still valid after considering the generalized uncertainty principle and do not introduce any assumption. In the whole process, the physics idea is clear and calculation is simple. It offers a new way for studying the corrections caused by the generalized uncertainty principle to the black hole thermodynamic quantity of the complicated spacetime.

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1. Introduction

One of the most remarkable achievements in gravitational physics was the realization that black holes have temperature and entropy [1–3]. There is a growing interest in the black hole entropy. Because entropy has statistical physics meaning in the thermodynamic system, it is related to the number of microstates of the system. However, in Einstein general relativity theory, the black hole entropy is a pure geometric quantity. If we compare the black hole with the thermodynamic system, we will find an important difference. A black hole is a vacancy with strong gravitation. But the thermodynamic system is composed of atoms and molecules. Based on the microstructure of thermodynamic systems, we can explain thermodynamic property by statistic mechanics of its microcosmic elements. Whether the black hole has interior freedom degree corresponding the black hole entropy [4]? Let us suppose that the Bekenstein–Hawking entropy can be attributed a definite statistical meaning. Then how might one go about identifying these microstates and, even more optimistically, counting them [5]? This is a key problem to study the black hole entropy.

Recently, string theory and loop quantum gravity have both had success at statistically explaining the entropy-area “law” [5]. However, who might actually prefer if there was only one fundamental theory? It is expected to choose it by quantum correction term of the black hole entropy. Therefore, studying the black hole entropy correction value becomes the focus of attention. Many ways of discussing the black hole entropy correction value have emerged [5–12]. But the exact value of coefficient of the logarithmic term in the black hole entropy correction term is not known.

Since we discuss the black hole entropy, we need study the quantum effect of the black hole. When we discuss radiation particles or absorption ones, we should consider the uncertainty principle. However, as gravity is turned on, the “conventional” Heisenberg relation is no longer completely satisfactory.
generalized uncertainty principle will replace it. There are many literatures to discuss the correction to the black hole entropy [5, 8,13–16]. However calculations of Refs. [5,8,14] are only valid for four-dimensional Schwarzschild black hole. Although calculations of Refs. [13,16] have discussed higher-dimensional spacetime, there are not logarithmic correction terms. In this Letter, we discuss the black hole entropy correction under the condition that the Bekenstein–Hawking area theorem is still valid after considering the generalized uncertainty principle. We obtain Cardy–Verlinde formula after correction. There is no restriction to spacetimes in the method given by us. So our result has general meaning. The Letter is organized as follows. Section 2 analyses Schwarzschild spacetime. Section 3 discusses higher-dimensional spacetimes. Section 4 gives the correction to Cardy–Verlinde formula. Section 5 provides a conclusion. We take the simple function form of temperature ($c = h = G = K_B = 1$).

2. Schwarzschild black hole

The linear element of Schwarzschild black hole spacetime:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{1}$$

Hawking radiation temperature $T$, horizon area $A$ and entropy $S$ are respectively

$$T = \frac{1}{4\pi r_H} = \frac{1}{8\pi M}, \quad A = 4\pi r_H^2 = 16\pi M^2,$$

$$S = \pi r_H^2 = 4\pi M^2, \tag{2}$$

where $r_H = 2M$ is the location of the black hole horizon.

Now for a black hole absorbing (radiating) particle of energy $dM \approx c d\rho$, the increase (decrease) in the horizon area can be expressed as

$$dA = 8\pi r_H dr_H = 32\pi M dM. \tag{3}$$

Because the discussed black hole radiation is a quantum effect, the particle of energy $dM$ should satisfies the following Heisenberg uncertainty relation.

At $\alpha = 0$, we express (6) by Taylor series and derive

$$\Delta p_i \geq \frac{1}{\Delta x_i} \left[1 + \left(\frac{2\alpha l_{pl}^2}{(\Delta x_i)^2}\right) + 2\left(\frac{2\alpha l_{pl}^2}{(\Delta x_i)^2}\right)^2 + \cdots \right]. \tag{7}$$

From (3) and (4), the change of the area of the black hole horizon can be written as follows:

$$dA = 8\pi r_H dr_H = 32\pi M dp = 32\pi M \frac{1}{\Delta x}. \tag{8}$$

According to the generalized uncertainty principle (7) and (3), the change of the area of the black hole horizon can be rewritten as follows:

$$dA_G = 8\pi r_H dr_H = 32\pi M dp$$

$$= 32\pi M \frac{1}{\Delta x} \left[1 + \left(\frac{2\alpha l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{2\alpha l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots \right] dA. \tag{9}$$

From (8) and (9), we have

$$dA_G = \left[1 + \left(\frac{2\alpha l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{2\alpha l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots \right] dA. \tag{10}$$

According to the view of Refs. [5,8], we take

$$\Delta x = 2r_H = 2\sqrt{\frac{\alpha}{4\pi}}. \tag{11}$$

Substituting (11) into (10) and integrating, we derive

$$A_G = A + \alpha l_{pl}^2 \pi \ln A - 2\left(\frac{\alpha l_{pl}^2}{\pi}\right)^2 \frac{1}{A} - \cdots. \tag{12}$$

Based on Bekenstein–Hawking area law, we take $S = A/4$. Therefore, we can derive the expression of entropy after considering the generalized uncertainty principle. That is, the correction to entropy is given by

$$S_G = S + \alpha l_{pl}^2 \pi \ln S - \left(\frac{\alpha l_{pl}^2}{\pi}\right)^2 \frac{1}{2S} - \cdots + K. \tag{13}$$

Where $S$ is Bekenstein–Hawking entropy, $K$ is an arbitrary constant. In the calculation, we can plus or minus an arbitrary constant $K$. From (13), we can calculate an arbitrary term of correction to entropy and obtain that the coefficient of the logarithmic correction term is positive. This result is different from that of Ref. [5].

3. $d$-dimensional Schwarzschild black hole

The metric of the $d$-dimensional Schwarzschild black hole in ($t, r, \theta_1, \theta_2, \ldots, \theta_{d-2}$) coordinates is [20–22]

$$ds^2 = -\left(1 - \frac{m}{r^{d-3}}\right) dt^2 + \left(1 - \frac{m}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2. \tag{14}$$

The mass of the black hole is given by $M = \frac{(d-2)\Omega_{d-2}^{-m}}{16\pi G_d}$, where $\Omega_{d-2} = \frac{2\pi^{(d-2)/2}}{\Gamma((d-2)/2)}$ is the area of a unit ($d-2$) sphere, $d\Omega_{d-2}$ is the linear element on the unit sphere $S_{d-2}$, and $G_d$ is Newton’s constant in $d$-dimensions. Its entropy and temperature are [19]:

$$S = \frac{\Omega_{d-2}^{d-2} \pi^{d-2}}{4G_d}. \tag{15}$$
\[ T = \frac{d-3}{4\pi r_+}. \]  
(16)

The mass is related to the horizon radius as:
\[ M = \frac{(d-2)\Omega_{d-2}r_+^{d-3}}{16\pi G_d}, \]  
(17)

where \( r_+ = m^{1/(d-3)} \) are the locations of outer horizons.
\[ A = \frac{\Omega_{d-2}}{d+1} r_+^{d-2}. \]  
(18)

Thus, we obtain
\[ dA = \frac{16\pi}{(d-3)} r_+ dM. \]  
(19)

Considering the generalized uncertainty principle, we have
\[ dAG = \left[ 1 + \frac{\alpha^2 l^2}{(\Delta x)^2} + 2 \frac{(\alpha^2 l^2)^2}{(\Delta x)^2} + \cdots \right] dA. \]  
(20)

Let \( \Delta x = 2r_+ \), (20) can be rewritten as
\[ dAG = \left[ 1 + \frac{\alpha^2 l^2 (\Omega_{d-2} \frac{\pi^2}{AG_d})^2}{4} + \frac{(\alpha^2 l^2)^2 (\Omega_{d-2} \frac{\pi^2}{AG_d})^2}{8} + \cdots \right] dA, \]  
(21)

where \( \chi_n \) is a constant.

When \( d \) is even number
\[ A_G = A + \sum_{n=1}^{d-2} \chi_n (\alpha l)^{2n} \frac{d-2}{d-2-2n} A \left( \frac{\Omega_{d-2}}{AG_d} \right)^{\frac{2n}{2n}} \]  
\[ + \chi(\alpha l)^{d-2} \frac{\Omega_{d-2}}{G_d} \ln A \]  
\[ - \sum_{n=1}^{d-2} \chi_n (\alpha l)^{2n} \frac{d-2}{2n-d+2} \left( \frac{\Omega_{d-2}}{AG_d} \right)^{\frac{2n}{2n}}. \]  
(22)

Based on Bekenstein–Hawking area law, we take \( S = A/4 \). Therefore, we can derive the expression of entropy after considering the generalized uncertainty principle. That is, the correction to entropy is given by
\[ S_G = S + \sum_{n=1}^{d-2} \chi_n (\alpha l)^{2n} \frac{d-2}{d-2-2n} S \left( \frac{\Omega_{d-2}}{SG_d} \right)^{\frac{2n}{2n}} \]  
\[ + \chi(\alpha l)^{d-2} \frac{\Omega_{d-2}}{G_d} \ln S \]  
\[ - \sum_{n=1}^{d-2} \chi_n (\alpha l)^{2n} \frac{d-2}{2n-d+2} S \left( \frac{\Omega_{d-2}}{SG_d} \right)^{\frac{2n}{2n}} \]  
\[ = S + \Delta S_G. \]  
(23)

In the above expression, to make it clear, we can add an arbitrary constant. \( \chi_{Gn} \) is a constant.

When \( d \) is an odd number
\[ S_G = S + \sum_{n=1}^{d-2} \chi_n (\alpha l)^{2n} \frac{d-2}{d-2-2n} S \left( \frac{\Omega_{d-2}}{SG_d} \right)^{\frac{2n}{2n}} + C \]  
\[ = S + \Delta S_2. \]  
(24)

Where \( C \) is an arbitrary constant. From (23), when \( d = 4 \), the result is consistent with (13). From (24), when \( d \) is an odd number, the logarithmic term does not exist in the correction to the black hole. It is different with the correction to entropy caused by fluctuation in Ref. [23].

From (16),
\[ dT = -\frac{4G_d dM}{(d-2)\Omega_{d-2} r_+^{d-2}}. \]  
(25)

According to (4), (7) and (25), after considering the generalized uncertainty principle, the change of temperature \( dT_G \) is
\[ dT_G = \left[ 1 + \frac{\alpha^2 l^2 (\Omega_{d-2} \frac{\pi^2}{AG_d})^2}{4} + \frac{(\alpha^2 l^2)^2 (\Omega_{d-2} \frac{\pi^2}{AG_d})^2}{8} + \cdots \right] dT. \]  
(26)

Let \( \Delta x = 2r_+ \), then \( (\Delta x)^2 = \frac{(d-3)^2}{4\pi^2 T^2} \), (26) can be rewritten as
\[ dT_G = \left[ 1 + \frac{\alpha^2 l^2 (4\pi^2)}{(d-3)^2} T^2 + 2 \frac{(\alpha^2 l^2)^2 (4\pi^2)}{(d-3)^2} T^4 + \cdots \right] dT. \]  
(27)

Integrating (27), we have
\[ T_G = \sum_{n=1}^{d-2} \chi_n \left( \frac{2\alpha l}{d-3} \right)^{2n} \frac{T^{2n-1}}{2n-1} = T + \Delta T, \]  
(28)

where \( \chi_n \) is a constant, \( \Delta T = \sum_{n=2}^{d-2} \chi_n \left( \frac{2\alpha l}{d-3} \right)^{2n-2} \frac{T^{2n-1}}{2n-1} \). From (16) and (17), we obtain
\[ E = M = \frac{(d-2)(d-3)^{d-3} \Omega_{d-2}}{4(4\pi)^{d-2} T^{d-3}}. \]  
(29)

From (29) and (28), the energy of the black hole \( E \) is a function about \( T \). After considering the generalized uncertainty principle, radiation temperature \( T \) becomes \( T_G \), so the energy is correspondingly as follows:
\[ E_G = M \left( 1 - \frac{\Delta T}{T} \right)^{d-3} = M + \Delta M, \]  
(30)

where \( \Delta M = M \sum_{n=1}^{d-3} \frac{1}{C_{d-3}} (-\frac{\Delta T}{T})^n \).

4. Generalized uncertainty principle corrections to the Cardy–Velinde formula

Recently, Cardy–Velinde formula [24] was generalized to asymptotically flat spacetime [25, 26]. In asymptotically flat spacetime Cardy–Velinde formula is given by:
\[ S_{\text{CFT}} = \frac{2\pi R}{d-2} \sqrt{2E_E}, \]  
(31)

where \( R \) is the radius of the system, \( E \) is the total energy and \( E_E \) is the Casimir energy, defined as
\[ E_E = (d-1)E - (d-2)TS. \]  
(32)
In this section we compute the generalized uncertainty principle corrections to the entropy of a $d$-dimensional Schwarzschild black hole described by the Cardy–Verlinde formula Eq. (31). The Casimir energy Eq. (32) now will be modified due to the uncertainty principle correction as

$$E_G^c = (d - 1)E_G - (d - 2)T_G S_G.$$  \hspace{1cm} (33)

It is easily seen that

$$2E_G E_G^c = 2(d - 1)E_G^c - 2(d - 2)E_G T_G S_G.$$  \hspace{1cm} (34)

When we only take $\alpha^2$ term, after considering the generalized uncertainty principle, $S_{CFT}^G$ is written as:

$$S_{CFT}^G = S_{CFT} \left[ 1 + \frac{(d - 2)}{2E_G^c} \left( \frac{2(d - 1)}{(d - 2)} E \Delta E - T S \Delta E - ES \Delta T - ET \Delta S \right) \right],$$  \hspace{1cm} (35)

where when $d = 4$,

$$\Delta S = \alpha^2 l_\text{pl}^2 4\pi^2 \ln S, \quad \Delta T = \frac{1}{3} \alpha^2 l_\text{pl}^2 4\pi^2 T^3,$$

$$\Delta E = - \frac{M \Delta T}{T}. \hspace{1cm} (36)$$

When $d > 4$

$$\Delta S = \alpha^2 l_\text{pl}^2 \frac{d - 2}{d - 4} S \left( \frac{\Omega_{d-2}}{SG_d} \right)^{\frac{2}{d-3}}, \quad \Delta T = \left( \frac{2\alpha l_\text{pl}\pi}{d - 3} \right)^{\frac{2}{3}} T^3,$$

$$\Delta M = - M(d - 3) \frac{\Delta T}{T}. \hspace{1cm} (37)$$

Ref. [16] has investigated a higher-dimensional spacetime, but the leading-order correction to the black hole entropy is negative. And when $d = 4$, there is not the logarithmic correction term. From (37), the leading-order corrections to the entropy caused by the generalized uncertainty principle are positive. When $d = 4$, the leading-order correction term is the logarithmic term.

5. Conclusion

In summary, we derive the correction term of higher-dimensional Schwarzschild black hole entropy by using the generalized uncertainty principle. From (13), (22) and (24), for even number dimensional Schwarzschild spacetime, coefficient of the logarithmic term in the black hole entropy correction terms is positive. For odd number dimensional Schwarzschild spacetime, there is not the logarithmic term in the black hole entropy correction term. However, studying the correction to the black hole entropy, we need consider many factors such as the generalized uncertainty principle, thermal fluctuation and retraction of the black hole. To simplify, in this Letter, we only consider the correction caused by the generalized uncertainty principle.

After deriving the correction to the black hole entropy due to the generalized uncertainty principle, we have calculated the other thermodynamic quantities. Further we obtain the corrections to the Cardy–Verlinde formula. Because our result is for arbitrary dimensional spacetime, our method is valid not only for four-dimensional spacetimes but also for higher-dimensional spacetimes. It offers a new way for studying the correction to entropy of the complicated spacetimes.

If we can obtain the exact value of the coefficient of the logarithmic term in the black hole entropy correction term by other method, we can determine the uncertainty number $\alpha$ in the generalized uncertainty principle.

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