THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS OF PEANO CONTINUA IN THE PLANE

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It is well known that if $X$ is one of an arc, a circle or a disk, $X$ does not admit an expansive homeomorphism. In this paper, we prove that there is no expansive homeomorphism on any (nondegenerate) Peano continuum in the plane.

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1. Introduction

If $X$ is a metric space with metric $d$ and $f: X \to X$ is a homeomorphism of $X$, then $f$ is said to be expansive provided that there is $c > 0$ such that if $x, y \in X$ and $x \neq y$, then there is an integer $n(x, y) \in \mathbb{Z}$ for which $d(f^n(x), f^n(y)) > c$. All spaces under consideration are assumed to be metric. A continuum is a compact connected nondegenerate space. It is well known that the Cantor set, the 2-adic solenoid and the 2-torus etc., admit expansive homeomorphisms (see [9, 10]). Bryant, Jakobsen and Utz showed that there are no expansive homeomorphisms on an arc, a circle or a disk (see [2, 3]). By using those results, Kawamura showed that if $X$ is a Peano continuum which contains a free arc, then $X$ does not admit an expansive homeomorphism (see [6]). Also, we showed that if $X$ is a Peano continuum which contains a 1-dimensional ANR neighborhood, then $X$ does not admit an expansive homeomorphism, and if $X$ is a dendroid (pathwise connected tree-like continuum), then $X$ does not admit an expansive homeomorphism (see [4, 5]). The following problem is interesting: Is it true that if $X$ is a continuum in the plane, then $X$ does not admit an expansive homeomorphism?

In this paper, we give a partial answer to this problem. More precisely, we prove that if $X$ is a Peano continuum in the plane, then $X$ does not admit an expansive homeomorphism.
We refer readers to [7] for the plane topology.

2. Preliminaries

In this section, we give some definitions and facts which we need. Let \( \delta > 0 \) be any positive number and let \( n \) be any natural number. Let \( A \) be an arc from \( p \) to \( q \) in a metric space \( Y \) with metric \( d \). Then the arc \( A \) is said to be \((n, \delta)\)-folding provided that there are points \( p = a_1 < b_1 < \cdots < a_n < b_n = q \) in \( A \) such that \( d(a_i, b_i) \geq \delta \) for each \( i = 1, \ldots, n \). A continuum \( X \) is called a Peano continuum if \( X \) is locally connected.

Lemma 2.1 [5, 2.21. Let \( f: X \to X \) be an expansive homeomorphism of a compactum \( X \). Then there exists \( \delta > 0 \) such that if \( A \) is a nondegenerate subcontinuum of \( X \), there exists a natural number \( n_0 \) such that one of the following conditions holds:

(a) diam \( f^n(A) \geq \delta \) for \( n \geq n_0 \);

(b) diam \( f^{-n}(A) \geq \delta \) for \( n \leq -n_0 \).

By using Lemma 2.1, we can easily see the following (cf. the proof of [5, 2.31]):

Lemma 2.2. Let \( f: X \to X \) be an expansive homeomorphism of a compactum \( X \). Let \( A \) be an arc from \( p \) to \( q \) in \( X \). Then there is \( \delta > 0 \) such that for any natural number \( n \), there is a natural number \( i(n) \) satisfying one of the following two conditions:

(a) \( f^m(A) \) is \((n, \delta)\)-folding for \( m \geq i(n) \);

(b) \( f^{-m}(A) \) is \((n, \delta)\)-folding for \( -m \leq -i(n) \).

We need the \( \theta \)-curve theorem as follows.

The \( \theta \)-curve theorem 2.3 [7, Theorem 2, p. 511]. If \( C \) is a \( \theta \)-curve in the plane \( E \) consisting of three arcs \( L_0, L_1, L_2 \) having, pairwise, only their end-points in common, then

\[
E - C = D_0 \cup D_1 \cup D_2, \quad Fr(D_j) = L_j \cup L_{j+1}(\text{mod } 3),
\]

where \( D_0, D_1, D_2 \) are the components of \( E - C \).

A locally connected continuum which contains no simple closed curve is called a dendrite. A continuum \( X \) is said to be regular [7] if for any point \( p \) of \( X \) and any \( \varepsilon > 0 \), there is an open set \( G \) such that \( p \in G \), \( \text{diam}(G) < \varepsilon \) and \( Fr(G) \) is a finite set. Clearly, if \( X \) is a regular, then \( \text{dim} X \leq 1 \).

By [7, Theorem 1, p. 283], we have:
Lemma 2.4. If a continuum $X$ is a regular, then $X$ is locally connected.

By [7, Fundamental Theorem 6, p. 531], we have:

Lemma 2.5. If $X$ and $X^*$ are two Janiszewski spaces which contain no separating points and which do not consist of single points, then $X$ is homeomorphic to $X^*$. In particular, $X$ is homeomorphic to the 2-sphere $S^2$.

Also, by [7, Theorem 4, p. 512] we have:

Lemma 2.6. Let $X$ be a Janiszewski space containing no separating points. If $C$ is a locally connected continuum in $X$, for any component $R$ of $X - C$, $\text{Fr}(R)$ is a regular continuum containing no $\theta$-curve.

Consequently, we can conclude that if $X$ is a locally connected continuum in the plane $E$, then for any component $U$ of $E - X$, $\text{Fr}(U)$ is a locally connected continuum and $\dim \text{Fr}(U) \leq 1$. Note that in the Euclidean 3-dimensional space $E^3$, one can easily construct a Peano continuum such that the boundary of a complementary domain is not locally connected.

3. Self-homeomorphisms of Peano continua in the plane

In this section we prove the following main result in this paper:

Theorem 3.1. If $X$ is a nondegenerate Peano continuum in the plane $E$, $X$ does not admit an expansive homeomorphism.

The following is easily proved by induction on $k$. We omit the proof.

Lemma 3.2. Let $X$ be a set and let $X_1, X_2, \ldots, X_k$ be subsets of $X$ such that $X = \bigcup X_i$. Then there is a sufficiently large natural number $n(k) (> 2k^2)$ such that for any sequence $a_1, b_1, a_2, b_2, \ldots, a_{n(k)}, b_{n(k)}$ of points of $X$, there are $i_1, i_2$ and $i_3$ such that $i_1 < i_2 < i_3$, $a_i$ and $a_j$ are contained in some $X_i$ and $b_j$ is contained in some $X_i$ which contains $b_j$.

Proof of Theorem 3.1. Suppose, on the contrary, that there exists an expansive homeomorphism $f$ on the Peano continuum $X$ in the plane $E$. Let $U$ be the component of $E - X$ such that $U$ is unbounded. By Lemmas 2.4-2.6, $\text{Fr}(U)$ is a locally connected continuum with $\dim \text{Fr}(U) = 1$. First, suppose that $\text{Fr}(U)$ does not contain a simple closed curve. Then $\text{Fr}(U)$ is a dendrite. By [1, Corollary 13.5, p. 138], $\text{Fr}(U)$ is an AR. By [1, Theorem 13.1, p. 132], $E - \text{Fr}(U)$ is connected. We shall show that $X$ is a dendrite. Suppose, on the contrary, that $X$ has a simple closed curve $S$. Let $W$ be the bounded component of $E - S$. Then $(E - \text{Fr}(U)) \cap W \neq \emptyset$ because $\dim \text{Fr}(U) = 1$. Since $E - \text{Fr}(U)$ is pathwise connected, there is an arc
A(=[p,q]) from a point p of U to a point q of W in \( E - \text{Fr}(U) \) such that \( p \not\in W \).

By the Jordan separation theorem, we can choose the point \( r \) of \( A \) such that \( r \in X \) and \([p, r] - \{r\}\) is contained in \( E - X \). Clearly, \([p, r] - \{r\}\) is contained in \( U \). Hence \( r \in \text{Fr}(U) \). This is a contradiction. Thus \( X \) is a dendrite. By [4], there is no expansive homeomorphism on \( X \). Hence we may assume that \( \text{Fr}(U) \) contains a simple closed curve \( S \). Next, suppose that \( \text{Fr}(U) \) contains a simple closed curve \( S \).

Let \( \delta > 0 \) be as in Lemma 2.2. Since \( X \) is a Peano continuum, there are subsets \( X_1, X_2, \ldots, X_k \) of \( X \) such that each \( X_i \) is a Peano continuum, \( \text{diam} \ X_i < \frac{1}{2} \delta \) and \( X = \bigcup X_i \). Choose a natural number \( n(k) \) as in Lemma 3.2. Let \( A \) be an arc from \( p \) to \( q \) in \( S \). By Lemma 2.2, for some integer \( m \in \mathbb{Z} \), \( f^m(A) \) is \((n(k), \delta)\)-folding. By Lemma 3.2, we can conclude that there are points \( f^m(p) \leq a < b < c < d \leq f^m(q) \) in \( f^m(A) \) such that \( d(a, b) \geq \delta \), \( a \) and \( c \) are contained in some \( X_i \), and \( b \) and \( d \) are contained in some \( X_j \). Since \( \text{diam} \ X_i < \frac{1}{2} \delta \), we see that \( X_i \cap X_j = \emptyset \). Since \( X_i \) is pathwise connected, there are an arc (which is homeomorphic to the unit interval) \( A_1 \) from \( a \) to \( c \) in \( X_i \) and an arc \( A_2 \) from \( b \) to \( d \) in \( X_j \) (Fig. 1). Note that \( A_1 \cap A_2 = \emptyset \).

Consider the sets \( S, A, f^{-m}(A_1) \) and \( f^{-m}(A_2) \). Since \( f^{-m}(A_1) \) and \( f^{-m}(A_2) \) are arcs, by the choice of the component \( U \) of \( E - X \), we can see that \( f^{-m}(A_1) \) and \( f^{-m}(A_2) \) are contained in \( \text{Cl} \ D \), where \( D \) is the bounded component of \( E - S \) (see Theorem 2.3). Also, by Theorem 2.3, we see that \( f^{-m}(A_1) \cap f^{-m}(A_2) \neq \emptyset \) (Fig. 2). This is a contradiction. This completes the proof. \( \square \)
The following problems remain open.

**Problem 1.** Is it true that if $X$ is a nondegenerate plane continuum, then $X$ does not admit an expansive homeomorphism?

**Problem 2.** Is it true that if $X$ is a nondegenerate 1-dimensional Peano continuum, then $X$ does not admit an expansive homeomorphism?

**Problem 3.** Is it true that if $X$ is a nondegenerate tree-like continuum, then $X$ does not admit an expansive homeomorphism?

**Note added in proof**

Problem 1 has a negative answer. Barge informed the author that there exists an indecomposable plane continuum which admits an expansive homeomorphism [11].

**References**


