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Trivial stationary solutions to the Kuramoto–Sivashinsky and certain nonlinear elliptic equations

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Abstract

We show that the only locally integrable stationary solutions to the integrated Kuramoto–Sivashinsky equation in $\mathbb{R}$ and $\mathbb{R}^2$ are the trivial constant solutions. We extend our technique and prove similar results to other nonlinear elliptic problems in $\mathbb{R}^N$.

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1. Introduction

The integrated Kuramoto–Sivashinsky equation (abbreviated hereafter as the KSE)

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \tag{1}$$

subject to appropriate initial and boundary conditions has been introduced in [15,16] and in [23,24] in studying phase turbulence and the flame front propagation in combustion theory.

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In the absence of any a priori estimates for the solutions of the scalar equation (1), most authors find it more convenient, for the mathematical study, to consider the differential form of the equation for $\mathbf{u} = \nabla\phi$

$$\mathbf{u}_t + \Delta^2\mathbf{u} + \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0. \tag{2}$$

The one-dimensional case has been studied by many authors (see, e.g., [2–6,8,9,11–14,18,20, 26,27] and references therein). It is known that the long-term dynamics of the 1-D equation (2) with periodic boundary condition, of period L, possesses a complicated global attractor $\mathcal{A}_L$ with finite dimension (see, e.g., [5,11,12,18,20,27] and references therein). The best upper bound for the dimension of the global attractor is of the order $O(L^{\frac{45}{40}})$ is obtained, based on the best available upper bound for the size of the absorbing ball, in [10]. Namely, the current estimates for the upper bound for the dimension of the global attractor depend explicitly on $R = \limsup_{t\to\infty}|u(\cdot,t)|_{L^2}$: if R is of the order $O(L^{\beta})$, then the upper bound for the Hausdorff and fractal dimensions of the global attractor satisfies $d_H(\mathcal{A}_L) \leqslant d_f(\mathcal{A}_L) \leqslant O(L^{\frac{30+10\beta}{40}})$. As mentioned above, the best estimate for R is given in [10]: $R \sim o(L^{\frac{3}{2}})$ (see also [2]). On the other hand, based on numerical simulations and physical arguments it is conjectured [21] that the upper bound for the dimension of the global attractor should behave like L. This conjectured estimate also matches the readily available lower bound for the dimension of the global attractor which is obtained by linearizing about the stationary solution $u \equiv 0$. To achieve this conjectured bound for the dimension of the global attractor requires, for instance, one to establish a uniform bound for the L^∞-norm of the solutions on the attractor, which is independent of the period L. Therefore, the question is: whether the L^∞-norm of all the solutions in the global attractor $\mathcal{A}_L$ is uniformly bounded, independent of L. The remarkable paper of Michelson [18] shows that all the bounded stationary (steady state) solutions of the one-dimensional KSE (2) on the whole line are uniformly bounded by a constant K_M. In a paper by Cheskidov and Foias [3], the authors consider the nonhomogeneous one-dimensional stationary KSE (2) subject to periodic boundary condition with zero spatial average: namely, the problem

$$u_{xxxx} + u_{xx} + uu_x = f(x), \qquad u(x) = u(x+L), \qquad \int_0^L u(x)\,dx = 0.$$

They obtain an explicit estimate for the Michelson constant K_M, namely, $K_M \leqslant 92.2$. Furthermore, they have also shown in [3] that the L^∞-norm of the infinite time averages of the solutions of the one-dimensional KSE (2) is uniformly bounded, independent of the period L. More precisely, let us denote by $H = \{u \in L^2\colon\ u(x) = u(x+L),\ \int_0^L u(x)\,dx = 0\}$, the phase space of the dynamical system induced by the solutions of the one-dimensional KSE (2), subject to periodic boundary conditions. And denote by $\mathcal{P}(H)$, the set of all time average invariant probability measures of this dynamical system (for details about stationary statistical solutions and time average invariant probability measures of dissipative dynamical systems see, e.g., [7]). It is shown in [3] that the L^∞-norm of the set

$$E = \left\{\bar{u} \in H\colon\ \bar{u} = \int_H u\mu(du),\ \text{where } \mu \in \mathcal{P}(H)\right\}$$

is uniformly bounded, independent of the period L. In particular, Michelson's result is a particular case of the above mentioned result of [3]. This is because every steady state solution of the one-dimensional KSE (2) belongs to the set E. To see that, let u_0 be a steady state solution of the one-dimensional KSE (2), then the measure $\mu(du) = \delta_{u_0}(du)$ is a trivial time average invariant probability measure, and hence it belongs to $\mathcal{P}(H)$. As a result we have

$$u_0 = \int_H u\delta_{u_0}(du) = \int_H u\mu(du) = \bar{u} \in E.$$

The question of global regularity of the Cauchy problem

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(x,0) = \phi_0(x) \tag{3}$$

or the periodic boundary condition case

$$\begin{gathered} \phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0, \\ \phi(x + Le_j, t) = \phi(x,t) \quad \text{for } j = 1, 2, \ldots, N, \qquad \phi(x,0) = \phi_0(x) \end{gathered} \tag{4}$$

is still an open question in dimensions two and higher cases (see, however, [22] for the case of thin two-dimensional domains for large, but restricted, initial data). Motivated by the question of global regularity of (3) or (4), the authors of [1] study what they call the hyper-viscous Hamilton–Jacobian-like equation for the scalar function u:

$$u_t + \Delta^2 u = |\nabla u|^p, \qquad u|_{\partial\Omega} = \nabla u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0(x), \tag{5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$.

In the case $p > 2$, they show that finite time blow-up will occur for special "large" initial conditions. It is remarked that the blow-up occurs in L^∞-norm, i.e., the derivative of the u remains finite as long as the solution exists and has finite L^∞-norm. In particular, there is essential difference in the structure of formation of singularities from that of generalized viscous Hamilton–Jacobi equations [25]:

$$u_t - \Delta u = |\nabla u|^p \quad \text{in } \Omega \times (0,\infty), \qquad u|_{\partial\Omega} = 0, \qquad u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{6}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$. Regardless of the value of p, $p \geqslant 1$, problem (6) satisfies a maximum principle, the L^∞-norm of the solutions to problem (6) remains bounded for as long as the solutions exist. Thus, the solutions to (6) that become singular in finite time must develop their singularities in one of their derivatives (see [25]). However, for the critical case of $p = 2$ in problem (5), it is still unknown whether, for $N \geqslant 2$, there is global regularity for all initial data, or there will be finite time blow-up for certain initial data. On the other hand, it is worth mentioning that for the case $p = 2$ the problem (6) is the viscous Burgers–Hopf equation which is globally well-posed for $N = 1, 2, 3$.

Motivated by the above discussion, we study in this paper the steady state problem (1) in $\mathbb{R}^N$ for $N \geqslant 1$. In particular, we show that in dimensions $N = 1, 2$ the only locally integrable steady solutions of (3) or (4) are the trivial solutions $\phi(x) = \text{constant}$. The techniques developed and used here are inspired by the work of Mitidieri and Pohozaev [19]. It is worth mentioning that for $N = 3$, Michelson [17] has established, using asymptotic methods, the existence of a nontrivial radial steady state solution of (3). This is consistent with our results which are restricted to dimensions $N = 1, 2$.

In Section 2, we will study the stationary solutions and present our main result. In Section 3, we extend our tools to certain nonlinear elliptic systems in $\mathbb{R}^N$ and show nonexistence of non-trivial solutions to those equations.

2. Steady state Kuramoto–Sivashinsky equation

In this section we consider the integrated version of the homogeneous steady state KSE in $\mathbb{R}^N$

$$\Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{7}$$

We emphasize the fact that we do not require ϕ to satisfy any specific boundary condition as $|x| \to \infty$.

Definition 1. A function $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is called a locally integrable solution of (7) in $\mathbb{R}^N$ if ϕ satisfies Eq. (7) in the distribution sense, i.e., in $\mathcal{D}'(\mathbb{R}^N)$.

Theorem 2. *For $N = 1, 2$, the only locally integrable solutions Eq.* (7) *are the trivial solution, i.e., $\phi =$* constant.

Proof. Consider the smooth radial cut-off function $\varphi_0(x) \in C_0^\infty(\mathbb{R}^N)$, such that $0 < \varphi_0(x) < 1$ whenever $1 < |x| < 2$, and

$$\varphi_0(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases}$$

Let

$$\varphi_R(x) = \varphi_0\left(\frac{x}{R}\right). \tag{8}$$

Suppose ϕ is a locally integrable solution of (7), taking action of (7) on the test function φ_R, we have

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\, dx = -2\left\langle \Delta^2\phi, \varphi_R\right\rangle - 2\langle \Delta\phi, \varphi_R\rangle. \tag{9}$$

We estimate the right-hand side of the above equality:

$$\begin{aligned} 2\left|\left\langle \Delta^2\phi, \varphi_R\right\rangle\right| &= 2\left|\int_{\mathbb{R}^N} -\nabla\phi(x)\cdot\nabla\big(\Delta\varphi_R(x)\big)\, dx\right| \leqslant 2\int_{R<|x|<2R} \left|\nabla\phi(x)\right|\left|D^3\varphi_R(x)\right| dx \\ &\leqslant 2\left(\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2\varphi_R(x)\, dx\right)^{\frac{1}{2}}\left(\int_{R<|x|<2R} \frac{|D^3\varphi_R(x)|^2}{\varphi_R(x)}\, dx\right)^{\frac{1}{2}}, \end{aligned}$$

where D^k denotes a generic expression of the form

$$D^k u = \sum_{|\alpha|=k} a_\alpha \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index and a_α are constants.

Also we have

$$\begin{aligned} 2\left|\langle \Delta\phi, \varphi_R\rangle\right| &= 2\left| -\int_{\mathbb{R}^N} \nabla\phi(x)\cdot\nabla\varphi_R(x)\,dx \right| \leqslant 2\Big(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|\left|D\varphi_R(x)\right|dx \Big) \\ &\leqslant 2\Big(\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \Big)^{\frac{1}{2}} \Big(\int_{R<|x|<2R} \frac{|D\varphi_R(x)|^2}{\varphi_R(x)}\,dx \Big)^{\frac{1}{2}}. \end{aligned}$$

The above estimates and (9) imply

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx &\leqslant 2\Big(\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \Big)^{\frac{1}{2}} \Big(\int_{R<|x|<2R} \frac{|D^3\varphi_R(x)|^2}{\varphi_R(x)}\,dx \Big)^{\frac{1}{2}} \\ &\quad + 2\Big(\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \Big)^{\frac{1}{2}} \Big(\int_{R<|x|<2R} \frac{|D\varphi_R(x)|^2}{\varphi_R(x)}\,dx \Big)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

By Young's inequality, we reach

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 8\Big(\int_{R<|x|<2R} \frac{|D^3\varphi_R(x)|^2}{\varphi_R(x)}\,dx + \int_{R<|x|<2R} \frac{|D\varphi_R(x)|^2}{\varphi_R(x)}\,dx \Big).$$

By our definition $\varphi_R(x) = \varphi_0(\frac{x}{R})$. Let us change the variables $x = R\xi$, then we obtain

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 8\Big(R^{N-6} \int_{1<|\xi|<2} \frac{|D^3\varphi_0(\xi)|^2}{\varphi_0(\xi)}\,d\xi + R^{N-2} \int_{1<|\xi|<2} \frac{|D\varphi_0(\xi)|^2}{\varphi_0(\xi)}\,d\xi \Big). \tag{11}$$

Now, we further specialize in the choice of the test function φ_0 such that the integrals on the right-hand side of (11) are finite. Then (11) implies

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 8C_0 R^{N-6} + 8C_1 R^{N-2}, \tag{12}$$

where

$$C_0 = \int\limits_{1<|\xi|<2} \frac{|D^3\varphi_0(\xi)|^2}{\varphi_0(\xi)}\,d\xi, \qquad C_1 = \int\limits_{1<|\xi|<2} \frac{|D\varphi_0(\xi)|^2}{\varphi_0(\xi)}\,d\xi. \tag{13}$$

Case $N = 1$. Let us first consider $N = 1$. Let $r > 0$ be fixed large enough, we consider R to be large enough such that $R > 4r$. From (12) we conclude that

$$\int\limits_{|x|<r} \left|\frac{d\phi(x)}{dx}\right|^2 dx \leqslant \int\limits_{\mathbb{R}} \left|\frac{d\phi(x)}{dx}\right|^2 \varphi_R(x)\,dx \leqslant 8C_0R^{-5} + 8C_1R^{-1}. \tag{14}$$

Passing to the limit as $R \to \infty$ in (14), we obtain that

$$\int\limits_{|x|<r} \left|\frac{d\phi(x)}{dx}\right|^2 dx = 0$$

for every $r > 0$. Therefore $\frac{d\phi}{dx} = 0$ and the assertion of the theorem is proved for $N = 1$.

Case $N = 2$. Now we consider the case $N = 2$. In this case, relation (12) implies that

$$\int\limits_{\mathbb{R}^2} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 8C_0R^{-4} + 8C_1.$$

Choose as before $r > 0$ fixed large enough, and let $R > 4r$. From the above we get

$$\int\limits_{|x|<r} \left|\nabla\phi(x)\right|^2 dx \leqslant \int\limits_{\mathbb{R}^2} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 8C_0R^{-4} + 8C_1.$$

Passing to the limit as $R \to \infty$, we obtain

$$\int\limits_{|x|<r} \left|\nabla\phi(x)\right|^2 dx \leqslant 8C_1 \tag{15}$$

for every $r > 0$. By the Lebesgue monotone convergence theorem, we conclude that

$$\nabla\phi \in L^2(\mathbb{R}^2) \quad \text{and} \tag{16}$$

$$\int\limits_{\mathbb{R}^2} \left|\nabla\phi(x)\right|^2 dx \leqslant 8C_1. \tag{17}$$

Now, let us return to Eq. (9). Note that

$$\operatorname{supp}\{D\varphi_R\} \subseteq \left\{x \in \mathbb{R}^N \mid R \leqslant |x| \leqslant 2R\right\}. \tag{18}$$

We estimate the right-hand side of relation (9):

$$
\begin{aligned}
2\left|\left\langle \Delta^2\phi, \varphi_R\right\rangle\right| &= 2\left|\int_{\mathbb{R}^2} -\nabla\phi(x)\cdot\nabla\big(\Delta\varphi_R(x)\big)\,dx\right| \leqslant 2\int_{R<|x|<2R} \left|\nabla\phi(x)\right|\left|D^3\varphi_R(x)\right|dx \\
&\leqslant 2\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2\varphi_R(x)\,dx\right)^{\frac{1}{2}}\left(\int_{R<|x|<2R} \frac{|D^3\varphi_R(x)|^2}{\varphi_R(x)}\,dx\right)^{\frac{1}{2}} \\
&\leqslant 2R^{-2}\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}}\left(\int_{1<|\xi|<2} \frac{|D^3\varphi_0(\xi)|^2}{\varphi_0(\xi)}\,d\xi\right)^{\frac{1}{2}} \\
&\leqslant 2C_0^{\frac{1}{2}}R^{-2}\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}},
\end{aligned}
$$

where C_0 is given in (13) and in the above we changed the variable $x = R\xi$ and applied (16). Similarly, for the other integral on the right-hand side of (9),

$$
\begin{aligned}
2\left|\langle \Delta\phi, \varphi_R\rangle\right| &= 2\left|\int_{\mathbb{R}^2} -\nabla\phi(x)\cdot\nabla\varphi_R(x)\,dx\right| \leqslant 2\int_{R<|x|<2R} \left|\nabla\phi(x)\right|\left|D\varphi_R(x)\right|dx \\
&\leqslant 2\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2\varphi_R(x)\,dx\right)^{\frac{1}{2}}\left(\int_{R<|x|<2R} \frac{|D\varphi_R(x)|^2}{\varphi_R(x)}\,dx\right)^{\frac{1}{2}} \\
&\leqslant 2\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}}\left(\int_{1<|\xi|<2} \frac{|D\varphi_0(x)|^2}{\varphi_0(x)}\,d\xi\right)^{\frac{1}{2}} \\
&\leqslant 2C_1^{\frac{1}{2}}\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}},
\end{aligned}
$$

where C_1 is given in (13) and in the above we changed the variable $x = R\xi$ and applied (17).

These estimates and (9) imply

$$
\begin{aligned}
\int_{\mathbb{R}^2} \left|\nabla\phi(x)\right|^2\varphi_R(x)\,dx \leqslant{}& 2C_0^{\frac{1}{2}}R^{-4}\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}} \\
&+ 2C_1^{\frac{1}{2}}\left(\int_{R<|x|<2R} \left|\nabla\phi(x)\right|^2dx\right)^{\frac{1}{2}}. \qquad (19)
\end{aligned}
$$

Passing to the limit as $R \to \infty$ in (19), by (15), (17), and the Lebesgue dominated convergence theorem we obtain that

$$\int_{\mathbb{R}^2} |\nabla\phi|^2\,dx = 0.$$

Hence, $\phi(x) = \text{constant}$. □

As a consequence of the above theorem we have the following corollary.

Corollary 3. *The only stationary solutions to* (4)*, i.e., the only periodic solutions of Eq.* (7)*, are the constants.*

Proof. Actually, one can prove this corollary in a direct trivial way and for all N. In this case the set of test functions $\mathcal{V}$ consists of all trigonometric polynomials. The function $\varphi(x) = 1 \in \mathcal{V}$ is a test function. Taking the action of (7) on φ in the region $\Omega = [0, L]^N$ will give us

$$\langle \Delta^2\phi, 1\rangle + \langle \Delta\phi, 1\rangle + \frac{1}{2}\int_{\Omega} \left|\nabla\phi(x)\right|^2 dx = 0.$$

By the periodicity of ϕ in Ω, we obtain

$$\int_{\Omega} |\nabla\phi|^2\,dx = 0.$$

It follows readily that $\nabla\phi = 0$, which implies that $\phi = \text{constant}$. □

Next we consider the nonhomogeneous steady state:

$$\Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = f, \tag{20}$$

where $f(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$.

Corollary 4. *Let* $N \leqslant 2$*, and* $\liminf_{R\to\infty} \int_{\mathbb{R}^N} f(x)\varphi_R(x)\,dx \leqslant \mu_0 < 0$ *for some constant* μ_0*. Here* φ_R *is specified in the manner of* (8) *and* (13)*. Then Eq.* (20) *has no locally integrable solutions, i.e., no solutions in* $H^1_{\text{loc}}(\mathbb{R}^N)$*.*

Proof. Taking action of (20) on the test function φ_R defined in (8) and (13), we get

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2\varphi_R(x)\,dx = \int_{\mathbb{R}^N} f(x)\varphi_R(x)\,dx - 2\langle \Delta^2\phi, \varphi_R\rangle - 2\langle \Delta\phi, \varphi_R\rangle.$$

By (10), we reach

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 \varphi_R(x)\,dx \leqslant 2\int_{\mathbb{R}^N} f(x)\varphi_R(x)\,dx + 8\left(\int_{R<|x|<2R} \frac{|D^3\varphi_R(x)|^2}{\varphi_R(x)}\,dx + \int_{R<|x|<2R} \frac{|D\varphi_R(x)|}{\varphi_R(x)}\,dx\right).$$

Since $\liminf_{R\to\infty}\int_{\mathbb{R}^N} f(x)\varphi_R(x)\,dx \leqslant \mu_0$, using the same argument as in the proof of Theorem 2, we obtain

$$\int_{\mathbb{R}^N} \left|\nabla\phi(x)\right|^2 dx \leqslant 2\mu_0 < 0$$

for $N = 1, 2$, which implies that we do not have any locally integrable solutions for Eq. (20). □

3. Generalization to other nonlinear elliptic problem

In this section we generalize the tools developed in the previous section and apply them to certain class of nonlinear elliptic problems. Consider the nonlinear elliptic equation

$$(-\Delta)^m u \pm \left|\nabla^l \Delta^n u\right|^p = 0 \tag{21}$$

defined in the whole space $\mathbb{R}^N$, where $l = 0$ or $l = 1$, $2n + l \geqslant 0$, $2m > 2n + l$ and $p > 1$.

Definition 5. A function $u \in W^{2n+l,p}_{\text{loc}}(\mathbb{R}^N)$ is called locally integrable solution of Eq. (21) if u satisfies Eq. (21) in the distribution sense.

We emphasize again that we do not require the solution u to satisfy any specific boundary condition as $|x| \to \infty$.

Remark 1. A solution u of Eq. (21) is said to be trivial if $\nabla^l \Delta^n u = 0$, and it is called nontrivial otherwise.

Theorem 6. *Let $N \leqslant \frac{(2m-(2n+l))p}{p-1}$, then the only locally integrable solutions of Eq.* (21) *are the trivial solutions.*

Proof. Taking the action of (21) on the test function φ_R defined in (8), we have

$$\left\langle(-\Delta)^m u, \varphi_R\right\rangle \pm \int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx = 0 \quad \text{or}$$

$$\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx = \mp\left\langle(-\Delta)^m u, \varphi_R\right\rangle.$$

By definition of distribution, we have

$$\left|\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx\right|$$

$$= \left| \mp \langle (-\Delta)^m u, \varphi_R \rangle \right| = \left| \int_{R<|x|<2R} \nabla^l \Delta^n u(x) \cdot D^{2m-(2n+l)} \varphi_R(x)\,dx \right|$$

$$\leqslant \left(\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx \right)^{\frac{1}{p}} \left(\int_{R<|x|<2R} \frac{|D^{2m-(2n+l)}\varphi_R(x)|^{p'}}{\varphi_R(x)^{p'-1}} \right)^{\frac{1}{p'}}, \tag{22}$$

where p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$. So, we have

$$\left(\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx \right) \leqslant \left(\int_{R<|x|<2R} \frac{|D^{2m-(2n+l)}\varphi_R(x)|^{p'}}{(\varphi_R(x))^{p'-1}}\,dx \right).$$

Again, we change the variables $x = R\xi$. If we further specify φ_0 such that

$$\widetilde{C}_0 = \int_{1<|\xi|<2} \frac{|D^{2m-(2n+l)}\varphi_0(\xi)|^{p'}}{(\varphi_0(\xi))^{p'-1}}\,d\xi < \infty. \tag{23}$$

We will have

$$\left(\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R\,dx \right) \leqslant \widetilde{C}_0 R^{\theta}, \tag{24}$$

where $\theta = N - (2m - (2n + l))p'$.

Case $N < \frac{(2m-(2n+l))p}{p-1}$. Let us first consider the case $N < \frac{(2m-(2n+l))p}{p-1}$, i.e., $\theta < 0$. Let $r > 0$ be fixed large enough. We consider R be large enough such that $R > 4r$. From (24) we conclude that

$$\int_{|x|<r} \left|\nabla^l \Delta^n u(x)\right|^p dx \leqslant \int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\,dx \leqslant \widetilde{C}_0 R^{\theta}. \tag{25}$$

Passing to the limit as $R \to \infty$ in (25), we obtain that

$$\int_{|x|<r} \left|\nabla^l \Delta^n u(x)\right|^p dx \leqslant 0$$

for every $r > 0$. By Lebesgue monotone convergence theorem, we conclude that

$$\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p dx = 0.$$

Then the assertion of the theorem is proved for $N < \frac{(2m-(2n+l))p}{p-1}$.

Case $N = \frac{(2m-(2n+l))p}{p-1}$. Next, we consider $N = \frac{(2m-(2n+l))p}{p-1}$, i.e., $\theta = 0$. Then relation (24) implies that

$$\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\, dx \leqslant \widetilde{C}_0.$$

Choose as before $r > 0$ fixed large enough and let $R > 4r$. From the above relation we obtain

$$\int_{|x|<r} \left|\nabla^l \Delta^n u(x)\right|^p dx \leqslant \int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\, dx \leqslant \widetilde{C}_0$$

for every $r > 0$. By Lebesgue monotone convergence theorem, we conclude that

$$\nabla^l \Delta^n u \in L^p\left(\mathbb{R}^N\right). \tag{26}$$

Now, let us return to inequality (22). Note that

$$\operatorname{supp}\left\{D^{2m-(2n+l)}\varphi_R\right\} \subseteq \left\{x \in \mathbb{R}^N \mid R \leqslant |x| \leqslant 2R\right\}. \tag{27}$$

Then relations (22) and (27) imply

$$\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p \varphi_R(x)\, dx \leqslant \tilde{C}_0^{\frac{1}{p'}} \left(\int_{R<|x|<2R} \left|\nabla^l \Delta^n u(x)\right|^p dx \right)^{\frac{1}{p}}, \tag{28}$$

where $\widetilde{C}_0$ is defined in the manner of (23). Passing to the limit as $R \to \infty$ in (28), by the absolute convergence of the integral $\int_{R<|x|<2R} |\nabla^l \Delta^n u(x)|^p\, dx$ and the Lebesgue dominated convergence theorem, we obtain that

$$\int_{\mathbb{R}^N} \left|\nabla^l \Delta^n u(x)\right|^p dx = 0,$$

which concludes our proof. □

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