Theoretical Computer Science 412 (2011) 1652–1668

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Inductive inference and computable numberings

Klaus Ambos-Spies [a,*], Serikzhan Badaev [b], Sergey Goncharov [c]

[a] University of Heidelberg, Institut für Informatik, Im Neuenheimer Feld 294, D-69120 Heidelberg, Germany
[b] Al-Farabi Kazakh National University, 71 Al-Farabi ave., Almaty 050038, Kazakhstan
[c] Sobolev's Math. Institute, 4 Koptyug ave., Novosibirsk 630090, Russia

ARTICLE INFO

Keywords:
Computably enumerable sets
Computable numberings
Inductive inference

ABSTRACT

It has been previously observed that for many TxtEx-learnable computable families of computably enumerable (c.e. for short) sets all their computable numberings are evidently $\mathbf{0}'$-equivalent, i.e. are equivalent with respect to reductions computable in the halting problem. We show that this holds for all TxtEx-learnable computable families of c.e. sets, and prove that, in general, the converse is not true. In fact there is a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are computably equivalent and $\mathcal{A}$ is not TxtEx-learnable. Moreover, we construct a computable family of c.e. sets which is not TxtBC-learnable though all of its computable numberings are $\mathbf{0}'$-equivalent. We also give a natural example of a computable TxtBC-learnable family of c.e. sets which possesses non-$\mathbf{0}'$-equivalent computable numberings. So, for the computable families of c.e. sets, the properties of TxtBC-learnability and $\mathbf{0}'$-equivalence of all computable numberings are independent.

1. Introduction

The theory of inductive inference has many relationships to the theory of computable numberings. Both these theories arose almost at the same time, in the 1960s, and since that time they have developed in a parallel way using similar methods and ideas. Sometimes, these theories had direct influence on each other. For instance, Kummer [14] suggested a solution for the famous problem of Ershov on characterizing the classes of computable functions with pairwise equivalent computable numberings. A criterion was given in terms of co-learning. We should note that another solution of that problem was given earlier by Goncharov [9] in the context of computable Abelian groups, see [10] for a direct proof of Goncharov's criterion in terms of computable numberings. As an example in the opposite direction, we can mention a criterion of Jain and Sharma [12] for a class of c.e. sets to be TxtEx-learnable which was given in terms of computable numberings.

It is not our goal to give a survey with a complete analysis of the interrelations of the theory of computable numberings and inductive inference theory. This is a too wide and deep subject and we do not consider ourselves leaders in the field of learning theory. We just wish to attract attention to these interrelations which we met during our research as well as during discussions with Frank Stephan and John Case. We are sure that the theory of inductive inference could pose natural questions and be useful for computable numbering theorists too.

We deal with language identification in the limit from positive data, namely, we use the classical learning model of Dana Angluin [8] and restrict ourselves to the two scenarios of identification: explanatory learning and behaviorally correct learning, i.e., TxtEx and TxtBC.

The theory of computable numberings started with the study of the sequences of sets of natural numbers which admit uniform algorithmic procedures for enumerating the elements of the sets of the sequence. If we identify this kind of sequence

* Corresponding author.
E-mail address: ambos@math.uni-heidelberg.de (K. Ambos-Spies).

0304-3975/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2010.12.041

with a mapping of the set of natural numbers onto the family of all sets from this sequence then we get the notion of computable numbering. The standard sequence $W_0, W_1, W_2, \ldots$ of c.e. sets is the most important computable numbering. Here indices are the codes of programs of the functions whose domains are the elements of the sequence.

It is a well known fact that a numbering $\alpha : \mathbb{N} \longrightarrow \mathcal{A}$ of a family $\mathcal{A}$ of c.e. sets is computable if and only if $\alpha(x) = W_{f(x)}$ for some computable function f and all x. Informally, each strategy of uniform algorithmic enumeration of a family $\mathcal{A}$ automatically finds a W-index of any set of the family by means of its approximation (or text) given by this strategy. And if, for every set in $\mathcal{A}$, we change our strategy finitely often (as we usually do in the priority constructions with finite injuries) then we will get a suitable W-index eventually. This informal consideration is a base for understanding the close connections between procedures of constructing computable numberings and language identification.

Most of the classes of c.e. sets considered in the theory of inductive inference are not computable. It seems that this is caused by the study of learning in the limit of families of computable functions. There, for a family of computable functions which has a computable numbering, the set of the least indices of the functions with respect to that numbering is computably enumerable. This immediately gives the existence of a learning strategy in the limit. Nevertheless, we could observe that special computable classes of c.e. sets are used as extensions of learnable (but not necessarily computable) classes (for instance, see the characterization of language identification in [12,11]). On the other hand, the study of non-computable families of c.e. sets is not too interesting for the theory of numberings, so in our paper we will deal only with computable families of c.e. sets.

It is well known [13] that computable families of finite sets, finite classes of computably enumerable sets, and some classes of the graphs of computable functions are TxtEx-learnable. On the other hand, the computable numberings of any of these classes are pairwise equivalent with respect to the reduction by $\mathbf{0}'$-computable functions. It might be that these observations led Frank Stephan to propose the following conjecture to one of the authors of this paper:

For every computable family $\mathcal{A}$ of c.e. sets, the following are equivalent.

(i) $\mathcal{A}$ is TxtEx-learnable.
(ii) All computable numberings of $\mathcal{A}$ are $\mathbf{0}'$-equivalent.

Our aim is to demonstrate that one of the directions of this statement is true, namely the direction (i) $\Rightarrow$ (ii) (see Theorem 3.1 below) and to prove that the converse fails. In fact, we show that there is a computable family of c.e. sets $\mathcal{A}$ such that all computable numberings of $\mathcal{A}$ are equivalent with respect to the reduction by computable functions and such that $\mathcal{A}$ is not TxtEx-learnable (see Theorem 4.1 below).

This leaves the question of whether we can restore Stephan's conjecture if we replace explanatory learnability by the weaker notion of behaviorally-correct learnability. We answer this question negatively too by constructing a computable family $\mathcal{A}$ of c.e. sets which is not TxtBC-learnable though all of its computable numberings are $\mathbf{0}'$-equivalent (see Theorem 6.1 below). We also give a natural example of a computable TxtBC-learnable family of c.e. sets which possesses non-$\mathbf{0}'$-equivalent computable numberings (see Theorem 5.1 below). So, for a computable family $\mathcal{A}$ of c.e. sets, the TxtBC-learnability and $\mathbf{0}'$-equivalence of all computable numberings are independent.

The outline of the paper is as follows. In Section 2 we introduce the basic concepts we will deal with. In Section 3 we prove Theorem 3.1. It should be noted that this result is not new but that Frank Stephan was aware of this fact when he communicated his conjecture to us. We first show that Theorem 3.1 is a consequence of some deep facts of learning theory. Then we give a simple self contained proof using some technique of the theory of numberings. In Sections 4–6 we prove Theorems 4.1, 5.1 and 6.1, respectively. The paper is finished with some concluding remarks in Section 7.

The present paper is an extended version of the presentation [1] given at the conference TAMC 2008. The results of Sections 5 and 6 are added to the results of that presentation. Moreover, the proof of Theorem 4.1 which was only sketched there is given in more detail here.

2. Preliminaries

We follow the monograph [5] in Russian and the survey papers [6,2] for the terminology and notations accepted in the theory of numberings. A mapping $\alpha : \mathbb{N} \longrightarrow \mathcal{A}$ of the set $\mathbb{N}$ of natural numbers onto a family $\mathcal{A}$ of computably enumerable sets is called a *computable numbering* of $\mathcal{A}$ if the set $\{\langle x, n\rangle : x \in \alpha(n)\}$ is c.e., and a family $\mathcal{A}$ of subsets of $\mathbb{N}$ is called *computable* if it has a computable numbering. In other words, a computable family $\mathcal{A}$ is a *uniformly c.e. class* of sets, and every computable numbering α of $\mathcal{A}$ defines a uniform c.e. sequence $\alpha(0), \alpha(1), \ldots$ of the members of $\mathcal{A}$. A numbering α is called *reducible* (*$\mathbf{0}'$-reducible*) to a numbering β if $\alpha = \beta \circ f$ for some computable (computable relative to the halting problem) function f. Numberings α, β are called *equivalent* (*$\mathbf{0}'$-equivalent*) if they are reducible ($\mathbf{0}'$-reducible) to each other.

We let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be a standard enumeration of the unary partial recursive functions, and let $W_0, W_1, W_2, \ldots$ be the corresponding standard enumeration of the family $\mathcal{E}$ of all c.e. sets where $W_e = \mathrm{dom}(\varphi_e)$. The *halting problem* is the set $K = \{e : \varphi_e(e) \downarrow\} = \{e : e \in W_e\}$.

The concepts of inductive inference we will deal with are as follows. (For more details see the monographs [13,16].) In our context a language L will be just a c.e. set. A *text for a set L* is a function $t : \mathbb{N} \to \mathbb{N}$ (i.e., an infinite sequence of natural numbers) such that $\mathrm{range}(t) = L$. Below we will identify a text t for L with an ascending sequence of finite sequences of natural numbers σ_n ($n \geq 0$), $\sigma_0 \sqsubset \sigma_1 \sqsubset \sigma_2 \sqsubset \ldots$, such that σ_n is an initial segment $t \restriction n'$ of t, and we let $\mathrm{content}(\sigma_n)$ denote the set of members of the sequence σ_n.

A learner M *identifies in the limit* or *learns in the limit* a language L from a text t for L if

1. $\lim_{n\to\infty} M(t \upharpoonright n)$ exists (we say in this case that M converges on the text t) and
2. $L = W_{\lim_{n\to\infty} M(t\upharpoonright n)}$;

and M TxtEx-*identifies in the limit* or TxtEx-*learns in the limit* L if M identifies in the limit L from every given text t for L. M TxtEx-*identifies* or TxtEx-*learns* a family $\mathcal{A}$ of languages if M TxtEx-identifies every language $L \in \mathcal{A}$. Finally, we say that a family $\mathcal{A}$ is TxtEx-*learnable* if it is TxtEx-identified by some computable learner. TxtEx denotes the class of all TxtEx-learnable families, [8].

A learner M *behaviorally correctly identifies* or *behaviorally correctly learns* a language L from a text t for L if

1. $M(t \upharpoonright n)\downarrow$ for all $n \geq 0$ and
2. $L = W_{M(t\upharpoonright n)}$ for all sufficiently large n;

and M TxtBC-*behaviorally correctly identifies* or TxtBC-*behaviorally correctly learns* L if M behaviorally correctly identifies L from every given text t for L. M TxtBC-*identifies* or TxtBC-*learns* a family $\mathcal{A}$ of languages if M TxtBC-identifies every language $L \in \mathcal{A}$. Finally, we say that a family $\mathcal{A}$ is TxtBC-*learnable* if it is TxtBC-identified by some computable learner.

Below we will use the following well known observation.

Proposition 2.1. *For every computable family $\mathcal{A}$ of c.e. sets, if $\mathcal{A}$ is* TxtEx-*learnable (*TxtBC-*learnable) then it is* TxtEx-*learnable (*TxtBC-*learnable) by some primitive recursive learner.*

3. Computable numberings of computable TxtEX-learnable families

In this section we will show that any two computable numberings of a computable TxtEx-learnable family of languages are $\mathbf{0}'$-equivalent.

Theorem 3.1. *Computable numberings of a computable* TxtEx-*learnable family of languages are pairwise $\mathbf{0}'$-equivalent.*

We first point out how this fact can be deduced from a sequence of some known facts. Then we give a more simple direct proof.

Let us recall the necessary notions and statements.

A sequence σ is called a *locking sequence* for a learner M on a language L [4] if $\mathrm{content}(\sigma) \subseteq L$, $W_{M(\sigma)} = L$, and, for every sequence σ', if $\sigma \sqsubseteq \sigma'$ and $\mathrm{content}(\sigma') \subseteq L$ then $M(\sigma') = M(\sigma)$.

A learner M is called *order-independent* [4] if, for every language L identified in the limit by M and for all texts t and t' for L, $M(t) = M(t')$.

A learner M is called *rearrangement-independent* [7,17] if, for all sequences σ and σ' such that $\mathrm{content}(\sigma) = \mathrm{content}(\sigma')$ and $|\sigma| = |\sigma'|$, $M(\sigma) = M(\sigma')$.

The following two propositions are crucial.

Proposition 3.1 ([7,17])**.** *For every computable learner M, there exists a computable learner M' such that*

- *for every language L, if M identifies in the limit L then M' also identifies in the limit L,*
- *M' is order-independent,*
- *M' is rearrangement-independent.*

Proposition 3.2 ([4])**.** *If M identifies in the limit L then there is a locking sequence for M on L.*

Now, given a computable family $\mathcal{A}$ identified in the limit by a learner M, then one can replace M by a learner M' as in Proposition 3.1 and try to find the least locking sequence (relative to some coding of sequences) for any given language $L \in \mathcal{A}$. By this crucial idea developed in [11,12] to characterize TxtEx-identifiable classes, it is easy to deduce the following statement from [11].

Proposition 3.3. *If M identifies in the limit a computable family $\mathcal{A}$ of c.e. sets then there is a $\mathbf{0}'$-computable function f such that for every $L \in \mathcal{A}$ and every two numbers i, j, if $W_i = L$, $W_j = L$ then $f(i) = f(j)$ and $W_{f(i)} = L$.*

Evidently, Proposition 3.3 implies that any two computable numberings of $\mathcal{A}$ are $\mathbf{0}'$-equivalent.

We now turn to our more simple, direct proof of Theorem 3.1. Our proof will use the following representation of computable numberings by Lachlan (see [3] for the details).

Let $\mathcal{A}$ be a computable family of c.e. sets. We say that a c.e. set A *represents* $\mathcal{A}$ if $\mathcal{A} = \{W_x : x \in A\}$. Now if A represents $\mathcal{A}$ then any computable function f enumerating A induces a computable numbering α_f of $\mathcal{A}$ where $\alpha_f(x) = W_{f(x)}$. Moreover, if f and g are computable functions enumerating A then the corresponding numberings α_f and α_g of $\mathcal{A}$ are equivalent. So, up to equivalence, any c.e. set A representing $\mathcal{A}$ induces a unique computable numbering of $\mathcal{A}$ in the way just described. Conversely, for any computable numbering α of $\mathcal{A}$, there is a c.e. set A representing $\mathcal{A}$ such that the numbering induced by A is equivalent to α. The latter follows from the following well known fact of the theory of numberings: $\alpha : \mathbb{N} \longrightarrow \mathcal{A} \subseteq \mathcal{E}$ is a computable numbering iff α is reducible to the standard numbering $W = \langle W_e : e \geq 0\rangle$, so A can be chosen as the range of a function which reduces α to W.

In the following we will refer to the above observations on representations as Lachlan's representation theorem. The following lemma gives a criterion for the numberings induced by two representations of a computable family to be $\mathbf{0}'$-equivalent.

Lemma 3.1. *Let A and B be c.e. sets such that*

$$\{W_a : a \in A\} = \{W_b : b \in B\} \tag{1}$$

and let α and β be the numberings induced by the sets A and B respectively. Then α is $\mathbf{0}'$-reducible to β if and only if there exists a computable function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for any $a \in A$,

$$\lim_{s\to\infty} g(a, s)\downarrow \ \& \ \lim_{s\to\infty} g(a, s) \in B \ \& \ W_a = W_{\lim_{s\to\infty} g(a,s)}. \tag{2}$$

Proof. The proof is a straightforward modification of the proof of Lemma 2.2 from [3]. □

Based on the above observations on numberings we can now present our proof of Theorem 3.1.

Proof. Let $\mathcal{A}$ be a computable TxtEx-learnable family, let M be a computable learner of $\mathcal{A}$, and let A and B be any c.e. sets such that

$$\mathcal{A} = \{W_a : a \in A\} = \{W_b : b \in B\}.$$

By Lachlan's representation theorem and by Lemma 3.1, it suffices to define a computable function g satisfying (2).

In order to do so, we will show that there is a computable function

$$\mathrm{state} : \mathbb{N}^3 \to \{0, 1\}$$

such that, for any numbers $a \in A$ and $b \in B$ the following hold.

$$W_a = W_b \ \Rightarrow\ \lim_{s\to\infty} \mathrm{state}(a, b, s)\downarrow \ \& \ \lim_{s\to\infty} \mathrm{state}(a, b, s) = 1 \tag{3}$$

$$W_a \neq W_b \ \Rightarrow\ \lim_{s\to\infty} \mathrm{state}(a, b, s)\downarrow \ \& \ \lim_{s\to\infty} \mathrm{state}(a, b, s) = 0. \tag{4}$$

If we define the function g by letting $g(a, s)$ be the least $b \in B_s$ such that $\mathrm{state}(a, b, s) = 1$ (if there is such a b and by letting $g(a, s) = 0$ otherwise) then g will have the desired properties. Namely, given $a \in A$, by (1), we may fix b minimal such that $b \in B$ and $W_a = W_b$. Then, given a stage s_0 such that $b \in B_{s_0}$, by (3) and (4), we may fix $s_1 \geq s_0$ such that, for $s \geq s_1$, $\mathrm{state}(a, b, s) = 1$ and $\mathrm{state}(a, b', s) = 0$ for all $b' < b$. So, for any stage $s \geq s_1$, $g(a, s) = b$.

The function $\mathrm{state}(a, b, s)$ is defined by induction on s. Simultaneously with $\mathrm{state}(a, b, s)$ we define a finite string $\sigma(a, b, s)$ over $\mathbb{N}$ and we let $\mathrm{content}(\sigma(a, b, s))$ be the set of numbers occurring in $\sigma(a, b, s)$.

For $s = 0$ we let $\sigma(a, b, 0) = \lambda$ and $\mathrm{state}(a, b, 0) = 1$. For the definition of $\sigma(a, b, s + 1)$ and $\mathrm{state}(a, b, s + 1)$ we distinguish the following mutually exclusive cases.

Case 1: $\mathrm{content}(\sigma(a, b, s)) \not\subseteq W_{a,s} \cap W_{b,s}$.

Then let $\sigma(a, b, s + 1) = \sigma(a, b, s)$ and $\mathrm{state}(a, b, s + 1) = 0$.

Case 2: $\mathrm{content}(\sigma(a, b, s)) \subseteq W_{a,s} \cap W_{b,s}$ and

$$M(\sigma(a, b, s)) = M(\sigma(a, b, s)W_{a,s}) = M(\sigma(a, b, s)W_{b,s}).$$

(Here $W_{a,s}$ is viewed as the string of the elements of $W_{a,s}$ in the order of enumeration (and, similarly, for $W_{b,s}$).)

Then let $\sigma(a, b, s + 1) = \sigma(a, b, s)$ and $\mathrm{state}(a, b, s + 1) = 1$.

Case 3: $\mathrm{content}(\sigma(a, b, s)) \subseteq W_{a,s} \cap W_{b,s}$ and $M(\sigma(a, b, s)) \neq M(\sigma(a, b, s)W_{a,s})$ or $M(\sigma(a, b, s)) \neq M(\sigma(a, b, s)W_{b,s})$.

Then let $\sigma(a, b, s + 1) = \sigma(a, b, s)W_{a,s}W_{b,s}$ if $M(\sigma(a, b, s)) \neq M(\sigma(a, b, s)W_{a,s})$ and $\sigma(a, b, s + 1) = \sigma(a, b, s)W_{b,s}W_{a,s}$ otherwise. In either case let $\mathrm{state}(a, b, s + 1) = 0$.

This completes the definition of σ and state.

To show that the function state satisfies (3) and (4), fix $a \in A$ and $b \in B$. Note that, by definition,

$$\sigma(a, b, s) \sqsubseteq \sigma(a, b, s + 1), \tag{5}$$

$$\mathrm{content}(\sigma(a, b, s)) \subseteq W_{a,s} \cup W_{b,s}, \tag{6}$$

and

$$\sigma(a, b, s) \sqsubset \sigma(a, b, s + 1) \ \Rightarrow\ \mathrm{content}(\sigma(a, b, s)) \subseteq W_{a,s} \cap W_{b,s}. \tag{7}$$

Next we will show that there are only finitely many stages s such that Case 3 applies in the definition of $\sigma(a, b, s + 1)$ and $\mathrm{state}(a, b, s + 1)$. For a contradiction assume that this happens infinitely often. Since, for $s < s'$ such that Case 3 applies to $s + 1$ and $s' + 1$, $\mathrm{content}(\sigma(a, b, s + 1)) = W_{a,s} \cup W_{b,s}$ and $\mathrm{content}(\sigma(a, b, s + 1)) \subseteq \mathrm{content}(\sigma(a, b, s')) \subseteq W_{a,s'} \cap W_{b,s'}$, it follows that $W_a \cup W_b \subseteq W_a \cap W_b$ hence $W_a = W_b$. So the infinite sequence $\lim_{s\to\infty} \sigma(a, b, s)$ is an enumeration (i.e., a text) of both W_a and W_b. Moreover, if Case 3 holds at stage $s + 1$ then the extension $\sigma(a, b, s + 1)$ of $\sigma(a, b, s)$ is chosen in such a way that $M(\sigma(a, b, s)) \neq M(\sigma)$ for some σ with $\sigma(a, b, s) \sqsubset \sigma \sqsubseteq \sigma(a, b, s + 1)$. So on the sequence $\lim_{s\to\infty} \sigma(a, b, s)$ the learner M changes its mind infinitely often, hence does not learn W_a contrary to assumption.

Now, since Case 3 applies only finitely often and since $\sigma(a, b, s)$ is only extended at a stage $s + 1$ at which Case 3 applies, we may fix s_0 such that $\sigma(a, b, s) = \sigma(a, b, s_0)$ for all $s > s_0$ and Case 3 does not apply after stage s_0. Distinguish the following two cases.

First assume that $\mathrm{content}(\sigma(a, b, s_0)) \nsubseteq W_a \cap W_b$. Then, by (6), $W_a \neq W_b$ and Case 1 applies to all stages $s > s_0$ hence $\lim_{s\to\infty} \mathrm{state}(a, b, s) = 0$. So (4) holds.

Finally, assume now that $\mathrm{content}(\sigma(a, b, s_0)) \subseteq W_a \cap W_b$. Then, for every $s_1 > s_0$ with $\mathrm{content}(\sigma(a, b, s_0)) \subseteq W_{a,s_1} \cap W_{b,s_1}$, Case 2 applies to all stages $s > s_1$ hence $\lim_{s\to\infty} \mathrm{state}(a, b, s) = 1$. It remains to show that $W_a = W_b$ in this case. Now, by definition, $M(\sigma(a, b, s_0)) = M(\sigma(a, b, s_0)W_{a,s})$ and $M(\sigma(a, b, s_0)) = M(\sigma(a, b, s_0)W_{b,s})$, so given the enumeration of W_a obtained by the initial segments $\sigma(a, b, s_0)W_{a,s}$ $(s \geq s_1)$, the learner M does not make any changes in his prediction after stage s_1. Since M learns W_a this implies that $M(\sigma(a, b, s_0))$ is an index of W_a. By a similar argument, $M(\sigma(a, b, s_0))$ is an index of W_b too, hence $W_a = W_b$.

This completes the proof of Theorem 3.1. □

4. A non-TxtEx-learnable class with only equivalent numberings

In this section we give a counterexample to the converse part of the conjecture of Frank Stephan. Indeed, we will prove the following stronger statement.

Theorem 4.1. *There exists a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are pairwise equivalent and $\mathcal{A}$ is not* TxtEx*-learnable.*

Proof. It suffices to construct a computable numbering α such that for the family $\mathcal{A} = \{\alpha(x) : x \in \mathbb{N}\}$ of c.e. sets the following hold.

(a) All computable numberings of $\mathcal{A}$ are equivalent to α.
(b) $\mathcal{A}$ is not TxtEx-learnable.

Indeed, we will build a *positive* numbering α, i.e., a numbering α such that the relation $\alpha(x) = \alpha(y)$ is c.e. in x and y. Any positive numbering is minimal under reduction, [15]. So, in order to ensure (a), it suffices to reduce all computable numberings of $\mathcal{A}$ to the numbering α. Moreover, in order to ensure (b), by Proposition 2.1 it suffices to guarantee that no primitive recursive learner TxtEx-identifies $\mathcal{A}$.

Requirements. Let $M_0, M_1, \dots$ be a computable sequence of all primitive recursive learners, and let $\gamma_0, \gamma_1, \dots$ be a uniformly computable sequence of all computable numberings of computable families of c.e. sets. Then the numbering α has to meet the following requirements for all $k, e \in \mathbb{N}$:

$\mathcal{P}$: α is a computable positive numbering.

$\mathcal{R}_k$: If γ_k is a numbering of $\mathcal{A}$ then γ_k is reducible to α.

$\mathcal{N}_e$: For some m, M_e fails to TxtEx-learn the set $\alpha(m)$.

We will refer to $\mathcal{R}_k$ as the kth *reduction requirement* and to $\mathcal{N}_e$ as the eth *nonlearning requirement*. The priority ordering among the requirements $\mathcal{R}_k$ and $\mathcal{N}_e$ is defined as usual by giving requirements with a smaller index higher priority and by giving $\mathcal{R}_n$ higher priority than $\mathcal{N}_n$. The requirement $\mathcal{P}$ is global.

Strategies. We identify each α-index n with a triple of numbers, $n = \langle e, i, j\rangle$, where the individual components of n have the following meaning:

- e means that the set $\alpha(n)$ might be used for diagonalizing against the learner M_e, i.e., for meeting the nonlearning requirement $\mathcal{N}_e$,
- i denotes the attempt number for trying to diagonalize against M_e (due to our strategy for meeting the higher priority reduction requirements, a single attempt might not suffice),
- j means that $\alpha(n)$ is the jth candidate in the ith attempt for diagonalizing against M_e.

We denote the components e, i, j of a triple $n = \langle e, i, j\rangle$ by $\pi_1(n)$, $\pi_2(n)$, and $\pi_3(n)$ respectively. Moreover, we refer to the sets $\alpha(n)$ with $\pi_1(n) = e$ and $\pi_2(n) = i$ as the sets in *section* (e, i) or sets in the ith e-section. (So sets in the ith e-section (e, i) are reserved for the ith attempt for meeting $\mathcal{N}_e$.)

For meeting the requirements $\mathcal{P}$ and $\mathcal{R}_k$ we have to take some precautions in the enumerations $\alpha^s(n)$ of the sets $\alpha(n)$, where $\alpha^s(n)$ denotes the finite part of $\alpha(n)$ enumerated by the end of stage s of the construction below. (As usual, for any parameter p used in the construction which depends on stage s, we assume that p retains its value at stage $s+1$, i.e., $p^{s+1} = p^s$, unless explicitly stated otherwise.)

Let $b(n) = 2n$ and let $a(n) = 2n + 1$. Initially, we let

$$\alpha^0(n) = \{b(n)\} \tag{8}$$

and call $b(n)$ the *base element* of $\alpha(n)$. So, for $n \neq m$, $\alpha^0(n) \neq \alpha^0(m)$ and $\alpha^0(n)$ can be positively distinguished from $\alpha^0(m)$ by its base element. In fact, sets in different sections will be permanently distinguishable by their base elements, i.e., the base element of a set $\alpha(n)$ will never be put into any set $\alpha(m)$ in a different section:

$$[\pi_1(m) \neq \pi_1(n) \text{ or } \pi_2(m) \neq \pi_2(n)] \Rightarrow b(n) \in \alpha(n) \setminus \alpha(m). \tag{9}$$

Numbers $a(s)$ may be enumerated into some sets $\alpha(m)$ in the course of the construction, where $a(s)$ will not enter any set $\alpha(m)$ before stage s. Moreover, for sets $\alpha(m)$ and $\alpha(n)$ in the *same* section (e, i) the conclusion of (9) may fail since our strategy for meeting the nonlearning requirements may force us to enumerate the base element of $\alpha(n)$ into $\alpha(m)$. (If this happens, the role of the base element of $\alpha(n)$ will be played by some new number $a(s)$ put into $\alpha(n)$ *before* $b(n)$ enters $\alpha(m)$ — unless we make $\alpha(n)$ and $\alpha(m)$ agree.)

After stage 0 numbers will be enumerated into the sets $\alpha(n)$ only by the strategies for meeting the nonlearning requirements. It will be convenient to let these strategies act at even stages only, while the strategies for meeting the reduction requirements will build the required reduction functions at odd stages. So, for $s, n \geq 0$,

$$\alpha^{2s+1}(n) = \alpha^{2s}(n). \tag{10}$$

Strategy for meeting $\mathcal{P}$. In order to make α positive we ensure that two sets $\alpha(m)$ and $\alpha(n)$ which agree at some stage will agree in the limit and, conversely, that sets which agree will agree from some stage on, i.e. for all stages s and numbers m and n,

$$\alpha^s(m) = \alpha^s(n) \Rightarrow \forall\, t \geq s\ (\alpha^t(m) = \alpha^t(n)) \tag{11}$$

and

$$\alpha(m) = \alpha(n) \Rightarrow \exists t\ (\alpha^t(m) = \alpha^t(n)). \tag{12}$$

So, in particular, $\alpha(m) = \alpha(n)$ if and only if $\alpha^s(m) = \alpha^s(n)$ for some stage s. By effectivity of the construction, this implies that $\{(m, n) : \alpha(m) = \alpha(n)\}$ is computably enumerable.

Strategy for meeting $\mathcal{R}_k$. The strategy for meeting the reduction requirements is more involved. Fix uniformly computable enumerations $\{\gamma_k^s(x)\}_{s\geq 0}$ of the sets $\gamma_k(x)$, $k, x \geq 0$, such that $\gamma_k^s(x)$ is finite and $\gamma_k^{2s+1}(x) = \gamma_k^{2s}(x)$. (I.e., just as for the sets $\alpha(n)$ we limit the enumeration of numbers into the sets $\gamma_k(x)$ to even stages.)

We say that k is *correct* if γ_k is a numbering of $\mathcal{A}$, i.e.,

$$\mathcal{A} = \{\gamma_k(x) : x \geq 0\}. \tag{13}$$

So, assuming that k is correct, we have to define a reduction g_k of γ_k to α, i.e., a computable function g_k such that

$$\forall\, x\ (\gamma_k(x) = \alpha(g_k(x))). \tag{14}$$

At stage s of the construction, g_k will be defined on a finite initial segment $0, \ldots, x_k^s - 1$ where $x_k^0 = 0$, and we let $g_k^s(x)$ denote the value of $g_k(x)$ for $0 \leq x < x_k^s$. The domain of g_k will be extended at odd stages only.

Note that, assuming (13), it follows from (8) that any set $\gamma_k(x)$ contains at least one base element. So, before defining $g_k(x)$ we can wait for a stage such that a base element has entered $\gamma_k(x)$. If s is the least stage such that $\gamma_k^s(x)$ contains a base element and $b(\langle e, i, j\rangle)$ is the least base element in $\gamma_k^s(x)$ then we call (e, i) the *target section* of $g_k(x)$ and denote it by $(e_{k,x}, i_{k,x})$. Note that, still assuming (13), it follows from (9) that $\gamma_k(x)$ is a member of section $(e_{k,x}, i_{k,x})$ hence we will have to set $g_k(x) = \langle e_{k,x}, i_{k,x}, j'\rangle$ for some number $j' \geq 0$.

In fact, if the target section $(e_{k,x}, i_{k,x})$ of $g_k(x)$ is connected to a higher priority nonlearning requirement, i.e., $e_{k,x} < k$, then the value of $g_k(x)$ will be specified only at the end of the construction (using some finite information on the outcomes of the strategies for meeting the higher priority nonlearning requirements $\mathcal{N}_e$, $e < k$). In the course of the construction we will only assign the temporary value $g_k(x) = \perp$.

For defining $g_k(x)$ if the target satisfies $e_{k,x} \geq k$, we have to introduce some more features of the construction first. So it will be crucial to note that at any stage of the construction the sets in a section (e, i) will be in one of the following states.

- Section (e, i) is *unused* at stage s. If unused at s, all sets in section (e, i) are still in their initial state, i.e. $\alpha^s(\langle e, i, j\rangle) = \alpha^0(\langle e, i, j\rangle) = \{b(\langle e, i, j\rangle)\}$ for $j \geq 0$. All sections are unused at stage 0.
- Section (e, i) is *active* at stage s. Once active, section (e, i) will stay active forever unless it will be eventually cancelled.
- Section (e, i) is *cancelled* at stage s. If (e, i) becomes cancelled at stage s first then s is even and (e, i) remains cancelled for ever. Moreover, for $n \geq 1$ and $j \leq n$,

$$\alpha^{s+2n}(\langle e, i, j\rangle) = \bigcup_{j' \leq n} \alpha^s(\langle e, i, j'\rangle) \cup \{a(0), \ldots, a(s+n)\}. \tag{15}$$

So, in the limit, all sets in section (e, i) are merged and trivialized, namely,

$$\alpha(\langle e, i, j\rangle) = \{b(n) : \pi_1(n) = e \;\&\; \pi_2(n) = i\} \cup \{a(n) : n \geq 0\} \tag{16}$$

for all $j \geq 0$. (Note that this procedure is consistent with (10)–(12).)

Now, if $g_k(x)$ has target $(e_{k,x}, i_{k,x})$ with $e_{k,x} \geq k$ then we ensure that the value $\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$ assigned to $g_k(x)$ is correct as follows. If we let $g_k(x) = \langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$ at stage 2s+1 then we choose $j_{k,x}$ such that $\gamma_k^{2s}(x) = \alpha^{2s}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)$. In order to preserve this agreement we limit the action of enumerating the base element $b(\langle e, i, j\rangle)$ of a set $\alpha(\langle e, i, j\rangle)$ in a section (e, i) into another set $\alpha(\langle e, i, j'\rangle)$ of this section as follows. Unless we do not merge the sets $\alpha(\langle e, i, j\rangle)$ and $\alpha(\langle e, i, j'\rangle)$, i.e., set $\alpha(\langle e, i, j\rangle) = \alpha(\langle e, i, j'\rangle)$, before we put $b(\langle e, i, j\rangle)$ into $\alpha(\langle e, i, j'\rangle)$ we put a new number $a(t)$ (taking over the role of the base element) into $\alpha(\langle e, i, j\rangle)$ which will never enter $\alpha(\langle e, i, j'\rangle)$ (unless $\alpha(\langle e, i, j\rangle)$ and $\alpha(\langle e, i, j'\rangle)$ are merged later). In fact we may iterate this game, i.e. we may put the element $a(t)$ – which has taken over the role of the base element of $\alpha(\langle e, i, j\rangle)$ – into $\alpha(\langle e, i, j'\rangle)$ or some other set $\alpha(\langle e, i, j''\rangle)$ later but we do not do this unless we have added a new distinguishing element $a(t')$ to $\alpha(\langle e, i, j'\rangle)$ before. (This will not happen here but in the proof of Theorem 6.1.) What will be crucial is that these procedures guarantee

$$\alpha(\langle e, i, j\rangle) \neq \alpha(\langle e, i, j'\rangle) \Rightarrow \forall s \exists y\, [y \in \alpha^{2s}(\langle e, i, j\rangle) \setminus \alpha^{2s+2}(\langle e, i, j'\rangle)]. \tag{17}$$

Moreover, if $\langle e, i, j\rangle$ has been assigned to some $g_k(x)$ prior to the stage $2s+2$ at which we want to put $b(\langle e, i, j\rangle)$ (or some other element of $\alpha(\langle e, i, j\rangle)$) into the set $\alpha(\langle e, i, j'\rangle)$ then this action is delayed until the new number y distinguishing $\alpha(\langle e, i, j\rangle)$ from the other sets in the section has shown up in $\gamma_k(x)$ too. (If k is correct this delay will be finite.)

Finally, we have to (further) slow down the enumeration of the sets in a given section (e, i) in order to ensure that, for correct k and for any x such that $g_k(x)$ has target (e, i), $g_k(x)$ can be eventually assigned a currently correct value.

We now formally define the strategy for meeting the reduction requirement $\mathcal{R}_k$ for given $k \geq 0$.

The restraint on enumerating numbers into the sets $\alpha(n)$ imposed by $\mathcal{R}_k$ is as follows.

For $x < x_k^{2s+1}$ call x *k-incorrect (on section $(e_{k,x}, i_{k,x})$)* at stage $2s+1$ if $k \leq e_{k,x}$, $(e_{k,x}, i_{k,x})$ is not cancelled at stage $2s+1$, and

$$\gamma_k^{2s+1}(x) \neq \alpha^{2s+1}(g_k(x)). \tag{18}$$

Call $x = x_k^{2s+1}$ *k-incorrect (on section $(e_{k,x}, i_{k,x})$)* at stage $2s + 1$ if the target $(e_{k,x}, i_{k,x})$ of $g_k(x)$ is defined at stage $2s + 1$, $k \leq e_{k,x}$, $(e_{k,x}, i_{k,x})$ is not cancelled at stage $2s + 1$,

$$\forall j \geq 0\, [\gamma_k^{2s+1}(x) \neq \alpha^{2s+1}(\langle e, i, j\rangle)], \tag{19}$$

and there is no pair (k', x') such that $\langle k', x'\rangle < \langle k, x\rangle$, $g_{k'}(x')$ has target $(e_{k,x}, i_{k,x})$ and $g_{k'}(x')$ becomes defined at stage $2s + 1$.

Call a number x *k-correct* at stage $2s + 1$ if x is not k-incorrect at stage $2s + 1$.

If there is a number x such that x is k-incorrect on (e, i) at stage $2s + 1$ then section (e, i) is *k-frozen* (via x) at stage $2s + 2$. Section (e, i) is *frozen* at stage $2s+2$ if (e, i) is k-frozen for some $k \leq e$. If (e, i) is frozen at stage $2s+2$ then no new elements are allowed to enter any of the sets in this section at stage $2s + 2$. (This is consistent with (15) since cancelled sections are not frozen.) So, by (10),

$$(e, i) \text{ frozen at stage } 2s + 2 \Rightarrow \forall j \geq 0\, [\alpha^{2s}(\langle e, i, j\rangle) = \alpha^{2s+2}(\langle e, i, j\rangle)]. \tag{20}$$

Now the domain of g_k is expanded at odd stages as follows. Given $s \geq k$, $g_k(x_k^{2s})$ becomes defined at stage $2s + 1$ (and $x_k^{2s+1} = x_k^{2s} + 1$) if the following hold.

(D_1) The target section $(e_{k,x_k^{2s}}, i_{k,x_k^{2s}})$ of $g_k(x_k^{2s})$ is defined at stage $2s$.

(D_2) If $e_{k,x_k^{2s}} \geq k$ and section $(e_{k,x_k^{2s}}, i_{k,x_k^{2s}})$ is not cancelled at stage $2s$ then there is a number j such that $\gamma_k^{2s}(x_k^{2s}) = \alpha^{2s}(\langle e_{k,x_k^{2s}}, i_{k,x_k^{2s}}, j\rangle)$.

(D_3) For all $x < x_k^{2s}$ there is a stage t such that $u_s \leq t < s$ and such that x is k-correct at stage $2t + 1$ where u_s is the least stage u such that $x_k^{2u} = x_k^{2s}$.

The value of $g_k(x_k^{2s})$ is determined by distinguishing the following cases. If $e_{k,x_k^{2s}} < k$, let $g_k^{2s+1}(x_k^{2s}) = \perp$. If $e_{k,x_k^{2s}} \geq k$ and section $(e_{k,x_k^{2s}}, i_{k,x_k^{2s}})$ is cancelled at stage $2s$, let $g_k^{2s+1}(x_k^{2s}) = \langle e_{k,x_k^{2s}}, i_{k,x_k^{2s}}, 0\rangle$. Finally, if $e_{k,x_k^{2s}} \geq k$ and section $(e_{k,x_k^{2s}}, i_{k,x_k^{2s}})$ is not cancelled at stage $2s$, let $g_k^{2s+1}(x_k^{2s}) = \langle e_{k,x_k^{2s}}, i_{k,x_k^{2s}}, j_{k,x_k^{2s}}\rangle$ where $j_{k,x_k^{2s}}$ is the least j such that $\gamma_k^{2s}(x_k^{2s}) = \alpha^{2s}(\langle e_{k,x_k^{2s}}, i_{k,x_k^{2s}}, j\rangle)$ holds.

To show that this guarantees that, for correct k, the reduction g_k is total and correct (whenever $g_k(x) \neq \perp$), we prove a series of claims based on the assumption that the enumeration of the sets $\alpha(n)$ satisfies (10), (11), (15) (for cancelled sections), (17) and (20). Fix k.

Claim 1. *Assume that $g_k(x)$ is defined (at stage $2s + 1$). Then the target section $(e_{k,x}, i_{k,x})$ of $g_k(x)$ is defined (at stage $2s$) and either $e_{k,x} < k$ and $g_k(x) = \perp$ or $e_{k,x} \geq k$ and $g_k(x) = \langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$ for some number $j_{k,x} \geq 0$. (Moreover, if $e_{k,x} \geq k$, section $(e_{k,x}, i_{k,x})$ is not cancelled at stage $2s$, and $g_k(x)$ becomes defined at stage $2s + 1$ then $\gamma_k^{2s}(x) = \alpha^{2s}(g_k(x))$.)*

Proof. Straightforward. $\square$

Claim 2. *Assume that k is correct. Then, for any x, the target section (e_x, i_x) of $g_k(x)$ is defined and $\gamma_k(x) = \alpha(\langle e_x, i_x, j\rangle)$ for some $j \geq 0$.*

Proof. Straightforward. $\square$

Claim 3. *Assume that k is correct. Then, for any x, the following hold.*

(a) $g_k(x)$ *is defined.*
(b) *If* $g_k(x) = \langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$ *then* $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)$.
(c) *There are infinitely many stages s such that x is k-correct at stage* $2s + 1$.

Proof. The proof is by induction. Fix x and assume that the claim holds for all $x' < x$. Let s_x be the least stage s such that $x_k^{2s} = x$. By Claim 2, fix $s' \geq max(s_x, k)$ such that the target section $(e_{k,x}, i_{k,x})$ of $g_k(x)$ is defined at stage $2s'$ and fix j_0 such that $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle)$.

Now, for a proof of (a), for a contradiction assume that $g_k(x)$ is not defined. Then $x_k^{2s} = x$ for all $s \geq s_x$. So condition (D_1) for defining $g_k(x_k^{2s})$ is satisfied for all $s \geq s'$. Moreover, by inductive hypothesis, there is a stage $s'' \geq s'$ such that (D_3) holds for all $s \geq s''$. So, since $g_k(x)$ is not defined, condition (D_2) fails for all $s \geq s''$. Hence $e_{k,x} \geq k$, section $(e_{k,x}, i_{k,x})$ is never cancelled, and

$$\forall\, s \geq s''\ \forall\, j\ [\gamma_k^{2s}(x) \neq \alpha^{2s}(\langle e_{k,x}, i_{k,x}, j\rangle)]. \tag{21}$$

So if we let s''' be the least number $s \geq s''$ such that, for any pair (k', x') such that $\langle k', x'\rangle < \langle k, x\rangle$ and $g_{k'}(x')$ is defined, $g_{k'}(x')$ is defined by stage $2s - 1$, then, for $s \geq s'''$, x is k-incorrect at stage $2s + 1$ hence section $(e_{k,x}, i_{k,x})$ is frozen at stage $2s + 2$. It follows, by (20), that $\alpha(\langle e_{k,x}, i_{k,x}, j\rangle) = \alpha^{2s'''}(\langle e_{k,x}, i_{k,x}, j\rangle)$ for all $j \geq 0$ hence, in particular, all sets in section $(e_{k,x}, i_{k,x})$ are finite. By $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle)$ this implies

$$\gamma_k^s(x) = \alpha^s(\langle e_{k,x}, i_{k,x}, j_0\rangle)$$

for all sufficiently large s. But this contradicts (21).

For a proof of parts (b) and (c), fix s'_x such that $g_k(x)$ becomes defined at stage $2s'_x + 1$ and distinguish the following three cases.

First assume that $g_k(x) = \perp$. Then (b) trivially holds and $e_{k,x} < k$ hence x is k-correct at all stages $2s + 1$ for $s > s'_x$.

Next assume that $g_k(x) = \langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$ and section $(e_{k,x}, i_{k,x})$ is eventually cancelled. By the latter and (16), all sets in section $(e_{k,x}, i_{k,x})$ are merged. So

$$\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle) = \alpha(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)$$

hence (b) holds. Moreover, once $(e_{k,x}, i_{k,x})$ is cancelled, x will become k-correct forever.

Finally, for the remainder of the proof, assume that $g_k(x) = \langle e_{k,x}, i_{k,x}, j_{k,x}\rangle$, $e_{,x} \geq k$, and section $(e_{k,x}, i_{k,x})$ is never cancelled. For a proof of (b) and (c) it suffices to show

$$\exists^\infty\, s\ [\gamma_k^{2s}(x) = \alpha^{2s}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)]. \tag{22}$$

For a contradiction assume that (22) fails. Since, by definition of $g_k(x)$ and choice of s'_x,

$$\gamma_k^{2s'_x}(x) = \alpha^{2s'_x}(g_k(x)) = \alpha^{2s'_x}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle),$$

we may fix $s'' \geq s'_x$ such that

$$\gamma_k^{2s''}(x) = \alpha^{2s''}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)\ \&\ \forall\, s > s''\ [\gamma_k^{2s}(x) \neq \alpha^{2s}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)]. \tag{23}$$

It follows that section (e, i) is frozen at all stages $\geq 2s'' + 4$ hence, in particular,

$$\alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle) = \alpha^{2s''+2}(\langle e_{k,x}, i_{k,x}, j_0\rangle). \tag{24}$$

By $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle)$ this implies that $\gamma_k^s(x) = \alpha^s(\langle e_{k,x}, i_{k,x}, j_0\rangle)$ for all sufficiently large s hence, by (23),

$$\alpha(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle) \neq \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle).$$

So, by (17), there is a number y such that

$$y \in \alpha^{2s''}(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle) \setminus \alpha^{2s''+2}(\langle e_{k,x}, i_{k,x}, j_0\rangle).$$

By (23) and (24), however, this implies that $y \in \gamma_k(x) \setminus \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle)$ contrary to $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_0\rangle)$.

This completes the proof of Claim 3. □

By the first two parts of Claim 3, for correct k the reduction function g is total and, for any x such that $g_k(x) \neq \perp$, the reduction is correct, i.e., $\gamma_k(x) = \alpha(g_k(x))$.

To show that the severe limitations on the enumeration of the sets $\alpha(n)$ caused by the $\mathcal{R}_k$ strategies will not interfere with the task of meeting the nonlearning requirements, the following observations will be crucial.

Claim 4. *Assume that the domain of* g_k *is infinite. Then, for any x,* $g_k(x)$ *is defined and there are infinitely many stages* $2s + 1$ *at which x is k-correct.*

Proof. The first part is straightforward. The second part follows from clause (D_3) in the definition of $g_k(x_k^{2s})$. □

We say that a section is *permanently frozen* if it is frozen at all sufficiently large even stages.

Claim 5. *For any number e there are only finitely many e-sections which are permanently frozen.*

Proof. Fix e. For k such that the domain of g_k is finite, let $x_k = \lim_{s\to\infty} x_k^s = \sup_{s\to\infty} x_k^s$, and fix i_0 such that, for any $k \leq e$ such that the domain of g_k is finite and for any $x \leq x_k$ such that $g_k(x)$ has target (e, i) for some i, $i < i_0$. We will show that, for $i \geq i_0$, section (e, i) is not permanently frozen.

For a contradiction assume that $i \geq i_0$ and (e, i) is permanently frozen, say (e, i) is frozen at all stages $2s$ with $s \geq s_0$. Then, by definition, section (e, i) is never cancelled and

$$\forall\, s \geq s_0\ \exists\, k \leq e\ \exists\, x\ [x\ k\text{-incorrect on } (e, i) \text{ at stage } 2s + 1]. \tag{25}$$

Moreover, by (20),

$$\forall\, s \geq s_0\ \forall\, j \geq 0\ [\alpha(\langle e, i, j\rangle) = \alpha^{2s}(\langle e, i, j\rangle) = \alpha^{2s_0}(\langle e, i, j\rangle)]. \tag{26}$$

In order to get the desired contradiction we will refute (25).

We start with some simple observations. Let I be the set of all (coded) pairs $\langle k, x\rangle$ such that x is k-incorrect on (e, i) at some odd stage. Then, for $\langle k, x\rangle \in I$, $k \leq e$ and $g_k(x)$ has target (e, i). By $i \geq i_0$, this implies that the domain of g_k is infinite. Hence, by Claim 4, g_k is total and x is k-correct at infinitely many odd stages.

By the latter, and since there are only finitely many pairs $\langle k, x\rangle$ in I such that $g_k(x)$ is defined by the end of stage $2s_0+1$, we may fix a stage $s_1 \geq s_0$ such that, for any such pair $\langle k, x\rangle$, there is a stage t, $s_0 \leq t \leq s_1$, such that x is k-correct at stage $2t + 1$.

Next fix $s \geq s_1$ and $\langle k, x\rangle$ in I such that $g_k(x)$ is defined by stage $2s + 1$. We will show that x is k-correct at stage $2s + 1$. Fix u such that $g_k(x)$ becomes defined at stage $2u + 1$. We first observe that there is a stage t, $max(s_0, u) \leq t \leq s$, such that x is k-correct at stage $2s + 1$. For $u \leq s_0$ this is immediate by choice of s_1 and by $s \geq s_1$. If $u > s_0$ then $t = u$ will do since, by clause (D_2) in the definition of g_k, x is k-correct at the stage at which $g_k(x)$ becomes defined. So, in order to show that x is k-correct at stage $2s + 1$, it suffices to show that, assuming that x is k-correct at stage $2t + 1$ where $t \geq s_0$ and $g_k^{2t+1}(x)$ is defined, x is k-correct at *all* stages $2s + 1 > 2t + 1$. Now, by choice of t, $g_k(x) = \langle e, i, j\rangle$ for some $j \geq 0$ and $\gamma_k^{2t+1}(x) = \alpha^{2t+1}(\langle e, i, j\rangle)$. It suffices to show that the latter equation holds for all $2s+1 > 2t+1$ in place of $2t+1$. Now, by $t \geq s_0$ and (26), $\alpha(\langle e, i, j\rangle)$ has reached its final value at stage $2t + 1$. So the equality will only be destroyed if a new number enters $\gamma_k(x)$ after stage $2t + 1$. In this case, however, x will be k-incorrect at *all* sufficiently large odd stages. But, as observed above, for $\langle k, x\rangle \in I$ this is impossible.

Now, by our last observation and by (25), for any stage $s \geq s_1$, there is a pair $\langle k, x\rangle$ in I such that x is k-incorrect on (e, i) at stage $2s+1$ and $g_k(x)$ is not yet defined at stage $2s+1$. Let $\langle k', x'\rangle$ be the least coded pair which has this property for some $s \geq s_1$. By $\langle k', x'\rangle \in I$, $g_{k'}$ is total hence there is a stage $v > s \geq s_1$ such that $g_{k'}(x')$ becomes defined at stage $2v+1$. It follows, by definition and by minimality of $\langle k', x'\rangle$, that no pair $\langle k, x\rangle$ such that $g_k(x)$ is undefined at stage $2v + 1$ is k-incorrect on (e, i) at stage $2v + 1$, a contradiction.

This completes the proof of Claim 5. □

Strategy for meeting $\mathcal{N}_e$. Given the ith e-section (e, i), in the ith attempt for meeting $\mathcal{N}_e$ we build a (finite or infinite) sequence of strings over $\mathbb{N}$, namely

$$\sigma_{e,i}^0 \sqsubset \sigma_{e,i}^1 \sqsubset \cdots \sqsubset \sigma_{e,i}^{j^*}$$

($j^* \geq 0$) or

$$\sigma_{e,i}^0 \sqsubset \sigma_{e,i}^1 \sqsubset \sigma_{e,i}^2 \sqsubset \cdots$$

such that – if the ith attempt is the successful one – either

$$\text{content}(\sigma_{e,i}^{j^*}) \subseteq \alpha(\langle e, i, 0\rangle) \cap \alpha(\langle e, i, j^* + 1\rangle)\ \&\ M_e \text{ fails to learn } \alpha(\langle e, i, 0\rangle) \text{ or } \alpha(\langle e, i, j^* + 1\rangle) \tag{27}$$

or

$$\bigcup_{j\in\mathbb{N}} \text{content}(\sigma_{e,i}^j) = \alpha(\langle e, i, 0\rangle)\ \&\ M_e \text{ fails to learn } \alpha(\langle e, i, 0\rangle) \text{ from the text } \sigma_{e,i} = \lim_{j\to\infty} \sigma_{e,i}^j. \tag{28}$$

This is achieved by induction on $j \geq 1$, where string $\sigma_{e,i}^j$ is defined in *cycle j* given below ($\sigma_{e,i}^0$ is the empty string).

When cycle 1 is started at stage $2s_1$ then $s_1 > 0$, section (e, i) was unused at stage $2s_1 - 1$, and section (e, i) becomes *active* at stage $2s_1$. Cycle j will affect $\alpha(\langle e, i, 0\rangle)$, called the *primary set* of section (e, i), and $\alpha(\langle e, i, j\rangle)$, called the *active set* (in cycle j). If cycle j is started at stage $2s_j$, the active set and the sets $\alpha(\langle e, i, j'\rangle)$ with $j' > j$, called *unused* sets, are still in their initial states. If cycle j is completed, the active set is *cancelled* and merged with the primary set. (So the cancelled sets agree with the primary set.) Moreover, at the start of cycle j, there will be a unique number, denoted by $\hat{b}_{j-1}(\langle e, i, 0\rangle)$ and called the *critical element* of the primary set, such that the primary set is the disjoint union

$$\alpha^{2s_j}(\langle e, i, 0\rangle) = \text{content}(\sigma_{e,i}^{j-1}) \,\dot\cup\, \{\hat{b}_{j-1}(\langle e, i, 0\rangle)\}. \tag{29}$$

For $j = 0$, $\hat{b}_0(\langle e, i, 0\rangle) = b(\langle e, i, 0\rangle)$ while, for $j \geq 1$, $\hat{b}_j(\langle e, i, 0\rangle) = a(j)$ where $a(j)$ is enumerated into the primary set at the end of cycle j. The critical element will positively distinguish the primary set from all other sets in section (e, i) which have not been merged with the primary set.

Now cycle j (started at stage $2s_j$) is as follows.

Cycle j ($j \geq 1$).

1. Wait for the least $s'_j > s_j$ such that section (e, i) is not frozen at stage $2s'_j + 2$.
2. At stage $2s'_j + 2$ enumerate the elements of the primary set with exception of the critical element into the active set:

$$\alpha^{2s'_j+2}(\langle e, i, j\rangle) = \alpha^{2s'_j}(\langle e, i, j\rangle) \cup [\alpha^{2s'_j}(\langle e, i, 0\rangle) \setminus \{\hat{b}_{j-1}(\langle e, i, 0\rangle)\}]. \tag{30}$$

1. Wait for the least stage $s''_j > 2s'_j + 2$ such that

$$M_e(\sigma^{j-1}_{e,i}\hat{b}_{j-1}(\langle e, i, 0\rangle)^{s''_j-(2s'_j+2)}) \neq M_e(\sigma^{j-1}_{e,i}) \tag{31}$$

or

$$M_e(\sigma^{j-1}_{e,i}b(\langle e, i, j\rangle)^{s''_j-(2s'_j+2)}) \neq M_e(\sigma^{j-1}_{e,i}). \tag{32}$$

3. Wait for the least $s'''_j > s''_j$ such that section (e, i) is not frozen at stage $2s'''_j + 2$.
4. At stage $2s'''_j + 2$ complete the cycle.
   Merge the active set with the primary set and add a new critical number to the primary set:

$$\forall j' \leq j\ (\alpha^{2s'''_j+2}(\langle e, i, j'\rangle) = \alpha^{2s'''_j}(\langle e, i, 0\rangle) \cup \alpha^{2s'''_j}(\langle e, i, j\rangle) \cup \{\hat{b}_j(\langle e, i, 0\rangle)\} \tag{33}$$

   where $\hat{b}_j(\langle e, i, 0\rangle) = a(j)$. (Note that once a set is merged with the primary set it will agree with the primary set at all later stages hence in the limit; i.e., if a number is enumerated in the primary set then it is tacitly simultaneously enumerated in all sets previously merged with the primary set.)
   Define $\sigma^j_{e,i}$ as follows.

$$\sigma^j_{e,i} = \begin{cases} \sigma^{j-1}_{e,i}\ \hat{b}_{j-1}(\langle e, i, 0\rangle)^{s''_j-(2s'_j+2)}\ b(\langle e, i, j\rangle) & \text{if (31) holds} \\ \sigma^{j-1}_{e,i}\ b(\langle e, i, j\rangle)^{s''_j-(2s'_j+2)}\ \hat{b}_{j-1}(\langle e, i, 0\rangle) & \text{otherwise.} \end{cases}$$

   Declare the active set $\alpha(\langle e, i, j\rangle)$ *cancelled* and start cycle $j + 1$ with the new active set $\alpha(\langle e, i, j + 1\rangle)$ at the next even stage, i.e., let $2s_{j+1} = 2s'''_j + 4$.

Note that it may happen that cycle j cannot be completed since we wait in step 1 or step 3 or step 4 forever. In this case the sequence $(\sigma^n_{e,i})$ is finite and $j^* = j - 1$.

Numbers are enumerated into sets of section (e, i) after stage 0 only in steps 2 and 5 of the above cycles (unless section (e, i) is cancelled and trivialized according to rule (15)). So, by a straightforward induction on j, the following hold.

(S_1) If the attack is in cycle j at stage s (i.e., $2s_j \leq s \leq 2s'''_j+2$) then the still unused sets $\alpha^s(\langle e, i, j'\rangle), j' > j$ are still in their initial states, and their base elements have not yet entered any other set, i.e, for $j' > j$ and $j'' \neq j'$, $b(\langle e, i, j'\rangle) \notin \alpha^s(\langle e, i, j''\rangle)$.

(S_2) If the attack is in cycle j at stage s and cycle j is not yet completed (i.e., $2s_j \leq s < 2s'''_j + 2$) then the base element of the active set $\alpha^s(\langle e, i, j\rangle)$ has not yet entered any other set, i.e, $b(\langle e, i, j\rangle) \notin \alpha^s(\langle e, i, j'\rangle)$ for $j' \neq j$. Moreover, $\alpha^s(\langle e, i, j\rangle)$ is in its initial state (if $s < 2s'_j + 2$) or

$$\alpha^s(\langle e, i, j\rangle) \setminus b(\langle e, i, j\rangle) = \alpha^s(\langle e, i, 0\rangle) \setminus \hat{b}_{j-1}(\langle e, i, 0\rangle) = \text{content}(\sigma^{j-1}_{e,i})$$

(if $s \geq 2s'_j + 2$).

(S_3) If the attack is in cycle j at stage s and cycle j is not yet completed (i.e., $2s_j \leq s < 2s'''_j + 2$) then the primary set agrees with all previously cancelled sets:

$$\forall j' < j\ \forall t \geq s\ (\alpha^t(\langle e, i, 0\rangle) = \alpha^t(\langle e, i, j'\rangle)).$$

Moreover, the critical element of the primary set has not entered the active set:

$$\hat{b}_{j-1}(\langle e, i, 0\rangle) \in \alpha^s(\langle e, i, 0\rangle) \setminus \alpha^s(\langle e, i, j\rangle).$$

(S_4) If cycle j is completed at stage s (i.e., $s = 2s'''_j + 2$) then the active set is cancelled and merged with the primary set, i.e., $\alpha^t(\langle e, i, j\rangle) = \alpha^t(\langle e, i, 0\rangle)$ for all $t \geq s$. Moreover, $\hat{b}_j(\langle e, i, 0\rangle) = a(j)$ and $\alpha^s(\langle e, i, 0\rangle) = \alpha^{2s_j}(\langle e, i, 0\rangle) \dot\cup \{\hat{b}_j(\langle e, i, 0\rangle)\}$.

This easily implies that the enumeration of the sets $\alpha(n)$ in section (e, i) obeys the rules (10)–(12), (17) and (20): if section (e, i) is unused, this is trivial. Moreover, once cancelled, (9)–(12) hold as observed above; (17) and (20) become trivial, since in a cancelled section all sets are made to agree and since a cancelled section cannot become frozen. So we may assume that (e, i) is active.

Then (10) and (20) hold since steps 2 and 5 in a cycle are limited to even stages at which section (e, i) is not frozen.

For a proof of (11) note that, by the above observations, for $j' < j''$, the sets $\alpha^s(\langle e, i, j'\rangle)$ and $\alpha^s(\langle e, i, j''\rangle)$ agree at stage s if and only if both sets have been cancelled by stage s or the first set is the primary set and the second set has been cancelled, and in either case the sets will agree forever. Similarly, for a proof of (12) note that, for $j' < j''$, the sets $\alpha(\langle e, i, j'\rangle)$ and $\alpha(\langle e, i, j''\rangle)$ agree in the limit if and only if both sets are eventually cancelled or the first set is the primary set and the second set is eventually cancelled.

For a proof of (17) fix $j' < j''$ and s such that $\alpha(\langle e,i,j'\rangle) \neq \alpha(\langle e,i,j''\rangle)$. We have to show that there are numbers y' and y'' such that

$$y' \in \alpha^{2s}(\langle e,i,j'\rangle) \setminus \alpha^{2s+2}(\langle e,i,j''\rangle) \;\&\; y'' \in \alpha^{2s}(\langle e,i,j''\rangle) \setminus \alpha^{2s+2}(\langle e,i,j'\rangle). \tag{34}$$

Since all cancelled sets are merged with the primary set, $\alpha(\langle e,i,j'\rangle) \neq \alpha(\langle e,i,j''\rangle)$ implies that j'' is never cancelled. So the attack gets stuck in some cycle j and either $j < j''$ or $j' < j = j''$. In the former case, by (S_1), $\alpha(\langle e,i,j''\rangle)$ will be in its initial state forever and its base element will not enter any other set in section (e,i). So (34) will hold for $y' = b(\langle e,i,j'\rangle)$ and $y'' = b(\langle e,i,j''\rangle)$. In the latter case, it follows from (S_2) that the base element of the active set $\alpha(\langle e,i,j\rangle)$ will never enter any other set in section (e,i) hence, by $j = j''$, $y'' = b(\langle e,i,j\rangle)$ will do. Similarly, by (S_3), the critical element $\hat{b}_{j-1}(\langle e,i,0\rangle)$ of the primary set will be an element of a set in section (e,i) if and only if $j' < j$. So $y' = \hat{b}_{j-1}(\langle e,i,0\rangle)$ will satisfy (34).

The above observations show that the $\mathcal{N}_e$ strategy is compatible with the $\mathcal{P}$ strategy and $\mathcal{R}_k$ strategies. So, in particular, requirement $\mathcal{P}$ is met and Claims 1–5 above are true.

To explain why the $\mathcal{N}_e$ strategy succeeds in meeting requirement $\mathcal{N}_e$ we first show that – assuming that no cycle gets stuck in step 1 or step 4 and that section (e,i) is never cancelled – the sequence $(\sigma^n_{e,i})$ built by the above described attack has property (27) (if finite) or (28) (if infinite), hence witnesses that M_e fails to TxtEx-learn $\mathcal{A}$.

First assume that the attack gets stuck in a cycle j. Then the sequence $(\sigma^n_{e,i})$ is finite and $j^* = j-1$. Moreover, by assumption, the attack gets stuck in step 3 of cycle j, i.e., we wait forever for a stage $s''_j \geq 2s'_j+2$ such that (31) or (32) holds. So

$$\forall\, n \geq 1\; [M_e(\sigma^{j-1}_{e,i}\hat{b}_{j-1}(\langle e,i,0\rangle)^n) = M_e(\sigma^{j-1}_{e,i} b(\langle e,i,j\rangle)^n) = M_e(\sigma^{j-1}_{e,i})]. \tag{35}$$

and, by (S_2) and (S_3),

$$\alpha(\langle e,i,0\rangle) = \alpha^{2s'_j+2}(\langle e,i,0\rangle) = \mathrm{content}(\sigma^{j-1}_{e,i}) \,\dot\cup\, \{\hat{b}_{j-1}(\langle e,i,0\rangle)\}$$

and

$$\alpha(\langle e,i,j\rangle) = \alpha^{2s'_j+2}(\langle e,i,j\rangle) = \mathrm{content}(\sigma^{j-1}_{e,i}) \,\dot\cup\, \{b(\langle e,i,j\rangle)\}$$

where $\hat{b}_{j-1}(\langle e,i,0\rangle) \neq b(\langle e,i,j\rangle)$.

So $\alpha(\langle e,i,0\rangle) \neq \alpha(\langle e,i,j\rangle)$, $\sigma^{j-1}_{e,i}\hat{b}_{j-1}(\langle e,i,0\rangle)^\omega$ is a text for $\alpha(\langle e,i,0\rangle)$, and $\sigma^{j-1}_{e,i} b(\langle e,i,j\rangle)^\omega$ is a text for $\alpha(\langle e,i,j\rangle)$. Finally, by (35), the learner M_e learns from both texts the index $M_e(\sigma^{j^*}_{e,i})$. So M_e fails to TxtEx-learn $\alpha(\langle e,i,0\rangle)$ or $\alpha(\langle e,i,j\rangle)$. It follows with $j^* = j-1$ that (27) holds.

Now assume that the attack gets never stuck, i.e., that all cycles are completed. Then the sequence $(\sigma^n_{e,i})$ is infinite. Moreover, given j, by completion of step 3 of cycle j, (31) or (32) holds, and in step 5 of cycle j the extension $\sigma^j_{e,i}$ of $\sigma^{j-1}_{e,i}$ is chosen so such that

$$M_e(\sigma^j_{e,i}) \neq M_e(\sigma^{j-1}_{e,i}).$$

Since, by (S_2) and (S_4),

$$\alpha^{2s_j}(\langle e,i,0\rangle) \subseteq \mathrm{content}(\sigma^j_{e,i}) \subseteq \alpha^{2s_{j+1}}(\langle e,i,0\rangle),$$

it follows that $\sigma_{e,i} = \lim_{j\to\infty} \sigma^j_{e,i}$ is a text for $\alpha(\langle e,i,0\rangle)$ and the learner M_e changes its mind on the text $\sigma_{e,i}$ infinitely often. So (28) holds.

By the above, in order to meet requirement $\mathcal{N}_e$, it suffices to ensure that there will be an attack which does not get stuck in step 1 or step 4 of any cycle and which is not cancelled. This is achieved by starting attacks on $\mathcal{N}_e$ as follows.

1. At stage $2e+2$ an attack on section $(e,0)$ is started.
2. If at stage s there is an attack on a section (e,i) which is in a cycle j and $s \in \{2s_j+2, 2s'_j+2, 2s''_j+2, 2s'''_j+2\}$ then fix i minimal with this property.

   Say that (the attack on) section (e,i) *acts* at stage s. Cancel all attacks on sections (e,i') with $i' > i$, i.e., cancel all sections (e,i') with $i' > i$ at stage s which were active at stage $s-2$. Moreover, if $s = 2s_j+2$ or $s = 2s''_j+2$ (i.e., the attack has reached step 1 or step 4 of cycle j and begins to wait for a stage at which section (e,i) is not frozen) then start a new attack on section (e,i') at stage s where i' is chosen minimal such that (e,i') is unused at stage $s-2$.

To show that this suffices to get a valid attack, for a contradiction assume that there is no attack on any e-section (e,i) such that the attack does not get stuck in step 1 or step 4 of any cycle and such that (e,i) is never cancelled. So, if there is an attack on a section (e,i) which is never cancelled, the attack gets stuck in step 1 or step 4 of some cycle. Obviously this implies that section (e,i) is permanently frozen. Since, by Claim 5, there are only finitely many $i \geq 0$ such that section (e,i) is permanently frozen, we get the desired contradiction by showing that, for any $i \geq 0$, there is a number $i' \geq i$ such that section (e,i') becomes active and is never cancelled. The proof is by induction. Fix i. If $i = 0$ then $i' = 0$ has the required properties since section $(e,0)$ becomes active but cannot become cancelled. If $i > 0$, by inductive hypothesis choose $i'' \geq i-1$ such that section (e,i'') has the required properties. If $i'' \geq i$ we are done. So assume $i'' = i-1$. Since the attack on $(e,i-1)$ is never cancelled it gets stuck at some stage s at the beginning of step 1 or step 4 of some cycle. So a new attack is started on a section

(e, i') where $i' \geq i$ is minimal such that (e, i') is unused at stage $s - 2$ and all sections (e, i''') with $i < i''' < i'$ are cancelled at stage s. By the latter, since (e, i) will not act after stage s and since (e, i) is never cancelled, (e, i') will never be cancelled.

This completes the construction of the sets $\alpha(n)$ and the proof that the nonlearning requirements $\mathcal{N}_e$ are met. By our previous discussion of the requirements $\mathcal{P}$ and $\mathcal{R}_k$, it only remains to show that, for a reduction requirement $\mathcal{R}_k$ such that k is correct, we can effectively replace the temporary values $g_k(x) = \perp$ assigned to the reduction function g_k in such a way such that (14) holds.

So, for the remainder of the proof, fix k such that k is correct, i.e., (13) holds. Now, if $g_k^s(x) = \perp$ then we have computed a target section $(e_{k,x}, i_{k,x})$ for $g_k(x)$ by stage s such that $e_x < k$ and $\gamma_k(x)$ is a set in section $(e_{k,x}, i_{k,x})$, i.e., $\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j_{k,x}\rangle)$ for some $j_{k,x}$. So, given $e < k$, it suffices to define a partial computable function j_e such that

$$\forall\, x\; [e_{k,x} = e \Rightarrow j_e(x)\downarrow\ \&\ \gamma_k(x) = \alpha(\langle e, i_{k,x}, j_e(x)\rangle)]. \tag{36}$$

For defining such a function j_e, we have to analyse the possible outcomes of the strategy for meeting nonlearning requirement $\mathcal{N}_e$.

Note that any e-section (e, i) is either permanently unused or *permanently active*, i.e., active from some stage on, or eventually cancelled. Moreover, if permanently active, then (e, i) is either *finitary*, namely the corresponding attack gets stuck in some cycle, or *infinitary*, namely the corresponding attack runs through all cycles.

Now, as observed above, there is a least number i such that section (e, i) is permanently active and, if finitary, the corresponding strategy gets stuck in step 3 (not in step 2 or step 4) of the final cycle. Now, if section (e, i) is infinitary, then it acts infinitely often and all sections (e, i') with $i < i'$ are eventually cancelled. On the other hand, if section (e, i) is finitary, then, at the last stage s at which (e, i) acts, all previously active sections (e, i') with $i < i'$ are cancelled at stage s. Moreover, since (e, i) does not get stuck in step 1 or step 4, no new e-section becomes active at stage s hence at the end of stage s all sections (e, i') with $i < i'$ are cancelled or unused. Moreover the status of such a section (e, i') cannot change later since, at the first change, some section (e, i'') with $i'' < i$ had to act and section (e, i) became cancelled contrary to assumption. Since by stage s only finitely many e-sections can become cancelled, it follows that almost all e-sections are permanently unused.

So, in any case, either almost all e-sections are cancelled or almost all e-sections are unused. This leaves the analysis of the structure of a permanently active section (e, i). If (e, i) is infinitary then all sets $\alpha(\langle e, i, j\rangle)$, $j \geq 1$, are eventually cancelled, hence merged with the primary set. If (e, i) is finitary and j is the final cycle then the active set $\alpha(\langle e, i, j\rangle)$ and any unused set $\alpha(\langle e, i, j'\rangle)$, $j' > j$, can be positively distinguished from the other sets in section (e, i) by its base element. The cancelled sets $\alpha(\langle e, i, j'\rangle)$, $j' < j$, are merged with the primary set, and $\alpha(\langle e, i, 0\rangle)$ can be positively distinguished by its final critical element $\hat{b}_{j-1}(\langle e, i, 0\rangle)$ from the uncancelled sets.

By the above analysis, the information on the final status of the e-sections (including the number j of the final cycle of a permanently active finitary section (e, i)) is finitely presentable, and, given this information, the partial function j_e can be defined as follows.

Fix x. By simulating the construction find the stage s (if any) at which the target $(e_{k,x}, i_{k,x})$ of $g_k(x)$ becomes defined. If $e_{k,x} \neq e$ there is no need to define $j_e(x)$. So assume $e_{k,x} = e$ and let $i = i_{k,x}$. Distinguish the following cases depending on i. If (e, i) is permanently unused then let $j_e(x)$ be the first number j such that $b(\langle e, i, j\rangle)$ shows up in $\gamma_k(x)$. (Since in an unused section all sets consist of their base element only, $\gamma_k(x) = \alpha(\langle e, i, j_e(x)\rangle)$.) If (e, i) is permanently active and infinitary or if (e, i) is eventually cancelled then let $j_e(x) = 0$. (Since all sets in such a section (e, i) are merged, $\gamma_k(x) = \alpha(\langle e, i, 0\rangle)$.) Finally, if (e, i) is permanently active and finitary and the attack on section (e, i) gets stuck in cycle j then enumerate $\gamma_k(x)$ until either the critical element $\hat{b}_{j-1}(\langle e, i, 0\rangle)$ or a base element $b(\langle e, i, j'\rangle)$ with $j' \geq j$ shows up in $\gamma_k(x)$, and let $j_e(x) = 0$ or $j_e(x) = j'$, respectively. (Again, by our analysis above, $\gamma_k(x) = \alpha(\langle e, i, j_e(x)\rangle)$.)

This completes the proof of Theorem 4.1. $\square$

5. A TxtBC-learnable class with nonequivalent numberings

Theorem 5.1. *There is a computable family $\mathcal{A}$ of c.e. sets which is* TxtBC*-learnable and which possesses computable numberings α and β which are not $\mathbf{0}'$-equivalent.*

Proof. Let $\mathcal{A} = \{K \cup D_n : n \geq 0\}$ where K is the (diagonal) halting problem and D_n is the finite set with canonical index n. Then, obviously, $\mathcal{A}$ is TxtBC-learnable: A learner M is obtained by letting $M(\sigma)$ be an index of $K \cup \mathrm{content}(\sigma)$.

So it suffices to give computable numberings α and β of $\mathcal{A}$ such that $\beta \not\leq_{\mathbf{0}'} \alpha$.

Let α be the obvious numbering of $\mathcal{A}$ given by

$$\alpha(n) = K \cup D_n.$$

For the definition of the numbering β, fix a Π_2^0-complete set C. β will code information on C in such a way that the existence of a $\mathbf{0}'$-reduction of β to α could be turned into a Turing reduction of C to the halting problem (thereby contradicting the Π_2^0-completeness of C).

Since $C \in \Pi_2^0$ there is a computable set R such that, for $n \geq 0$,

$$n \in C \Leftrightarrow \exists^{\infty} x\ (\langle n, x\rangle \in R).$$

Define $V_{\langle n,k\rangle}$ by

$$V_{\langle n,k\rangle} = \begin{cases} \mathbb{N} & \text{if } \exists^{\geq k} x\ (\langle n, x\rangle \in R) \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the sets $V_{\langle n,k\rangle}$, $n, k \geq 0$, are uniformly c.e. So there is a computable function f such that

$$W_{f(\langle n,k\rangle)} = V_{\langle n,k\rangle}. \tag{37}$$

Moreover, by choice of R and by definition of $V_{\langle n,k\rangle}$,

$$n \in C \Rightarrow \forall\, k \geq 0\ (V_{\langle n,k\rangle} = \mathbb{N}) \tag{38}$$

and

$$n \notin C \Rightarrow \exists\, k_n \geq 0\ [\forall\, k < k_n\ (V_{\langle n,k\rangle} = \mathbb{N})\ \&\ \forall\, k \geq k_n\ (V_{\langle n,k\rangle} = \emptyset)]. \tag{39}$$

Now define β by letting

$$\beta(2n) = K \cup \{f(\langle n, 0\rangle)\} \cup \{f(\langle n, k+1\rangle) : k \geq 0\ \&\ V_{\langle n,k\rangle} \neq \emptyset\} \tag{40}$$

and

$$\beta(2n+1) = \alpha(n). \tag{41}$$

Obviously, β is a computable numbering. Moreover, β codes C as follows.

$$n \in C \Rightarrow \beta(2n) = K \tag{42}$$

and

$$n \notin C \Rightarrow \exists\, x_n \notin K\ (\beta(2n) = K \cup \{x_n\}). \tag{43}$$

(For a proof of (42) fix $n \in C$. By (37) and (38), $W_{f(\langle n,k\rangle)} = V_{\langle n,k\rangle} = \mathbb{N}$ for all $k \geq 0$. So, in particular, $f(\langle n, k\rangle) \in W_{f(\langle n,k\rangle)}$, i.e., $f(\langle n, k\rangle) \in K$ for all $k \geq 0$. Hence $\beta(2n) = K$ by (40). For a proof of (43) fix $n \notin C$ and fix k_n as in (39). Then, by (39) and (40), $\beta(2n) = K \cup \{f(\langle n, k\rangle) : k \leq k_n\}$. Moreover, as above, $f(\langle n, k\rangle) \in K$ for $k < k_n$ whereas $f(\langle n, k_n\rangle) \notin K$ since $W_{f(\langle n,k_n\rangle)} = V_{\langle n,k_n\rangle} = \emptyset$. So $\beta(2n) = K \cup \{x_n\}$ for $x_n = f(\langle n, k_n\rangle) \notin K$.)

Since, by (42) and (43), $\{\beta(2n) : n \geq 0\} \subseteq \mathcal{A}$, it follows with (41) that β is a numbering of $\mathcal{A}$.

It remains to show that $\beta \not\leq_{\mathbf{0}'} \alpha$. For a contradiction assume that $\beta \leq_{\mathbf{0}'} \alpha$ via g, i.e., $\beta(n) = \alpha(g(n))$ where $g \leq_T K$. Then, by (42) and (43) and by definition of α,

$$n \in C \Leftrightarrow \beta(2n) \subseteq K \Leftrightarrow \alpha(g(2n)) \subseteq K \Leftrightarrow D_{g(2n)} \subseteq K.$$

Since $g \leq_T K$, and since, for given m, $D_m \subseteq K$ can be decided relative to K, it follows that $C \leq_T K$. But this contradicts our assumption that C is Π^0_2-complete.

This completes the proof. □

6. A non-TxtBC-learnable class with only 0′-equivalent numberings

In the proof of Theorem 4.1 we used the fact [15] that any positive numbering is minimal under reduction by computable functions. The same is true for reduction by $\mathbf{0}'$-computable functions.

Proposition 6.1. *Let α and β be computable numberings of a computable family $\mathcal{A}$ of c.e. sets such that α is positive and β is $\mathbf{0}'$-reducible to α. Then α and β are $\mathbf{0}'$-equivalent.*

Proof. Fix a function $f \leq_T K$ such that $\beta \leq_{\mathbf{0}'} \alpha$ via f, i.e., $\beta(n) = \alpha(f(n))$ for all numbers n. Since α and β are numberings of $\mathcal{A}$, the function g defined by

$$g(m) = \mu_n(\alpha(f(n)) = \alpha(m))$$

is total and, for any number m, $\alpha(m) = \beta(g(m))$. Moreover, since α is positive, the relation

$$R(n, m) \Leftrightarrow \alpha(n) = \alpha(m)$$

is computably enumerable hence $R \leq_T K$. Since g is computable relative to f and R, it follows that $g \leq_T K$ too. So $\alpha \leq_{\mathbf{0}'} \beta$ via g. □

Theorem 6.1. *There exists a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are pairwise $\mathbf{0}'$-equivalent and $\mathcal{A}$ is not* TxtBC*-learnable.*

Proof. Since the proof resembles the proof of Theorem 4.1 and uses many of the features of this previous proof, we only give a sketch. In particular, we adopt the terminology of the proof of Theorem 4.1.

We construct a computable numbering α such that the family $\mathcal{A} = \{\alpha(x) \mid x \in \mathbb{N}\}$ of c.e. sets has the required properties. It suffices to meet the following requirements.

Requirements. Let $M_0, M_1, \ldots$ be a computable sequence of all primitive recursive learners, and let $\gamma_0, \gamma_1, \ldots$ be a uniformly computable sequence of all computable numberings of computable families of c.e. sets. Then the numbering α has to meet the following requirements for all $k, e \in \mathbb{N}$:

$\mathcal{P}$: α is a computable positive numbering.

$\mathcal{R}_k$: If γ_k is a numbering of $\mathcal{A}$ then γ_k is $\mathbf{0}'$-reducible to α.

$\mathcal{N}_e$: For some m, M_e fails to TxtBC-learn the set $\alpha(m)$.

By Proposition 6.1, the global requirement $\mathcal{P}$ together with the *reduction requirements* $\mathcal{R}_k$ ensure that all computable numberings of $\mathcal{A}$ are pairwise $\mathbf{0}'$-equivalent. By Proposition 2.1, the *nonlearning requirements* $\mathcal{N}_e$ guarantee that $\mathcal{A}$ is not TxtBC-learnable.

Strategies for meeting $\mathcal{P}$ and $\mathcal{R}_k$. The strategies for meeting the global requirement $\mathcal{P}$ and the reduction requirements $\mathcal{R}_k$ are directly adopted from the proof of Theorem 4.1.

In order to meet $\mathcal{P}$ we satisfy the conditions (11) and (12) thereby ensuring that two sets $\alpha(m)$ and $\alpha(n)$ agree in the limit if and only if they agree at some stage of the effective construction given below.

For meeting $\mathcal{R}_k$ we use the machinery introduced in the proof of Theorem 4.1 for meeting the corresponding requirement. Recall that there, assuming that k is correct, we constructed the required computable reduction function g_k satisfying (14) in two steps. In the course of the construction, we specified the value of $g_k(x)$ only if $\gamma_k(x)$ was a member of a section reserved for a lower priority nonlearning requirement. If $\gamma_k(x)$ was a member of a section reserved for a higher priority nonlearning requirement then, in the course of the construction, we assigned the temporary value $\perp$ to $g_k(x)$, and only after completion of the construction, using some finite (but nonuniform) information on the outcomes of the finitely many higher priority nonlearning requirements we replaced $\perp$ by the correct value.

Now here the first part of the definition of g_k is exactly as in the previous proof, i.e., we will ensure that the enumeration of numbers into the sets $\alpha(n)$ by the nonlearning requirements will follow the rules established in the proof of Theorem 4.1 and we will inductively define g_k as described there. So Claims 1–5 established there will hold here too. In particular, the part of g_k defined in the course of the construction will be computable and – assuming that k is correct – the final values assigned to $g_k(x)$ in the course of the construction will be correct. The reason, why here the function g_k will be computable in the halting problem only, is found in the second part of the definition. If, for a number x, only a temporary value $g_k(x) = \perp$ will be assigned in the course of the construction and the actual value of $g_k(x)$ will be determined only after completion of the construction then the specification of this value will require some information on the outcomes of the higher priority nonlearning requirements. For some numbers x this information can be only obtained by using the halting problem as an oracle.

Strategy for meeting $\mathcal{N}_e$. It will be convenient to split the numbers $a(n) = 2n + 1$ into the numbers $c(n) = 4n + 1$ and $d(n) = 4n + 3$, called *coding* numbers and *diagonalization* numbers, respectively. (The nth coding number $c(n)$ essentially plays the same role as number $a(n)$ in the previous proof.)

Given the ith e-section (e, i), in the ith attempt for meeting $\mathcal{N}_e$ we build a (finite or infinite) sequence of strings over $\mathbb{N}$, namely

$$\sigma^0_{e,i} \sqsubset \sigma^1_{e,i} \sqsubset \cdots \sqsubset \sigma^{j^*}_{e,i}$$

($j^* \geq 0$) or

$$\sigma^0_{e,i} \sqsubset \sigma^1_{e,i} \sqsubset \sigma^2_{e,i} \sqsubset \cdots$$

such that $\mathrm{content}(\sigma^j_{e,i}) \subseteq \alpha(\langle e, i, 0\rangle)$ and such that – if the ith attempt is the successful one – either there is a sequence of diagonalization numbers $\hat{d}_1 < \hat{d}_2 < \cdots$ such that, for $\tau_m = \sigma^{j^*}_{e,i}\, b(\langle e, i, j^* + 1\rangle)\, \hat{d}_1 \ \ldots\ \hat{d}_m$ ($m \geq 0$),

$$\bigcup_{m \geq 0} \mathrm{content}(\tau_m) = \alpha(\langle e, i, j^* + 1\rangle) \ \&\ \forall m \geq 0\ [W_{M_e(\tau_m)} \neq \alpha(\langle e, i, j^* + 1\rangle)] \tag{44}$$

or

$$\bigcup_{j \geq 0} \mathrm{content}(\sigma^j_{e,i}) = \alpha(\langle e, i, 0\rangle) \ \&\ \forall j \geq 0\ \exists \sigma\ [\sigma^j_{e,i} \sqsubset \sigma \sqsubset \sigma^{j+1}_{e,i} \ \&\ W_{M_e(\sigma)} \neq \alpha(\langle e, i, 0\rangle)]. \tag{45}$$

So either M_e does not TxtBC-learn the set $\alpha(\langle e, i, j^* + 1\rangle)$ from the text for $\alpha(\langle e, i, j^* + 1\rangle)$ defined by the sequence $\tau_m, m \geq 0$, or M_e does not TxtBC-learn the set $\alpha(\langle e, i, 0\rangle)$ from the text for $\alpha(\langle e, i, 0\rangle)$ defined by the sequence $\sigma^j_{e,i}, j \geq 1$.

The above is achieved by induction on $j \geq 1$, where string $\sigma^j_{e,i}$ is defined at the end of *cycle* j below ($\sigma^0_{e,i}$ is the empty string).

When cycle 1 is started at stage $2s_1$ then $s_1 > 0$, section (e, i) was unused at stage $2s_1 - 1$, and section (e, i) becomes *active* at stage $2s_1$. Cycle j will affect the *primary* set $\alpha(\langle e, i, 0\rangle)$ and the (currently) *active* set $\alpha(\langle e, i, j\rangle)$. If cycle j is started at stage $2s_j$, then – as in the previous construction – the active set and the *unused* sets $\alpha(\langle e, i, j'\rangle)$, $j' > j$, are still in their initial states. Moreover, there will be a critical element of the primary set, $\hat{b}_{j-1}(\langle e, i, 0\rangle)$, such that, at stage $2s_j$, the primary set is the disjoint union of $\mathrm{content}(\sigma_{e,i}^{j-1})$ and $\{\hat{b}_{j-1}(\langle e, i, 0\rangle)\}$ (i.e., (29) holds), and $\hat{b}_{j-1}(\langle e, i, 0\rangle)$ is not a member of any other uncancelled set in section (e, i). For $j = 1$, $\hat{b}_{j-1}(\langle e, i, 0\rangle) = b(\langle e, i, 0\rangle)$ while, for $j > 1$, $\hat{b}_{j-1}(\langle e, i, 0\rangle) = c(j-1)$ and $c(j-1)$ is enumerated into the primary set at the end of cycle $j-1$. If cycle j is completed, the active set is *cancelled* and some (proper) part of the active set is enumerated into the primary set. All cancelled sets are made to agree but, in contrast to the previous proof, the cancelled sets will *not* agree with the primary set.

Now cycle j (started at stage $2s_j$) is as follows.

Cycle j ($j \geq 1$).

1. Wait for the least $s'_j > s_j$ such that section (e, i) is not frozen at stage $2s'_j + 2$.
2. At stage $2s'_j + 2$ enumerate the elements of the primary set with the exception of the critical element into the active set:

$$\begin{aligned}\alpha^{2s'_j+2}(\langle e, i, j\rangle) &= \alpha^{2s'_j}(\langle e, i, j\rangle) \cup [\alpha^{2s'_j}(\langle e, i, 0\rangle) \setminus \{\hat{b}_{j-1}(\langle e, i, 0\rangle)\}] \\ &= \alpha^{2s'_j}(\langle e, i, j\rangle) \cup \mathrm{content}(\sigma_{e,i}^{j-1}).\end{aligned} \tag{46}$$

Let $\tau_0 = \sigma_{e,i}^{j-1} b(\langle e, i, j\rangle)$, and start the following subcycle $(j, 1)$, i.e., let $s_{j,1} = s'_j + 1$.

Subcycle (j, m) ($m \geq 1$).

(a) Wait for the least $s'_{j,m} > s_{j,m}$ such that section (e, i) is not frozen at stage $2s'_{j,m} + 2$.

(b) At stage $2s'_{j,m} + 2$ enumerate the least diagonalization number which has not previously entered any set in section (e, i), say $\hat{d}_m$, into $\alpha(\langle e, i, j\rangle)$. Set

$$\tau_m = \tau_{m-1}\hat{d}_m,$$

and, for $n < m$, enumerate $W_{M_e(\tau_n)}$ for up to m steps. If, for some such n, a number $\hat{d}_p$ with $n < p \leq m$ shows up in $W_{M_e(\tau_n)}$ then let n_m be the least such n, stop the subcycle at step m, let $s''_j = s'_{j,m} + 1$, and continue with step 3 of cycle j. Otherwise, start subcycle $(j, m+1)$ at the next even stage, i.e., let $s_{j,m+1} = s'_{j,m} + 1$.

3. Wait for the least $s'''_j > s''_j$ such that section (e, i) is not frozen at stage $2s'''_j + 2$.
4. At stage $2s'''_j + 2$, for m and n_m as above, enumerate $\mathrm{content}(\tau_{n_m}) \cup \{c(j)\}$ into $\alpha(\langle e, i, 0\rangle)$, let $\hat{b}_j(\langle e, i, 0\rangle) = c(j)$, and let $\sigma_{e,i}^j = \tau_{n_m}\hat{b}_{j-1}(\langle e, i, 0\rangle)$. Declare the active set $\alpha(\langle e, i, j\rangle)$ *cancelled*, merge it with all previously cancelled sets,

$$\forall j' \left[1 \leq j' \leq j \Rightarrow \alpha^{2s'''_j+2}(\langle e, i, j'\rangle) = \bigcup_{1 \leq j'' \leq j} \alpha^{2s'''_j}(\langle e, i, j''\rangle)\right], \tag{47}$$

and start cycle $j+1$ with the new active set $\alpha(\langle e, i, j+1\rangle)$ at the next even stage, i.e., let $2s_{j+1} = 2s'''_j + 4$.

Note that in step 1 of cycle j, where a part of the primary set is enumerated into the active set, the current critical number does not enter the active set. Similarly, when a part of the active set is enumerated into the primary set at step 4 of the attack then there is a diagonalization number $\hat{d}_p$, $n_m < p \leq m$ which has entered the active set in subcycle (j, p) of step 2 and which will never enter the primary set. This easily implies that condition (17) in the proof of Theorem 4.1 is satisfied. Since, moreover, we only enumerate new numbers in the sets in section (e, i) at even stages at which (e, i) is not frozen, one can easily show that the above action obeys the rules made by the reduction strategies. So Claims 1–5 in the proof of Theorem 4.1 hold here too.

Moreover, if – whenever we start to wait for an unfrozen stage (in step 1 or 3 of cycle j or step (a) of subcycle (j, m)) – we start a new attack on a new unused section (e, i'), $i' > i$; and if – whenever we make some progress in our attack – we cancel all e-sections (e, i') with $i' > i$ on which we had started an attack before, then as in the proof of Theorem 4.1 we can argue that there will be a permanently active section (e, i) such that the attack on this section never gets stuck in a step waiting for (e, i) not to be frozen.

In order to show that the above strategy succeeds, fix such a section (e, i). Note that there are two possible outcomes of the attack on this section. First we may get stuck in step 2 of some cycle j since we run through all subcycles (j, m), $m \geq 1$. Then we build an infinite sequence $\tau_0 \sqsubset \tau_1 \sqsubset \tau_2 \ldots$ where

$$\mathrm{content}(\tau_0) \subset \mathrm{content}(\tau_1) \subset \mathrm{content}(\tau_2) \subset \cdots$$

and

$$\bigcup_{m \geq 0} \mathrm{content}(\tau_m) = \alpha(\langle e, i, j\rangle).$$

Moreover, for any $n \geq 0$,

$$W_{M_e(\tau_n)} \cap (\alpha(\langle e, i, j\rangle) \setminus \mathrm{content}(\tau_n)) = \emptyset$$

(since otherwise for sufficiently large $m \geq n$ the subcycle will be left). So

$$W_{M_e(\tau_n)} \neq \alpha(\langle e, i, j \rangle)$$

for all n hence M_e will fail to learn $\alpha(\langle e, i, j \rangle)$ (according to (44) for $j^* = j - 1$).

Otherwise we run through all cycles j. Since we do not get stuck in step 2 of cycle j, for m and n_m as given there,

$$\sigma_{e,i}^{j-1} \sqsubset \tau_{n_m} \sqsubset \sigma_{e,i}^{j}$$

and $W_{M_e(\tau_{n_m})}$ contains one of the diagonalization numbers $\hat{d}_p$ with $n_m < p \leq m$. Note that all such numbers are not in $\mathrm{content}(\sigma_{e,i}^{j})$ and will never enter $\alpha(\langle e, i, 0 \rangle)$. Hence $W_{M_e(\tau_{n_m})} \neq \alpha(\langle e, i, 0 \rangle)$. Moreover,

$$\mathrm{content}(\sigma_{e,i}^{j-1}) \subseteq \alpha^{2s_j}(\langle e, i, 0 \rangle) \subseteq \mathrm{content}(\sigma_{e,i}^{j})$$

hence

$$\alpha(\langle e, i, 0 \rangle) = \bigcup_{j \geq 0} \mathrm{content}(\sigma_{e,i}^{j}).$$

Hence, if we complete all cycles j, the construction will ensure that (45) holds.

So in either case the strategy succeeds in meeting $\mathcal{N}_e$.

In order to complete the proof, given k such that k is correct, we have to give a $\mathbf{0}'$-effective procedure for replacing temporary values $g_k(x) = \perp$ in the definition of the reduction g_k in such a way such that (14) holds. In fact, given $e < k$, it suffices to define a partial $\mathbf{0}'$-computable function j_e such that (36) holds.

The definition of j_e is based on the following analysis of the final effect which the strategy for meeting the nonlearning requirement $\mathcal{N}_e$ has on the e-sections.

As observed above, we may fix i_0 minimal such that the attack on section (e, i_0) is permanently active and never gets stuck in a step waiting for a stage at which section (e, i_0) is unfrozen. It follows that section (e, i_0) acts infinitely often hence, for $i > i_0$ section (e, i) will eventually be cancelled and all sets in section (e, i) are identified. So, in particular, no e-section is permanently unused and all but finitely many e-sections are trivialized by cancellation. It remains to analyze permanently active e-sections (e, i).

First assume that (e, i) is permanently active and *finitary*, i.e., the attack on section (e, i) gets stuck in some cycle j (either by waiting forever for a stage at which (e, i) is unfrozen or by running through all subcycles (j, m), $m \geq 1$). Then the sets $\alpha(\langle e, i, 1 \rangle), \ldots, \alpha(\langle e, i, j-1 \rangle)$ are cancelled, hence agree, and these are the only sets in section (e, i) which agree. Moreover, from j we can compute numbers $y_{j'}, j' \geq 0$, such that $y_{j'}$ positively distinguishes $\alpha(\langle e, i, j' \rangle)$ from the other sets in the section (which do not agree with $\alpha(\langle e, i, j' \rangle)$). Namely, the final critical element $\hat{b}_{j-1}(\langle e, i, 0 \rangle)$ of the primary set is unique to this set, i.e., for $y_0 = \hat{b}_{j-1}(\langle e, i, 0 \rangle)$, $y_0 \in \alpha(\langle e, i, j'' \rangle)$ if and only if $j'' = 0$. The greatest diagonalization number $\hat{d}$ which enters $\alpha(\langle e, i, j-1 \rangle)$ in cycle $j-1$ is an element of the cancelled sets but of none of the other sets. So if we let $y_{j'} = \hat{d}$ for $1 \leq j' < j$ then $y_{j'} \in \alpha(\langle e, i, j'' \rangle)$ if and only if $\alpha(\langle e, i, j' \rangle)$ is cancelled, i.e., if and only if $1 \leq j'' < j$. Finally, for $j' \geq j$, the base element $b(\langle e, i, j' \rangle)$ of $\alpha(\langle e, i, j' \rangle)$ is unique to this set. So if we let $y_{j'} = b(\langle e, i, j' \rangle)$ for $j' \geq j$ then $y_{j'} \in \alpha(\langle e, i, j'' \rangle)$ if and only if $j'' = j'$.

Now assume that (e, i) is permanently active and *infinitary*, i.e., all cycles j are completed. Then all sets $\alpha(\langle e, i, j \rangle), j \geq 1$ are eventually cancelled, hence agree; i.e.,

$$\forall j \geq 1\ [\alpha(\langle e, i, j \rangle) = \alpha(\langle e, i, 1 \rangle)]. \tag{48}$$

Moreover, the primary set $\alpha(\langle e, i, 0 \rangle)$ is a *proper* subset of $\alpha(\langle e, i, 1 \rangle)$. Namely, for any $j \geq 1$,

$$\alpha^{2s_j}(\langle e, i, 0 \rangle) \subseteq \alpha^{2s'_{j+1}+2}(\langle e, i, j+1 \rangle)$$

hence, by (48), $\alpha(\langle e, i, 0 \rangle) \subseteq \alpha(\langle e, i, 1 \rangle)$. This inclusion is proper since, for instance, the greatest diagonalization number $\hat{d}$ which enters $\alpha(\langle e, i, 1 \rangle)$ in cycle 1 never enters $\alpha(\langle e, i, 0 \rangle)$. So here – assuming that we know that (e, i) is permanently active and all cycles j are completed – we can find a number y which positively distinguishes the merged sets $\alpha(\langle e, i, j \rangle)$, $j \geq 1$, from the primary set $\alpha(\langle e, i, 0 \rangle)$, but the primary set cannot be positively distinguished from the other sets in the section.

Now given the finite list of the indices i such that (e, i) is permanently active together with the information whether or not (e, i) is finitary or not, and if so the number j of the final cycle, we can define j_e as follows.

Given x, wait that the target section $(e_{k,x}, i_{k,x})$ for $g_k(x)$ is defined. By construction,

$$\exists j \geq 0\ [\gamma_k(x) = \alpha(\langle e_{k,x}, i_{k,x}, j \rangle)]. \tag{49}$$

If $e_{k,x} \neq e$ then $j_e(x)$ may be undefined. So assume $e_{k,x} = e$ and let $i = i_{k,x}$. Now if (e, i) is not permanently active then section (e, i) is cancelled and all sets in section (e, i) agree. So, by (49), $\gamma_k(x) = \alpha(\langle e, i, 0 \rangle)$ and we may let $j_e(x) = 0$. Otherwise, first assume that (e, i) is finitary. Then we can compute numbers $y_{j'}$ as above and enumerate $\gamma_k(x)$ up to the first stage at which such a number $y_{j'}$ shows up in $\gamma_k(x)$. Then $\gamma_k(x) = \alpha(\langle e, i, j' \rangle)$ and we may let $j_e(x) = j'$. Finally, if (e, i) is infinitary then we can compute the greatest diagonalization number $\hat{d}$ which has entered $\alpha(\langle e, i, 1 \rangle)$ in cycle 1. Now, using the halting problem as an oracle, we may decide whether $\hat{d} \in \gamma_k(x)$. (Note that this is the only place where in the definition of g_k oracle $\mathbf{0}'$ is used!) If so, we let $j_e(x) = 1$; if not, we let $j_e(x) = 0$.

This completes the sketch of the proof of Theorem 6.1. □

Frank Stephan has pointed out to us an alternative proof of Theorem 6.1 by giving an explicit example of a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are pairwise $\mathbf{0}'$-equivalent and $\mathcal{A}$ is not TxtBC-learnable. Still we think that our proof describing the construction of a family $\mathcal{A}$ with this property might be of interest for looking at possible strengthenings of Theorem 6.1 (see Open Problem 2 below).

7. Conclusion

The starting point of our research was the conjecture of Frank Stephan: a computable family of c.e. sets is TxtEx-learnable if and only if its computable numberings are pairwise $\mathbf{0}'$-equivalent. We refuted this conjecture in one direction by constructing a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are computably equivalent and $\mathcal{A}$ is not TxtEx-learnable.

Open Problem 1. Is there a learning scenario $\mathcal{S}$ such that, for every computable family $\mathcal{A}$ of c.e. sets, $\mathcal{A}$ is $\mathcal{S}$-learnable iff and only if all computable numberings of $\mathcal{A}$ are computably equivalent?

We have also shown that, for a computable family $\mathcal{A}$ of c.e. sets, the TxtBC-learnability and $\mathbf{0}'$-equivalence of its computable numberings are independent. We do not know whether Theorem 6.1 can be improved like the statement of Theorem 4.1.

Open Problem 2. Is there a computable family $\mathcal{A}$ of c.e. sets such that all computable numberings of $\mathcal{A}$ are *computably* equivalent and $\mathcal{A}$ is not TxtBC-learnable?

Acknowledgements

The second author's research was partially supported by the State Grants of Kazakhstan ''Best Teacher of Higher Education" for 2006 and 2007. The third author was partially supported by the Grant RFBR-08-01-00336 and the Grant for Leading Scientific Schools SS-4413 for 2006.

We are grateful to Frank Stephan and John Case for useful discussions.

References

[1] K. Ambos-Spies, S.A. Badaev, S.S Goncharov, On a question of Frank Stephan, in: Proceedings of 5th International Conference, TAMC 2008, Xi'an, China, April 2008, in: Lecture Notes of Computer Science, vol. 4978, Springer-Verlag, 2008, pp. 423–432.
[2] S.A. Badaev, S.S. Goncharov, The theory of numberings: open problems, in: P.A. Cholak, S. Lempp, M. Lerman, R.A. Shore (Eds.), Computability Theory and its Applications. Current Trends and Open Problems, in: Contemporary Mathematics, vol. 257, Amer. Math. Soc., Providence, 2000, pp. 23–38.
[3] S.A. Badaev, S.S. Goncharov, S.Yu. Podzorov, A. Sorbi, Algebraic properties of Rogers semilattices of arithmetical numberings, in: S.B. Cooper, S.S. Goncharov (Eds.), Computability and Models, Kluwer/Plenum Publishers, New York, 2003, pp. 45–77.
[4] L. Blum, M. Blum, Toward a mathematical theory of inductive inference, Information and Control 28 (1975) 125–155.
[5] Yu.L. Ershov, Theory of Numberings, Nauka, Moscow, 1977.
[6] Yu.L. Ershov, Theory of numberings, in: Handbook of Computability Theory, North-Holland, Amsterdam, 1999, pp. 473–503.
[7] M. Fulk, 1985. A Study of Inductive Inference Machines. Ph.D. Thesis, SUNY/Buffalo.
[8] E.M. Gold, Language identification in the limit, Information and Control 10 (1967) 447–474.
[9] S.S. Goncharov, Autostability of models and Abelian groups, Algebra and Logic 19 (1) (1980) 13–27.
[10] S.S. Goncharov, S.A. Badaev, Families with one-element Rogers semilattice, Algebra and Logic 37 (1) (1980) 21–34.
[11] S. Jain, A. Sharma, Characterizing language identification by standardizing operations, Journal of Computer and System Sciences 49 (1) (1994) 96–107.
[12] S. Jain, A. Sharma, Characterizing language identification in terms of computable numberings, Annals of Pure and Applied Logic 84 (1) (1997) 51–72.
[13] S. Jain, D. Osherson, J.S. Royer, A. Sharma, Systems that Learn. An Introduction to Learning Theory, 2nd edn., MIT Press, Cambridge Massachusetts, 1999.
[14] M. Kummer, A learning-theoretic characterization of discrete families of recursive functions, Information Processing Letters 54 (1995) 205–211.
[15] A.I. Mal'cev, Positive and negative numberings, Soviet Math. Dokl. 160 (1965) 75–77.
[16] P.G. Odifreddi, Classical recursion theory. Vol. II, in: Studies in Logic and the Foundations of Mathematics, vol. 143, North-Holland, Amsterdam, 1999.
[17] G. Schäfer-Richter, 1984. Über Eingabeabhängigkeit und Komplexität von Inferenzstrategien. Ph.D. Thesis, RWTH Aachen.