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Application of weighting functions to the ranking of fuzzy numbers

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ABSTRACT

This paper proposes a new ranking method for fuzzy numbers, which uses a defuzzification of fuzzy numbers and a weighting function. Following Saeidifar and Pasha (2008), first, we define a weighted distance measure on fuzzy numbers, and then, by minimizing this distance, the weighted interval and point approximations of fuzzy numbers are obtained. These indices are applied to rank the fuzzy numbers. This method is new and interesting for ranking fuzzy numbers, and it can be applied for solving and optimizing engineering and economics problems in a fuzzy environment.

1. Introduction

Ranking fuzzy numbers plays an important role in fuzzy decision making problems; therefore, deriving the final efficiency and powerful ranking are helpful to decision makers when solving fuzzy problems. Selecting a good ranking method can apply to choosing a desired criterion in a fuzzy environment. In recent years, many ranking methods have been introduced by researchers; some of these ranking methods have been compared and reviewed by Bortolan and Degani [1]. Wang and Kerre [2,3] proposed some axioms as reasonable properties to determine the rationality of a fuzzy ranking method and systematically compared a wide array of existing fuzzy ranking methods. Almost all methods, however, have pitfalls in some aspect, such as inconsistency with human intuition, indiscrimination, and difficulty of interpretation. What seems to be clear is that there exists no uniquely best method for comparing fuzzy numbers, and different methods may satisfy different criteria. Among the existing ranking methods of fuzzy numbers, a number of them are based on area measurements with the integral value of the membership function of fuzzy numbers. A commonly used ranking technique for fuzzy numbers is the centroid-based ranking method. Some other methods use statistical techniques such as simulation and hypothesis and quadratic fuzzy regression.

In the following, we first introduce the developments of centroid-based fuzzy number ranking methods. Yager [4] proposed the centroid index ranking method with a weighting function. Chen and others have proposed a centroid index ranking method that calculates the distance between the centroid point of each fuzzy number and the original point to improve some of the ranking methods [5–8]. They also proposed a coefficient of variation (CV index) to improve Lee and Li's method [9]. Chu and Tsao [7] proposed a ranking method of fuzzy numbers by using the area between the centroid and the original point. Chen and Chen [10] proposed a ranking index based on the centroid point and standard deviations to overcome some of the drawbacks of previous centroid point indices. Lee [11] proposed a fuzzy number ranking method with user viewpoints. Yager and Filve [12] proposed a ranking method with parameterized valuation functions. Detyniecki and Yager [13] proposed a fuzzy number ranking method with an α weighting function. Tran and Duckstein [14] proposed a weighting function that represents the decision maker's attitude. Lee and Li [9] introduced a ranking method for normalized trapezoidal fuzzy numbers (NTFNs). Tang [15] showed that Lee and Li's method for ranking fuzzy numbers is inconsistent. Liu and Han [16] proposed a method to rank fuzzy numbers with preference weighting function expectation. Cheng [6]

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has proposed the distance method for ranking fuzzy numbers. Goetschel and Voxman [17] introduced a method for ranking fuzzy numbers: their definition for ordering fuzzy numbers was motivated by the desire to give less importance to the lower levels of fuzzy numbers. Deng et al. [18] introduced the ranking of fuzzy numbers by an area method using the radius of gyration (ROG). Wang et al. [8] improved the correct centroid formula for ranking fuzzy numbers that justified them from the viewpoint of analytical geometry. Abbasbandy and Asady [19] proposed the ranking of fuzzy numbers by sign distance. Therefore, the essential subject of paper is the weighted ranking of fuzzy numbers.

The rest of this paper is organized as follows. In Section 2, we recall some of the basic definitions and notions. In Section 3, we obtain the nearest weighted interval and point approximations of a fuzzy number. In Section 4, we introduce a new method for ranking fuzzy numbers by the weighting mean, and its properties are mentioned. The last section (Section 5) is devoted to a discussion and conclusion.

2. Basic definitions and notions

Let $\mathbb{R}$ be the set of all real numbers. We assume a fuzzy number A that can be expressed for all $x \in \mathbb{R}$ in the form

$$A(x) = \begin{cases} A_L(x) & x \in [a, b], \\ 1 & x \in [b, c] \\ A_R(x) & x \in [c, d], \\ 0 & otherwise, \end{cases} \tag{I}$$

where a, b, c, and d are real numbers such that $a < b \leq c < d$, A_L is real-valued function that is increasing and right continuous, and A_R is a real-valued function that is decreasing and left continuous. Notice that (I) is an **L**–**R** fuzzy number with strictly monotone shape function, as proposed by Dubois and Prade in 1981, and also described in [20]. Each fuzzy number A described by (I) has the following γ-level sets (γ-*cuts*): $[A]_\gamma = [A_L^{-1}(\alpha), A_R^{-1}(\gamma)] = [\underline{a}(\gamma), \overline{a}(\gamma)]$ for all $\gamma \in [0, 1]$. The family of fuzzy numbers will be denoted by $\mathcal{F}$.

Definition 1. A fuzzy number $A = (a, b, c, d)$ is called a trapezoidal fuzzy number if its membership function $A(x)$ has the following form:

$$A(x) = \begin{cases} \dfrac{x-a}{b-a} & x \in [a, b], \\ 1 & x \in [b, c], \\ \dfrac{d-x}{d-c} & x \in [c, d], \\ 0 & otherwise. \end{cases} \tag{II}$$

Definition 2. For two arbitrary fuzzy numbers A and B with γ-*cuts* $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ and $[B]_\gamma = [\underline{b}(\gamma), \overline{b}(\gamma)]$, respectively, the quantity

$$d(A, B) = \left[\int_0^1 (\underline{a}(\gamma) - \underline{b}(\gamma))^2 \mathrm{d}\gamma + \int_0^1 (\overline{a}(\gamma) - \overline{b}(\gamma))^2 \mathrm{d}\gamma\right]^{\frac{1}{2}} \tag{1}$$

is the distance between A and B. This metric is a particular member of the family of distance $\delta_{p,q}$, defined as follows:

$$\delta_{p,q} = \left[\int_0^1 (1-q)|\underline{a}(\gamma) - \underline{b}(\gamma)|^p \mathrm{d}\gamma + \int_0^1 q|\overline{a}(\gamma) - \overline{b}(\gamma)|^p \mathrm{d}\gamma\right]^{\frac{1}{p}},$$

where $1 \leq p \leq \infty$ and $0 \leq q \leq 1$ (see [21,22]).

We also recall some concepts and results on the possibility space and fuzzy variables.

Let Θ be a nonempty set, and $P(\Theta)$ the power set of Θ. A function Pos is called a possibility measure if

(i) $Pos\{\Theta\} = 1$,
(ii) $Pos\{\phi\} = 0$,
(iii) $Pos\{\bigcup_i A_i\} = \sup_i Pos\{A_i\}$ for any collection $\{A_i\}$ in $P(\Theta)$.

Then, the triplet $(\Theta, P(\Theta), Pos)$ is called a possibility space.

Definition 3 (*[23]*)**.** A fuzzy variable ξ is defined as a function from a possibility space $(\Theta, P(\Theta), Pos)$ to the set of real numbers.

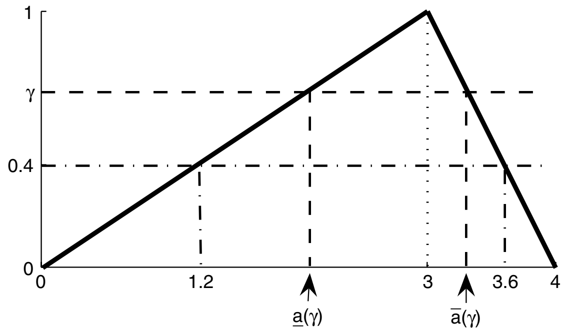

Fig. 1. Fuzzy number A from Example 2.

Definition 4 ([23]). A fuzzy variable ξ is said to be nonnegative, denoted by $\xi \geq 0$, if $Pos\{\xi < 0\} = 0$. If ξ is a fuzzy variable with membership function $A(x)$, then the possibility measure of a fuzzy event $\xi \leq r$ $(r \in \mathbb{R})$ is defined as

$$Pos\{\xi \leq r\} = \sup_{x \leq r} A(x).$$

Also it follows that, like probability, the possibility measure on finite or countably infinite sets is determined by its behavior on singletons:

$$Pos\{u\} = \max_{\theta \in u} Pos(\{\theta\}).$$

Example 1. Let $\Theta = \{a, b, c, d, e, f\}$, and let $\Pi = \{1, 1, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\}$ be a possibility distribution over Θ. Then

$$\Pi\{a, b\} = \Pi\{b, c\} = \Pi\{b\} = 1, \qquad \Pi\{c, d\} = \frac{1}{2}, \qquad \Pi\{d, e, f\} = \frac{3}{4},$$

and, if $U = \{a, c, d\}$, then

$$\Pi(\theta \in U) = \sup_{\theta \in U} \pi(\{\theta\}) = 1.$$

Example 2 ([24]). Let A be a fuzzy number with the following membership function (Fig. 1).

$$A(x) = \begin{cases} \dfrac{x}{3} & 0 \leq x \leq 3, \\ 4 - x & 3 \leq x \leq 4. \end{cases}$$

Then, we have $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)] = [3\gamma, 4 - \gamma]$, and hence

$$Pos(A \leq \underline{a}(\gamma)) = \sup_{x \leq \underline{a}(\gamma)} A(x) = \gamma,$$
$$Pos(A \geq \overline{a}(\gamma)) = \sup_{x \geq \overline{a}(\gamma)} A(x) = \gamma.$$

3. The nearest weighted interval and point approximations

In this section, we propose an interval operator of a fuzzy number, which is called the nearest weighted possibilistic interval approximation. First, an f-weighted distance quantity is introduced on the fuzzy numbers, and then the interval and point approximations are obtained for a fuzzy number (see [22,25,24]).

Definition 5 ([24]). Suppose that $A \in \mathcal{F}$ is a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$. A possibilistic distance quantity of A is defined as

$$d(A, C_d(A)) = \left[\int_0^1 Pos(A \leq \underline{a}(\gamma))(\underline{a}(\gamma) - C_L)^2 \mathrm{d}\gamma + \int_0^1 Pos(A \geq \overline{a}(\gamma))(\overline{a}(\gamma) - C_U)^2 \mathrm{d}\gamma\right]^{\frac{1}{2}}$$

$$d(A, C_d(A)) = \sqrt{\int_0^1 \gamma(\underline{a}(\gamma) - C_L)^2 \mathrm{d}\gamma + \int_0^1 \gamma(\overline{a}(\gamma) - C_U)^2 \mathrm{d}\gamma}, \tag{2}$$

where $C_d(A) = [C_L, C_U]$ is an interval of support function, i.e. $[C_d(A)]_\gamma = [C_L, C_U]$. In fact, the relationship (2) is a type expected distance between the endpoints of its level sets and two points of support function fuzzy number. In [24], it is shown that the interval

$$C_d(A) = [C_L, C_U] = \left[2\int_0^1 \gamma \underline{a}(\gamma)\mathrm{d}\gamma, 2\int_0^1 \gamma \overline{a}(\gamma)\mathrm{d}\gamma\right]$$

is the nearest interval to A with respect to (2). Similarly, we consider the f-weighted distance quantity as

$$d_f(A, C_d^f(A)) = \sqrt{\frac{1}{2}\int_0^1 \left[\underline{f}(\gamma)(\underline{a}(\gamma) - C_L^f(A))^2 + \overline{f}(\gamma)(\overline{a}(\gamma) - C_U^f(A))^2\right]\mathrm{d}\gamma}, \tag{3}$$

where $C_d^f(A) = [C_L^f, C_U^f]$ is an interval of support function, and $f = (\underline{f}, \overline{f}) : ([0, 1], [0, 1]) \to (\mathbb{R}, \mathbb{R})$ is a weighting function such that the functions $\underline{f}, \overline{f}$ are non-negative, monotone increasing, and satisfy the following normalization condition:

$$\int_0^1 \underline{f}(\gamma)\mathrm{d}\gamma = 1, \qquad \int_0^1 \overline{f}(\gamma)\mathrm{d}\gamma = 1.$$

Note that if $g = (\underline{g}, \overline{g}) : ([0, 1], [0, 1]) \to (\mathbb{R}, \mathbb{R})$ is a function that is non-negative and monotone increasing, then we can consider

$$\underline{f}(\gamma) = \frac{\underline{g}(\gamma)}{\int_0^1 \underline{g}(\gamma)\mathrm{d}\gamma}, \qquad \overline{f}(\gamma) = \frac{\overline{g}(\gamma)}{\int_0^1 \overline{g}(\gamma)\mathrm{d}\gamma}.$$

Now we minimize (3) with respect to C_L^f, C_U^f. In order to minimize $d_f(A, C_d^f(A))$ it suffices to minimize the function $D(C_L^f, C_U^f) = d_f^2(A, C_d^f(A))$. Thus we have to find partial derivatives

$$\frac{\partial D(C_L^f, C_U^f)}{\partial C_L^f} = -\int_0^1 \underline{f}(\gamma)(\underline{a}(\gamma) - C_L^f)\mathrm{d}\gamma = -\int_0^1 \underline{f}(\gamma)\underline{a}(\gamma)\mathrm{d}\gamma + C_L^f$$

$$\frac{\partial D(C_L^f, C_U^f)}{\partial Cu} = -\int_0^1 \overline{f}(\gamma)(\overline{a}(\gamma) - C_U^f)\mathrm{d}\gamma = -\int_0^1 \overline{f}(\gamma)\overline{a}(\gamma)\mathrm{d}\gamma + C_U^f.$$

By solving the equations $\frac{\partial D(C_L^f, C_U^f)}{\partial C_L^f} = 0$ and $\frac{\partial D(C_L^f, C_U^f)}{\partial C_U^f} = 0$, we get

$$C_L^f = \int_0^1 \underline{f}(\gamma)\underline{a}(\gamma)\mathrm{d}\gamma, \qquad C_U^f = \int_0^1 \overline{f}(\gamma)\overline{a}(\gamma)\mathrm{d}\gamma. \tag{4}$$

Moreover, since $det \begin{pmatrix} \frac{\partial D^2(C_L^f, C_U^f)}{\partial (C_L^f)^2} & \frac{\partial D^2(C_L^f, C_U^f)}{\partial C_L^f \partial C_U^f} \\ \frac{\partial D^2(C_L^f, C_U^f)}{\partial C_U^f \partial C_L^f} & \frac{\partial D^2(C_L^f, C_U^f)}{\partial (C_U^f)^2} \end{pmatrix} = det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 > 0$, and $\frac{\partial D^2(C_L^f, C_U^f)}{\partial (C_L^f)^2} = 1 > 0$, $\frac{\partial D^2(C_L^f, C_U^f)}{\partial (C_U^f)^2} = 1 > 0$, then C_L^f and C_U^f given by (4) actually minimize $D(C_L^f, C_U^f)$ and simultaneously minimize $d_f(A, C_d^f(A))$. Therefore, the interval $[C_L^f, C_U^f]$ is indeed the nearest weighted interval approximation to fuzzy number A with respect to the f-weighted distance quantity.

Definition 6. Let $A \in \mathcal{F}$ be a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ and $f(\gamma) = (\underline{f}(\gamma), \overline{f}(\gamma))$ be a weighting function. We define the nearest $f = (\underline{f}, \overline{f})$-weighted interval approximation of A as

$$NWIA_f(A) = [C_L^f, C_U^f] = \left[\int_0^1 \underline{f}(\gamma)\underline{a}(\gamma)\mathrm{d}\gamma, \int_0^1 \overline{f}(\gamma)\overline{a}(\gamma)\mathrm{d}\gamma\right], \tag{5}$$

where C_L^f is the nearest lower weighted point approximation ($NLWPA_{\underline{f}}(A)$) and C_U^f is the nearest upper weighted point approximation ($NUWPA_{\overline{f}}(A)$) of fuzzy number A.

Theorem 1. *Let $A \in \mathcal{F}$ be a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ and $f(\gamma) = (\underline{f}(\gamma), \overline{f}(\gamma))$ be a weighted function. Then, the interval $NUWPA_{\overline{f}}(A) = [NLWPA_{\underline{f}}(A), NUWPA_{\overline{f}}(A)]$ is the nearest weighted interval approximation to fuzzy number A.*

Definition 7. Let $A \in \mathcal{F}$ be a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ and $f(\gamma) = (\underline{f}(\gamma), \overline{f}(\gamma))$ be a weighting function. We define the f-weighted mean of A as

$$\bar{M}_f(A) = \int_0^1 \frac{\underline{f}(\gamma)\underline{a}(\gamma) + \overline{f}(\gamma)\overline{a}(\gamma)}{2}\mathrm{d}\gamma. \tag{6}$$

In fact, $\bar{M}_f(A)$ is the weighting mean of fuzzy number A,

$$\bar{M}_f(A) = \frac{1}{2}\int_0^1 (\underline{f}(\gamma)\underline{a}(\gamma)\mathrm{d}\gamma + \bar{f}(\gamma)\bar{a}(\gamma))\mathrm{d}\gamma$$
$$= \frac{\int_0^1 \underline{f}(\gamma)\underline{a}(\gamma)\mathrm{d}\gamma + \int_0^1 \bar{f}(\gamma)\bar{a}(\gamma)\mathrm{d}\gamma}{\int_0^1 \underline{f}(\gamma)\mathrm{d}(\gamma) + \int_0^1 \bar{f}(\gamma)\mathrm{d}(\gamma)}. \tag{7}$$

Therefore, we have the following theorems.

Theorem 2. *Let $A \in \mathcal{F}$ be a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \bar{a}(\gamma)]$ and $f(\gamma) = (\underline{f}(\gamma), \bar{f}(\gamma))$ be a weighted function. Then $\bar{M}_f(A)$ is the nearest f-weighted point approximation to fuzzy number A which is unique.*

Proof. For the proof of theorem it suffices that we replace $C_L^f = C_U^f = \bar{M}_f(A)$ in (3), and then minimize function $d_f(A, \bar{M}_f(A))$ with respect to $\bar{M}_f(A)$. □

This theorem shows that $\bar{M}_f(A)$ is the nearest f-weighted point approximation to $A \in \mathcal{F}$ which is unique. The nearest f-weighted point to fuzzy number A belongs to support function, and this theorem, which is the main result of this section, is a new and interesting justification for the definition of the weighted mean of a fuzzy number.

Theorem 3. *Let $A, B \in \mathcal{F}$, let $f(\gamma) = (\underline{f}(\gamma), \bar{f}(\gamma))$ be a weighting function, and let $\lambda \in \mathbb{R}$. Then we have*

$$NLWPA_{\underline{f}}(A+B) = NLWPA_{\underline{f}}(A) + NLWPA_{\underline{f}}(B),$$
$$NLWPA_{\underline{f}}(\lambda A) = \lambda NLWPA_{\underline{f}}(A)$$
$$\bar{M}_f(A+B) = \bar{M}_f(A) + \bar{M}_f(A), \qquad \bar{M}_f(\lambda A) = \lambda \bar{M}_f(A).$$

Corollary 1. *Let $A = (a, b, c, d)$ be a trapezoidal fuzzy number, and let $f(\gamma) = (\underline{f}(\gamma), \bar{f}(\gamma))$ be a weighting function. Then the following hold.*

(1) *For $f(\gamma) = (1, 1)$, we have*

$$NWIA_f(A) = \left[\frac{a+b}{2}, \frac{c+d}{2}\right], \qquad \bar{M}_f(A) = \frac{a+b+c+d}{4}. \tag{8}$$

(2) *For $f(\gamma) = (2\gamma, 2\gamma)$,*

$$NWIA_f(A) = \left[\frac{a+2b}{3}, \frac{2c+d}{3}\right], \qquad \bar{M}_f(A) = \frac{a+2(b+c)+d}{6}. \tag{9}$$

(3) *For $f(\gamma) = (n\gamma^{n-1}, n\gamma^{n-1})$, $n \in \mathcal{N}$ (natural numbers),*

$$NWIA_f(A) = \left[\frac{a+nb}{n+1}, \frac{nc+d}{n+1}\right], \qquad \bar{M}_f(A) = \frac{a+n(b+c)+d}{2n+2}. \tag{10}$$

(4) *For $f(\gamma) = (n\gamma^{n-1}, m\gamma^{m-1})$, $n, m \in \mathcal{N}$,*

$$NWIA_f(A) = \left[\frac{a+nb}{n+1}, \frac{mc+d}{m+1}\right], \qquad \bar{M}_f(A) = \frac{a+nb}{2n+2} + \frac{d+mc}{2m+2}. \tag{11}$$

Proof. The proof is simple.

Corollary 2. *Let $A = (a, b, c, d)$ be a trapezoidal fuzzy number, and let $f(\gamma) = (m\gamma^{m-1}, n\gamma^{n-1})$, $m, n \in \mathcal{N}$ be a weighting function. Then, for $m, n \to \infty$,*

$$NWIA_f(A) = \left[\frac{a+mb}{m+1}, \frac{d+nc}{n+1}\right] \to [b, c],$$
$$\bar{M}_f(A) \to \frac{b+c}{2}.$$

The above corollary shows that, for large values m and n, the interval $[b, c]$ and the point $\frac{b+c}{2}$ are the nearest weighted interval and point to the trapezoidal fuzzy number, respectively (see Fig. 2).

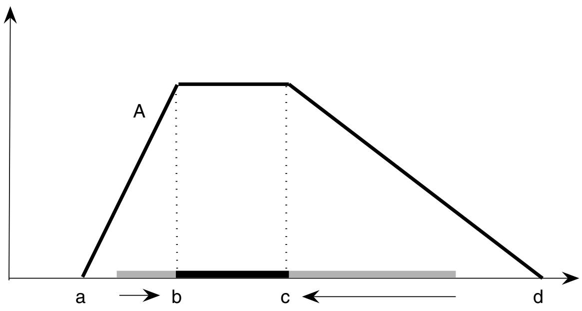

Fig. 2. Fuzzy number A and its intervals.

When the variables are fuzzy in nature, in order to find the degree of relationship between them, a measure of correlation coefficient is required that can compute the relation value between fuzzy variables. Here, we consider the correlation coefficient between two fuzzy number $A, B \in \mathcal{F}$ as

$$\rho_{f_A,f_B}(A,B) = \frac{NLWPA_{\underline{f}}(A)NLWPA_{\underline{f}}(B) + NUWPA_{\overline{f}}(A)NUWPA_{\overline{f}}(B)}{\sqrt{(NLWPA_{\underline{f}}(A))^2 + (NUWPA_{\overline{f}}(A))^2}\sqrt{(NLWPA_{\underline{f}}(B))^2 + (NUWPA_{\overline{f}}(B))^2}}.$$

The above-mentioned measure of correlation coefficient depends on the nearest f-weighted interval approximation to $A \in \mathcal{F}$. Also the choice of the weighting function $f(\gamma) = (\underline{f}(\gamma), \overline{f}(\gamma))$ plays a key role in the conclusions. Thus, for different choices of the weighting function, the value of the correlation coefficient will differ [24].

Example 3. Let $A = (a_1, b_1, c_1, d_1)$ and $B = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers, and let $f_A = (m_1\gamma^{m_1-1}, n_1\gamma^{n_1-1})$ and $f_B = (m_2\gamma^{m_2-1}, n_2\gamma^{n_2-1})$ be two weighting functions. Then we obtain the correlation coefficient between A and B as follows:

$$\rho_{f_A,f_B}(A,B) = \frac{\frac{a_1+m_1b_1}{m_1+1}\frac{a_2+m_2b_2}{m_2+1} + \frac{d_1+n_1c_1}{n_1+1}\frac{d_2+n_2c_2}{n_2+1}}{\sqrt{\left(\frac{a_1+m_1b_1}{m_1+1}\right)^2 + \left(\frac{d_1+n_1c_1}{n_1+1}\right)^2}\sqrt{\left(\frac{a_2+m_2b_2}{m_2+1}\right)^2 + \left(\frac{d_2+n_2c_2}{n_2+1}\right)^2}}. \tag{12}$$

For large values of m_1, n_1, m_2, n_2 we have

$$\rho_{f_A,f_B}(A,B) = \frac{b_1b_2 + c_1c_2}{\sqrt{b_1^2 + c_1^2}\sqrt{b_2^2 + c_2^2}}.$$

Therefore, the above results show that the weighting functions f_A, f_B can affect the correlation coefficient between two fuzzy numbers. It will also be a good thing to define a measure of correlation coefficient that is devoid of the γ-*cut* values.

Example 4. Let A be a fuzzy number with the following membership function and $f_1(\gamma) = (2\gamma, 2\gamma)$ be a weighting function.

$$A(x) = \begin{cases} 1 - \dfrac{(x-5)^2}{4} & 3 \le x \le 7, \\ 0 & otherwise. \end{cases}$$

Then we have

$$[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)] = [5 - 2\sqrt{1-\gamma}, 5 + 2\sqrt{1-\gamma}], \quad \gamma \in (0, 1],$$

and hence the nearest weighted interval to A is

$$NWIA_{f_1}(A) = [NLWPA_{f_1}(A), NUWPA_{f_1}(A)] = \left[\frac{59}{15}, \frac{91}{15}\right],$$

and its membership grade is at least 0.715. The nearest weighted point to fuzzy number A is $\bar{M}_f(A) = 5$ such that its membership grade is 1.

Also, let $f_2(\gamma) = (4\gamma^3, 4\gamma^3)$ be a weighting function. Then we obtain

$$NWIA_{f_2}(A) = [NLWPA_{f_2}(A), NUWPA_{f_2}(A)] = \left[\frac{1319}{315}, \frac{1831}{315}\right].$$

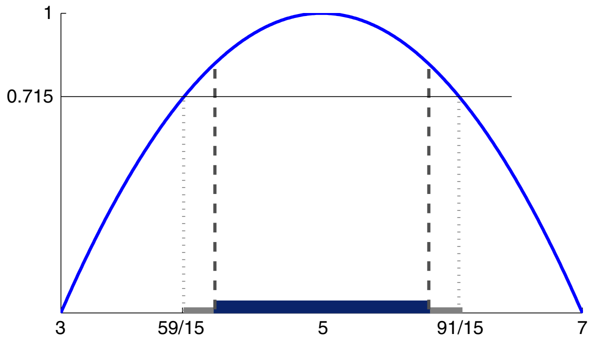

Fig. 3. Fuzzy number A from Example 4.

This example shows that weighting functions have an effect on the fuzzy intervals, namely $NWIA_{f_2}(A) \subseteq NWIA_{f_1}(A)$ (see Fig. 3).

4. Ranking fuzzy numbers by the weighting mean

This section proposes a new ranking method by the weighting mean of a fuzzy number.

Definition 8. For two fuzzy numbers $A, B \in \mathcal{F}$, and the weighting function f, we define the ranking of A and B by $\bar{M}_f(A)$, i.e.,

$\bar{M}_f(A) < \bar{M}_f(B)$ if and only if $A \prec B$,
$\bar{M}_f(A) = \bar{M}_f(B)$ if and only if $A \sim B$,
$\bar{M}_f(A) > \bar{M}_f(B)$ if and only if $A \succ B$.

Then we formulate the order $\preceq$ and $\succeq$ as

$A \preceq B$ if and only if $A \prec B$ or $A \sim B$,
$A \succeq B$ if and only if $A \succ B$ or $A \sim B$.

We consider the following reasonable properties for the ordering approaches (see [3]).

A_1: For an arbitrary finite subset Γ of $\mathcal{F}$ and $A \in \Gamma$, $A \succeq A$.
A_2: For an arbitrary finite subset Γ of $\mathcal{F}$ and $(A, B) \in \Gamma^2$, $A \succeq B$ and $B \succeq A$, we should have $A \sim B$.
A_3: For an arbitrary finite subset Γ of $\mathcal{F}$ and $(A, B, C) \in \Gamma^3$, $A \succeq B$ and $B \succeq C$, we should have $A \succeq C$.
A_4: For an arbitrary finite subset Γ of $\mathcal{F}$ and $(A, B) \in \Gamma^2$, $infsup(A) \geq infsup(B)$, we should have $A \succeq B$.
A'_4: For an arbitrary finite subset Γ of $\mathcal{F}$ and $(A, B) \in \Gamma^2$, $infsup(A) > infsup(B)$, we should have $A \succ B$.
A'_5: Let Γ and Γ' be two arbitrary finite subset of $\mathcal{F}$; also, A and B are in $\Gamma \cap \Gamma'$. We obtain the ranking order $A \succ B$ by $\bar{M}_f(.)$ on Γ' if and only if $A \succ B$ by $\bar{M}_f(.)$ on Γ.
A_6: Let $A, B, A + C$ and $B + C$ be elements of $\mathcal{F}$. If $A \succeq B$, then $A + C \succeq B + C$.
A'_6: Let $A, B, A + C$, and $B + C$ be elements of $\mathcal{F}$. If $A > B$, then $A + C \succ B + C$.
A_7: For an arbitrary finite subset Γ of $\mathcal{F}$ and $A \in \Gamma$, $\bar{M}_f(A)$ must belong to its support.

Theorem 4. *The function $\bar{M}_f$ has the properties $A_1, A_2, \ldots, A_7$.*

Proof. It is easy to verify that the properties A_1–A_6 hold. For the proof of A_7 we consider the fuzzy number $[A]_\gamma = [\underline{a}(\gamma), \bar{a}(\gamma)]$, and the weighting function $f(\gamma) = (\underline{f}(\gamma), \bar{f}(\gamma))$. For all $\gamma \in [0, 1]$, we have $a \leq \underline{a}(\gamma) \leq \bar{a}(\gamma) \leq d$; hence

$$\frac{\underline{f}(\gamma)a + \bar{f}(\gamma)a}{2} \leq \frac{\underline{f}(\gamma)\underline{a}(\gamma) + \bar{f}(\gamma)\bar{a}(\gamma)}{2} \leq \frac{\underline{f}(\gamma)d + \bar{f}(\gamma)d}{2},$$

so

$$\int_0^1 \frac{\underline{f}(\gamma)a + \bar{f}(\gamma)a}{2} \mathrm{d}\gamma \leq \int_0^1 \frac{\underline{f}(\gamma)\underline{a}(\gamma) + \bar{f}(\gamma)\bar{a}(\gamma)}{2} \mathrm{d}\gamma \leq \int_0^1 \frac{\underline{f}(\gamma)d + \bar{f}(\gamma)d}{2} \mathrm{d}\gamma,$$

or

$$\frac{a}{2} \int_0^1 (\underline{f}(\gamma) + \bar{f}(\gamma))\mathrm{d}\gamma \leq \int_0^1 \frac{\underline{f}(\gamma)\underline{a}(\gamma) + \bar{f}(\gamma)\bar{a}(\gamma)}{2} \mathrm{d}\gamma \leq \frac{d}{2} \int_0^1 (\underline{f}(\gamma) + \bar{f}(\gamma))\mathrm{d}\gamma,$$

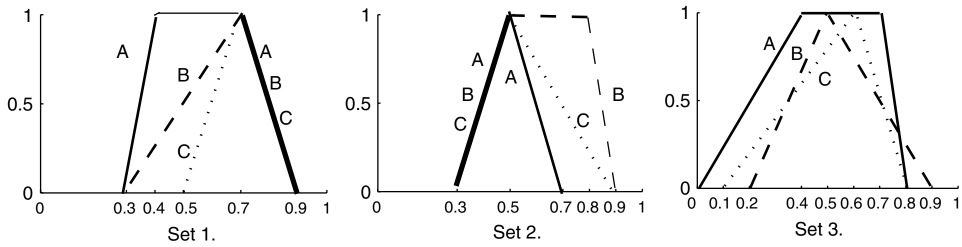

Fig. 4. Fuzzy numbers A, B, C in sets 1, 2, 3.

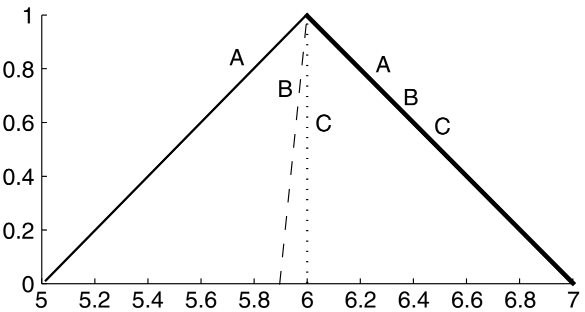

Fig. 5. Fuzzy numbers A, B, C from Example 6.

and this implies that

$$a \leq \bar{M}_f(A) \leq d. \quad \square$$

Example 5. Consider the following sets (Fig. 4); see [26,19].

Set 1: $A = (0.3, 0.4, 0.7, 0.9)$, $B = (0.3, 0.7, 0.9)$, $C = (0.5, 0.7, 0.9)$.

Set 2: $A = (0.3, 0.5, 0.7)$, $B = (0.3, 0.5, 0.8, 0.9)$, $C = (0.3, 0.5, 0.9)$.

Set 3: $A = (0, 0.4, 0.7, 0.8)$, $B = (0.2, 0.5, 0.9)$, $C = (0.1, 0.6, 0.8)$.

Table 1 gives the results.

In set 1, the ranking result by our method and by nine other methods is $A \prec B \prec C$. One can see that this result is more desirable with respect to the other six methods (see Fig. 4, set 1).

In set 2, the ranking result for the four methods is $A \prec B \prec C$. Our method has the same result as in other eleven papers ($A \prec C \prec B$). We conclude that $A \prec C \prec B$ is better than $A \prec B \prec C$ (see Fig. 4, set 2).

In set 3 (Fig. 4, set 3), our method has the same result as in four papers of Choobineh and Li, Yager, Chen, and Goetschel and Voxman. The ranking result by Baldwin and Guild, Yao and Wu, Abbasbandy and Asady, and Asady and Zendehnam is $A \prec B \sim C$. By Cheng's distance method and that of Chu and Taso, the ranking order is $A \prec C \prec B, B \prec C \prec A$. For the CV index, the ranking order is $C \prec A \prec B$. By the Wang distance method, the Wang et al. centroid method, and the Deng et al. area method, the ranking order is $C \prec B \prec A$. However some of methods have a shortcoming.

Example 6. Consider the three triangular fuzzy numbers $A = (5, 6, 7)$, $B = (5.9, 6, 7)$, $C = (6, 6, 7)$ (Fig. 5). The results are given in Table 2.

In Table 2, the ranking results by the Chu and Tsao method, the Cheng CV index, the Wang et al. centroid method, and the Deng et al. area method are unreasonable. By the proposed method, the ranking result is $A \prec B \prec C$ (see Fig. 5).

Example 7. Consider fuzzy number A_n (Fig. 6) as follows:

$$A_n(x) = \begin{cases} \left(\dfrac{x-a}{b-a}\right)^n & x \in [a, b), \\ 1 & x \in [b, c), \\ \left(\dfrac{d-x}{d-c}\right)^n & x \in [c, d), \\ 0 & otherwise. \end{cases}$$

Table 1
Comparative results of Example 5.

Authors	Fuzzy number	Set 1	Set 2	Set 3
	A	0.458	0.333	0.50
Choobineh and Li	B	0.583	0.4167	0.5833
	C	0.667	0.5417	0.6111
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
	A	0.5778	0.5	0.4336
Yager	B	0.6333	0.6222	0.5353
	C	0.8571	0.6986	84
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
	A	0.4315	0.375	0.52
Chen	B	0.5625	0.425	0.57
	C	0.625	0.55	0.625
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
	A	0.27	0.27	0.40
Baldwin and Guild	B	0.27	0.37	0.42
	C	0.37	0.45	0.42
Results		$A \sim B \prec C$	$A \prec B \prec C$	$A \prec B \sim C$
	A	0.2847	0.25	0.24402
Chu and Tsao	B	0.32478	0.31526	0.26243
	C	0.350	0.27475	0.2619
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
	A	0.7577	0.7071	0.7106
Cheng distance	B	0.8149	0.8037	0.7256
	C	0.8602	0.7458	0.7241
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
	A	0.2568	0.1778	0.1967
Wang et al. centroid	B	0.2111	0.2765	0.1778
	C	0.2333	0.1889	0.1667
Results		$B \prec C \prec A$	$A \prec C \prec B$	$C \prec B \prec A$
	A	0.7289	0.6009	0.6284
Wang distance	B	0.7157	0.7646	0.6289
	C	0.7753	0.6574	0.6009
Results		$B \prec A \prec C$	$A \prec C \prec B$	$C \prec A \prec B$
	A	0.575	0.5	0.475
Yao and Wu	B	0.65	0.625	0.525
	C	0.7	0.55	0.525
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
	A	0.3169	0.2369	0.2523
Deng et al. area method	B	0.3240	0.3503	0.2495
	C	0.3240	0.2549	0.2473
Results		$A \prec B \sim C$	$A \prec C \prec B$	$C \prec B \prec A$
	A	0.0328	0.0133	0.0693
Cheng CV uniform distribution	B	0.0246	0.0304	0.0385
	C	0.0095	0.0275	0.0433
Results		$C \prec B \prec A$	$A \prec C \prec B$	$B \prec C \prec A$
	A	0.026	0.008	0.0471
Cheng CV proportional distribution	B	0.0146	0.0234	0.0236
	C	0.0057	0.0173	0.0255
Results		$C \prec B \prec A$	$A \prec C \prec B$	$B \prec C \sim A$
	A	0.5667	0.5	0.5
Goetschel and Voxman	B	0.6667	0.6333	0.5167
	C	0.7	0.5333	0.55
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
	A	1.15	1	0.95
Abbasbandy and Asady sign distance	B	1.3	1.25	1.05
	C	1.4	1.1	1.05
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
	A	0.575	0.5	0.475
Asady and Zendehnam	B	0.65	0.625	0.525
	C	0.7	0.55	0.525

Table 1 (*continued*)

Authors	Fuzzy number	Set 1	Set 2	Set 3
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
	A	0.5667	0.5	0.5
Propose method $f(\gamma) = (2\gamma, 2\gamma)$	B	0.6667	0.6333	0.5167
	C	0.7	0.5333	0.55
Results		$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$

Table 2
Comparative results of Example 6.

Fuzzy number	A	B	C
Proposed method, $f(\gamma) = (1, 1)$	6	6.225	6.25
Result	$A \prec B \prec C$		
Proposed method, $f(\gamma) = (2\gamma, 2\gamma)$	6	6.15	6.1667
Result	$A \prec B \prec C$		
Proposed method, $f(\gamma) = (3\gamma^2, 3\gamma^2)$	6	6.1125	6.875
Result	$A \prec B \prec C$		
Chu and Tsao	3	3.125	3.085
Result	$A \prec C \prec B$		
CV index	0.028	.0098	0.0089
Result	$C \prec B \prec A$		
Deng et al. area method (ROG)	2.6719	2.5786	2.5874
Result	$B \prec C \prec A$		
Wang et al. centroid	2	2.1	1.44
Results	$C \prec A \prec B$		

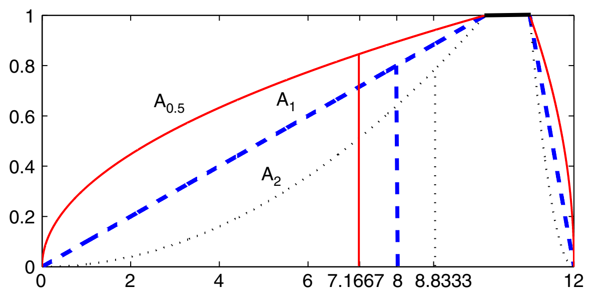

Fig. 6. Fuzzy numbers $A_{0.5}, A_1, A_2$ from Example 7.

We obtain that

$$\text{if } f(\gamma) = (1, 1) \text{ then, } \bar{M}(A_{0.5}) = 7.1667, \bar{M}(A_1) = 8, \bar{M}(A_2) = 8.8333,$$

and hence the ranking order is $A_{0.5} \prec A_1 \prec A_2$;

$$\text{if } f(\gamma) = (2\gamma, 2\gamma) \text{ then, } \bar{M}(A_{0.5}) = 8, \bar{M}(A_1) = 8.8333, \bar{M}(A_2) = 9.5,$$

and hence $A_{0.5} \prec A_1 \prec A_2$.

Note that decision makers can select other suitable functions for ranking fuzzy numbers. Therefore this method of ranking fuzzy numbers is new and flexible.

Example 8. Let $[A]_\gamma = [3\gamma, 4-\gamma]$, $[B]_\gamma = [1+\gamma, 5-3\gamma]$, and $[C]_\gamma = [2+\frac{1}{2}\gamma, 3-\frac{1}{2}\gamma]$ be three triangular fuzzy numbers (Fig. 7).

If $f(\gamma) = (1, 1)$, then $\bar{M}_f(A) = 2.5$, $\bar{M}_f(B) = 2.5$, $\bar{M}_f(C) = 2.5$; hence, the ranking result is $A \sim B \sim C$.
If $f(\gamma) = (2\gamma, 2\gamma)$, then,

$$\bar{M}_f(A) = 2.667, \qquad \bar{M}_f(B) = 2.333, \qquad \bar{M}_f(C) = 2.5;$$

hence, the ranking result is $B \prec C \prec A$.

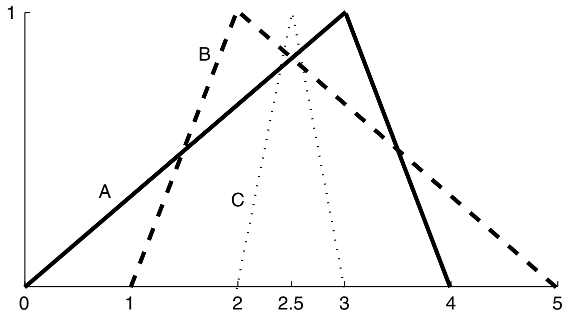

Fig. 7. Fuzzy numbers A, B, C from Example 8.

If $f(\gamma) = (2\gamma, 3\gamma^2)$, then

$$\bar{M}_f(A) = 2.625, \qquad \bar{M}_f(B) = 2.208, \qquad \bar{M}_f(C) = 2.479;$$

therefore, the fuzzy numbers are ranked as $B \prec C \prec A$.

Note that the function f is selected by decision maker and it depends on different problems in fuzzy environments.

We give an interesting example that the nearest weighted interval and point of a fuzzy number can be applied for a variety of problems in a fuzzy environment, namely the fuzzy analytical hierarchical process (FAHP). These approximations can apply for transforming the fuzzy comparison matrices into the interval and point comparison matrices.

Example 9. Consider the following fuzzy comparison matrix:

$$\widetilde{M} = \begin{bmatrix} (1,1,1) & (1,2,3) & (3,4,5) & (7,8,9) \\ & (1,1,1) & (1,2,3) & (3,4,5) \\ & & (1,1,1) & (1,2,3) \\ & & & (1,1,1) \end{bmatrix},$$

where its elements are triangular fuzzy numbers.

Suppose that $A = (1, 2, 3)$, $B = (3, 4, 5)$, and $C = (7, 8, 9)$ are triangular fuzzy numbers. And let $f_A = (1, 1), f_B = (2\gamma, 2\gamma), f_c = (3\gamma^2, 3\gamma^2)$ be weighting functions for the fuzzy numbers A, B, C; respectively. We obtain the interval and point comparison matrices as follows:

$$IM_f = \begin{bmatrix} [1,1] & \left[\frac{3}{2}, \frac{5}{2}\right] & \left[\frac{11}{3}, \frac{13}{3}\right] & \left[\frac{15}{4}, \frac{17}{4}\right] \\ & [1,1] & \left[\frac{3}{2}, \frac{5}{2}\right] & \left[\frac{11}{3}, \frac{13}{3}\right] \\ & & [1,1] & \left[\frac{3}{2}, \frac{5}{2}\right] \\ & & & [1,1] \end{bmatrix},$$

$$PM_f = \begin{bmatrix} 1 & 2 & 4 & 8 \\ & 1 & 2 & 4 \\ & & 1 & 2 \\ & & & 1 \end{bmatrix},$$

and therefore the fuzzy comparison matrix $\widetilde{M}$ is transformed into a interval comparison matrix IM_f and a crisp point comparison matrix PM_f. Also we get the ranking result of fuzzy numbers as $A \prec B \prec C$, and

$$\rho_{f_A,f_B}(A, B) = 0.9869, \qquad \rho_{f_B,f_C}(B, C) = 0.9987, \qquad \rho_{f_A,f_C}(A, C) = 0.9834.$$

One can see that the correlation coefficient between the two fuzzy numbers A and B is greater than the correlation coefficient between the two fuzzy numbers A and C.

Furthermore, if $A = (1, 2, 3)$, $B = (2, 3, 4)$, and $C = (2, 4, 5)$ are three triangular fuzzy numbers and $f_A = (2\gamma, 2\gamma)$, $f_B = (2\gamma, 3\gamma^2)$, $f_C = (3\gamma^2, 4\gamma^3)$ are weighting functions, then we have

$$IM_f = \begin{bmatrix} [1,1] & \left[\frac{5}{3}, \frac{7}{3}\right] & \left[\frac{8}{3}, \frac{13}{4}\right] & \left[\frac{7}{2}, \frac{21}{5}\right] \\ & [1,1] & \left[\frac{5}{3}, \frac{7}{3}\right] & \left[\frac{8}{3}, \frac{13}{4}\right] \\ & & [1,1] & \left[\frac{5}{3}, \frac{7}{3}\right] \\ & & & [1,1] \end{bmatrix},$$

$$PM_f = \begin{bmatrix} 1 & 2 & \frac{71}{24} & \frac{77}{20} \\ & 1 & 2 & \frac{71}{24} \\ & & 1 & 2 \\ & & & 1 \end{bmatrix}.$$

Therefore, the ranking result is $A \prec B \prec C$, and we have

$$\rho_{f_A,f_B}(A, B) = 0.9978, \qquad \rho_{f_B,f_C}(B, C) = 0.9999, \qquad \rho_{f_A,f_C}(A, C) = 0.9872.$$

The above correlation coefficients show that the weighting functions have an effect on the ranking fuzzy numbers. However, comparison matrices can apply in the industry engendering problems, especially in the FAHP.

5. Discussion and conclusion

In this paper the following properties are discussed.

1. We have applied a weighted distance in the metric space of fuzzy numbers. According to this distance, the nearest weighed interval and point approximations to a fuzzy number are obtained, such that the approximations are unique.

2. The function $\bar{M}_f(.)$ is applied for ranking fuzzy numbers and this topic is not mentioned in previous papers. The properties of $\bar{M}_f(.)$ are given by theorems and corollaries.

3. For weighting function $f(\gamma) = (1, 1)$, $\bar{M}_f(A)$ is the index of ranking fuzzy numbers that was introduced by Asady and Zendehnam [26] as

$$\bar{M}_f(A) = \frac{\int_0^1 \underline{a}(\gamma) + \overline{a}(\gamma)\mathrm{d}\gamma}{2}.$$

4. For weighting function $f(\gamma) = (2\gamma, 2\gamma)$, $\bar{M}_f(A)$ is the index of ranking fuzzy numbers that was introduced by Goetschel and Voxman [17] as

$$\bar{M}_f(A) = \int_0^1 \gamma(\underline{a}(\gamma) + \bar{a}(\gamma))\mathrm{d}\gamma.$$

5. The flexibility is one of the most important properties of our ranking method, because decision makers can select different weighting functions as $f = (\underline{f}, \overline{f})$ such that functions $\underline{f}, \overline{f} : [0, 1] \to R$ are weighting functions for the lower and upper γ-*cuts* sets of a fuzzy number, respectively. This means that the functions $\underline{f}(\gamma)$ and $\overline{f}(\gamma)$ can be treated as subjective weights indicating neutral, optimistic, or pessimistic preferences of the decision maker. Therefore, our method is more general and interesting for ranking fuzzy numbers.

6. For weighting function $\underline{f}(\gamma) = \overline{f}(\gamma)$, $\bar{M}_f(A)$ is defined as the f-weighted possibilistic mean of the fuzzy number A [27,24]. Also, the maximum entropy of weighting function $(f(\gamma))$ is discussed in [28]. It can be used to choose a suitable weighting function.

In general, the aim of this paper is threefold. The first aim is to find out the nearest weighted interval approximation of a fuzzy number. The second one is to obtain the nearest weighted point approximations of a fuzzy number. The last one is a new flexibility ranking method of the fuzzy numbers by their the f-weighted mean.

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