Journal of Mathematical Analysis and Applications **246,** 217–229 (2000)
doi:10.1006/jmaa.2000.6783, available online at http://www.idealibrary.com on IDEAL®

Geometry of Three-Homogeneous Polynomials on Real Hilbert Spaces

Bogdan C. Grecu

Submitted by William F. Ames

Received June 28, 1999

Let H be a two-dimensional real Hilbert space. We give a characterisation of the extreme and the smooth points of the unit ball of the space of three-homogeneous polynomials on H in terms of the coefficients of the polynomial. We also determine the smooth points of the unit ball of the predual of the space of three-homogeneous polynomials.

1. INTRODUCTION

In this paper we describe the geometry of the unit ball of the space of three-homogeneous polynomials on a two-dimensional real Hilbert space. In the last few years some work has been done in the geometry of spaces of polynomials on low-dimensional spaces. Recently Choi and Kim [2] studied the two-homogeneous polynomials on a two-dimensional Hilbert space and obtained characterisations of smooth and extreme polynomials in terms of the coefficients of the polynomial. The geometry of spaces of polynomials has also been the subject of the papers by Boyd and Ryan [1], Ryan and Turett [7], Sundaresan [8], and the author [5].

We start by presenting a general result about smooth polynomials in finite dimensions. Then we deal with the three-homogeneous polynomials. First we prove that the angle between any two unit vectors x and y at which a three-homogeneous polynomial of unit norm P takes the value 1 is smaller than $2\pi/3$. Then we move on to space of the three-homogeneous polynomials on a two-dimensional Hilbert space H and find a characterisation of the extreme and smooth points of its unit ball by the number of the points at which a polynomial takes the value 1. Using the

0022-247X/00 $35.00

form of the extreme polynomials thus found, we give a characterisation of the smooth points of the unit ball of the predual of the space of three-homogeneous polynomials.

2. NOTATION, TERMINOLOGY, AND GENERAL RESULTS

We say that P is an n-homogeneous polynomial on a (real or complex) normed space X if there exists an n-linear form B on the product X^n such that $P(x) = B(x, \dots, x)$ for every x in X. We denote by $\mathcal{P}(^n X)$ the space of all continuous n-homogeneous polynomials on X endowed with the natural norm $\|P\| = \sup\{|P(x)| : \|x\| = 1\}$ and by $\mathcal{L}_s(^n X)$ the space of continuous symmetric n-linear forms on X with the sup norm. According to the polarization formula [3] for each P there exists a unique A in $\mathcal{L}_s(^n X)$ such that $P(x) = A(x, x, \dots, x)$. We have

$$\|P\| \le \|A\| \le \frac{n^n}{n!}\|P\|,$$

so these two spaces are isomorphic. Furthermore if X is a Hilbert space then $\|P\| = \|A\|$, so we have an isometry. $\mathcal{P}(^n X)$ is always a dual space. Indeed $\mathcal{L}_s(^n X)$ is the dual of the n-fold symmetric projective tensor product. The space $\otimes_s^n X$ can be renormed such that $\mathcal{P}(^n X)$ becomes its dual [6]. Denote by x^n the tensor $x \otimes x \otimes \cdots \otimes x$. If u belongs to the uncompleted n-fold symmetric tensor product, then u can be expressed as a linear combination of tensors of the form x^n. Define the norm

$$\|u\|_x = \inf\left\{ \sum_{j=1}^{k} |\lambda_j| \, \|x_j\|^n : u = \sum_{j=1}^{k} \lambda_j x_j^n \right\}.$$

We denote by $\hat{\otimes}_{s,\pi}^n X$ the complemented tensor product endowed with this norm. Then $\mathcal{P}(^n X)$ is the dual space of $\hat{\otimes}_{s,\pi}^n X$, the duality being given by $P(u) = \sum_{j=1}^k \lambda_j P(x_j)$. The unit ball of $\hat{\otimes}_{s,\pi}^n X$ is the closed absolutely convex hull of $\{x^n : x \in B_X\}$. If X is finite dimensional then so is the n-fold tensor product, so its unit ball is the absolutely convex hull of that set. For more details on polynomials and symmetric tensor products see [3, 4].

A unit vector x in a normed space X is an extreme point of B_X if x is not the midpoint of a nontrivial segment lying in B_X. A unit vector x is a smooth point of B_X if there exists exactly one linear functional ϕ in B_{X^*} such that $\phi(x) = 1$.

We are interested in finding the extreme and the smooth three-homogeneous polynomials on a two-dimensional real Hilbert space. We start with a general result about smooth n-homogeneous polynomials on a finite-dimensional normed space.

PROPOSITION 1. *Let X be a finite-dimensional (complex or real) normed space and let P be a norm one n-homogeneous polynomial on X. The polynomial P is a smooth point of $B_{\mathscr{P}(^nX)}$ if and only if for any two linearly independent unit vectors x and y in X we cannot have $|P(x)| = |P(y)| = 1$.*

Proof. Let P be a smooth polynomial. Since X is finite dimensional, there exists x in B_X such that $P(x) = \lambda$ with $|\lambda| = 1$. Thus $\bar{\lambda} x^n \in \otimes_s^n X$ is a norm one linear functional in $\mathscr{P}(^nX)^* = (\otimes_{s,\pi}^n X)^{**} = \otimes_{s,\pi}^n X$ which norms P. Suppose there is another y in B_X such that $P(y) = \mu$ with $|\mu| = 1$. Then $\bar{\mu} x^n(P) = 1$ and since P is smooth, $\bar{\lambda} x^n = \bar{\mu} y^n$; hence x and y are linearly dependent.

Conversely, suppose that the second assertion holds. Since X is finite dimensional there is a unit vector x such that $P(x) = \mu$ with $|\mu| = 1$. Then $\bar{\mu} x^n(P) = 1$. We will show that this is the only norm one element u of $B_{\otimes_{s,\pi}^n X}$ such that $u(P) = 1$. Indeed let $u \in B_{\otimes_{s,\pi}^n X}$. It can be written as

$$u = \lambda x^n + \sum_{i=1}^{k} \lambda_k x_k^n,$$

with x_k elements of B_X, all of them independent of x and $\Sigma |\lambda_k| + |\lambda| = 1$. Then $u(P) = \lambda P(x) + \Sigma \lambda_k P(x_k)$. If one of the λ_k's is nonzero, then $|u(P)| < |\lambda| + \Sigma |\lambda_k| = 1$. Thus $|u(P)| = 1$ if and only if all the λ_k's are zero, so $u = \lambda x^n$. Then $u(P) = \lambda \mu = 1$ if and only if $\lambda = \bar{\mu}$ so $u = \bar{\mu} x^n$. Hence P is smooth. ∎

Remark 2. Sundaresan [8] obtained that a two-homogeneous polynomial on an n-dimensional real Hilbert space is smooth if and only if just one of the eigenvalues of the $n \times n$ symmetric matrix associated with P has absolute value 1. This can be easily deduced from the proposition we have just proved.

Let us note that the sufficient condition in the proposition above is equivalent to $|P|$ taking the value 1 on B_X only at the vectors λx for some $x \in B_X$ and $\lambda^n = \pm 1$.

COROLLARY 3. *Let X be a finite-dimensional real normed space and let P be a norm one n-homogeneous polynomial with n odd. Then P is smooth if and only if there is only one vector x in B_X such that $P(x) = 1$.*

Proof. According to the proposition P is smooth if and only if the points in B_X where $|P|$ takes the value 1 are of the form λx with $\lambda^n = \pm 1$ for some unit vector x. Since n is odd, $\lambda^n = \pm 1$ has only the solutions 1 and -1. Suppose that $P(x) = 1$. Then $P(-x) = -1$, so there is just one unit vector x such that $P(x) = 1$. ▮

3. THREE-HOMOGENEOUS POLYNOMIALS ON A REAL HILBERT SPACE

Let P be a three-homogeneous polynomial on a real Hilbert space H and let A be its associated symmetric three-linear form. By the Riesz representation theorem there exists a symmetric bounded bilinear mapping B: $H \times H \to H$ such that $A(x, y, z) = \langle B(x, y), z \rangle$. Furthermore $\|B\| = \|A\| = \|P\|$. Let Q: $H \to H$ be the two-homogeneous vector valued polynomial defined by $Q(x) = B(x, x)$. Since we are working on a Hilbert space, $\|Q\| = \|B\|$ (see [3]). Therefore if we start with a norm one polynomial P we obtain a norm one vector valued polynomial Q such that $P(x) = \langle Q(x), x \rangle$. Thus if x is a unit vector then $P(x) = 1$ is equivalent to $Q(x) = x$ and $P(x) = -1$ to $Q(x) = -x$.

We are going to work on a two-dimensional real Hilbert space and we shall classify the extreme and the smooth polynomials according to the number of unit vectors where $P(x) = 1$. We shall see that we can have at most three such distinct points.

Now let us consider a three-homogeneous polynomial P of unit norm on a real Hilbert space H. We have:

PROPOSITION 4. *Let P be a three-homogeneous polynomial of unit norm on a real Hilbert space H. If there exist elements x and y of B_H such that $P(x) = P(y) = 1$ then $\langle x, y \rangle \geq -1/2$ and so the angle between x and y is at most $2\pi/3$.*

Proof. Since $P(x) = P(y) = 1$ we have $Q(x) = x$ and $Q(y) = y$. Then

$$\begin{aligned} P(x+y) &= A(x+y, x+y, x+y) \\ &= P(x) + 3A(x,x,y) + 3A(x,y,y) + P(y) \\ &= P(x) + 3\langle Q(x), y\rangle + 3\langle Q(y), x\rangle + P(y) \\ &= 2 + 6\langle x, y\rangle. \end{aligned}$$

At the same time $|P(x+y)| \leq \|x+y\|^3 = (2 + 2\langle x, y\rangle)^{3/2}$. Let $a = \langle x, y \rangle$. We must have

$$|2 + 6a| \leq (2 + 2a)^{3/2},$$

so

$$4 + 24a + 36a^2 \leq 8 + 24a + 24a^2 + 8a^3,$$

which yields

$$8a^3 - 12a^2 + 4 = 4(a-1)^2(2a+1) \geq 0.$$

Thus $2a + 1 \geq 0$; hence $\langle x, y \rangle \geq -1/2$. ∎

From now on we restrict our attention to a two-dimensional real Hilbert space that we denote by H.

4. EXTREME POINTS OF THE UNIT BALL OF $\mathcal{P}(^3H)$

Let P be a three-homogeneous polynomial of unit norm on the two-dimensional real Hilbert space H. There exists a unit vector w such that $P(w) = 1$. Consider an orthonormal basis $\{e_1, e_2\}$ of H such that $e_1 = w$. Since $P(e_1) = 1$ we have $Q(e_1) = e_1$, so $A(e_1, e_1, x) = \langle Q(e_1), x \rangle = \langle e_1, x \rangle$ for all x in H. Thus $A(e_1, e_1, e_2) = 0$ and the expression of the polynomial P in the basis $\{e_1, e_2\}$ is

$$P(x) = x_1^3 + 3A(e_1, e_2, e_2)x_1x_2^2 + P(e_2)x_2^3.$$

For the sake of simplicity let $b = A(e_1, e_2, e_2)$ and $c = P(e_2)$. Of course $|b| \leq 1$ and $|c| \leq 1$ but the conditions $\|P\| = 1$ and P being extreme will impose further restrictions on b and c.

We will associate a function $f\colon [-\pi, \pi] \to \mathbf{R}$ with the polynomial P in the following way: each unit norm x can be written $(\cos\alpha, \sin\alpha)$ with $\alpha \in [-\pi, \pi]$. Thus

$$P(x) = f(\alpha) = \cos^3\alpha + 3b\cos\alpha\sin^2\alpha + c\sin^3\alpha.$$

Since $P(-x) = -P(x)$ we have $f(\alpha - \pi) = -f(\alpha)$, so it is enough to study the behaviour of f on $[0, \pi]$.

Let us show that $\|P\| = 1$ implies $b \leq 1/2$. Indeed

$$\begin{aligned} f'(\alpha) &= 3\sin\alpha(2b\cos^2\alpha - \cos^2\alpha - b\sin^2\alpha + c\sin\alpha\cos\alpha) \\ &= 3\sin\alpha\, g(\alpha). \end{aligned}$$

Since $|f(\alpha)| \leq \|P\| = 1$ for all α and $f(0) = 1$, the function f must be decreasing after zero, so the derivative $f'(\alpha)$ must be negative for α in an interval $[0, \delta)$. Since $\sin\alpha \geq 0$ for such α, we need to have $g(\alpha) \leq 0$ in this interval, so $\lim_{\alpha \to 0_+} g(\alpha) \leq 0$. Thus $2b - 1 \leq 0$; hence $b \leq 1/2$. The same condition can be obtained by studying the behaviour of f around π.

We will see that $b = 1/2$ is a very special case. In this case the condition $\|P\| = 1$ implies that $c = 0$. Indeed since $b = 1/2$ we have

$f'(\alpha) = 3\sin^2\alpha(c\cos\alpha - 1/2\sin\alpha)$. As above $f'(\alpha) \le 0$ as $\alpha \to 0_+$, so $-1/2\sin\alpha + c\cos\alpha \le 0$ as $\alpha \to 0_+$; hence $c \le 0$. Since $f(\pi) = -1$ the function f must be decreasing before π, so we must have $f'(\alpha) \le 0$ as $\alpha \to \pi_-$. This gives $-1/2\sin\alpha + c\cos\alpha \le 0$ as $\alpha \to \pi$; hence $c \ge 0$. Thus $c = 0$, so $P(x) = x_1^3 + 3/2 x_1 x_2^2$.

PROPOSITION 5. *The three-homogeneous polynomial on the two-dimensional real Hilbert space H given by*

$$P(x) = x_1^3 + \frac{3}{2}x_1 x_2^2$$

is an extreme point of $B_{\mathscr{P}(^3H)}$.

Proof. In this case $f'(\alpha) = -3/2\sin^3\alpha < 0$ on $(0, \pi)$ so $-1 < f(\alpha) < 1$ on $(0, \pi)$; hence $\|P\| = 1$.

If there exist three-homogeneous polynomials of unit norm $P_i(x) = P_i(e_1)x_1^3 + 3a_i x_1^2 x_2 + 3b_i x_1 x_2^2 + c_i x_2^3$, with $i = 1, 2$ such that $P = (P_1 + P_2)/2$ then $P_1(e_1) = P_2(e_1) = 1$ and consequently $a_1 = a_2 = 0$ and $b_i \le 1/2$. But $b_1 + b_2 = 1$, so $b_1 = b_2 = 1/2$. According to the remarks preceding the proposition, $c_i = 0$, so $P_1 = P_2 = P$; hence P is extreme. ∎

From now on we concentrate on the case $b < 1/2$. In this case $g(0) = g(\pi) = 2b - 1 < 0$, so for $\alpha \ne 0, \pi$ the condition $f'(\alpha) = 0$ which is the same as $g(\alpha) = 0$ is equivalent to $g(\alpha)/\sin^2\alpha = 0$, i.e.,

$$(2b - 1)\cot^2\alpha + c\cot\alpha - b = 0.$$

Thus we can have at most two more points where $g'(\alpha) = 0$ at which points f and implicitly P could have extreme values. These extreme values must be in $[-1, 1]$ and this is where the restriction for b and c will come from. This also shows that a three-homogeneous polynomial of unit norm on a two-dimensional Hilbert space can take the value 1 at no more than three unit vectors. Three situations could arise:

(1) There are no more solutions. In this case $c^2 + 4b(2b - 1) < 0$ and since $b < 1/2$ we get that necessarily $b > 0$.

We show that in this case P is not extreme. Since $\lim_{\alpha \to 0} g(\alpha) = \lim_{\alpha \to \pi} g(\alpha) = 2b - 1 < 0$ and g is continuous, there is $\delta > 0$ such that $g(\alpha) < b - 1/2 < 0$ on $[0, \delta) \cup (\pi - \delta, \pi]$, $f(\alpha) > 0$ on $[0, \delta)$, and $f(\alpha) < 0$ on $(\pi - \delta, \pi]$.

On $[\delta, \pi - \delta]$ we have $|f(\alpha)| < 1$ for all α, so $|f(\alpha)| \le m < 1$.

Choose an $\varepsilon > 0$ such that $\varepsilon < 1 - m$ and $b - 1/2 + \varepsilon < 0$.

Let $P_1(x) = P(x) + \varepsilon x_2^3$ and $P_2(x) = P(x) - \varepsilon x_2^3$. Thus $f_1(\alpha) = f(\alpha) + \varepsilon\sin^3\alpha$ and $f_2(\alpha) = f(\alpha) - \varepsilon\sin^3\alpha$. On $[\delta, \pi - \delta]$, we have $|f_i(\alpha) -$

$f(\alpha)| = \varepsilon \sin^3 \alpha \leq \varepsilon < 1 - m$, so $|f_i(\alpha)| < 1$. If $\alpha \in [0, \delta)$ then

$$\begin{aligned} f_1'(\alpha) &= f'(\alpha) + 3\varepsilon \sin^2 \alpha \cos \alpha \\ &= 3 \sin \alpha (g(\alpha) + \varepsilon \sin \alpha \cos \alpha) \\ &\leq 3 \sin \alpha (b - 1/2 + \varepsilon) < 0; \end{aligned}$$

hence $-1 \leq f_1(\alpha) \leq f_1(0) = 1$ on $[0, \delta)$. On $(\pi - \delta, \pi]$ we have $1 \geq f_1(\alpha) \geq f(\alpha) \geq -1$. Thus $|f_1(\alpha)| \leq 1$ on $[0, \pi]$. Thus $\|P_1\| = 1$. Likewise $\|P_2\| = 1$. Obviously $P_1 \neq P_2$ and $P = (P_1 + P_2)/2$ which shows that P is not extreme.

(2) The equation has one more solution, β. In this case β is necessarily a double zero, so $c^2 + 4b(2b - 1) = 0$ which gives $b \geq 0$. Of course $f'(\beta) = 0$ but on $(0, \pi)$ we have $f'(\alpha) < 0$ for $\alpha \neq \beta$. We are in the same position as above, so the polynomial P is not extreme.

(3) The equation has two more distinct solutions, γ and β with $0 < \gamma < \beta < \pi$. Then f is decreasing on $[0, \gamma]$ and $[\beta, \pi]$ and increasing on (γ, β). Let $m = f(\gamma)$ and $M = f(\beta)$. Suppose b and c are chosen so that $\|P\| = 1$. This forces $|m|, |M| \leq 1$.

(a) If $|m| < 1$ and $|M| < 1$ then as in the first case the polynomial is not extreme.

(b) $M = 1$. Let $x_\beta = (\cos \beta, \sin \beta)$. Then $P(x_\beta) = 1$, so $Q(x_\beta) = x_\beta$. According to Proposition 4, we have $\beta \leq 2\pi/3$. If we go back to the definition of Q we see that in this case

$$Q(x) = \left(x_1^2 + b x_2^2\right) e_1 + \left(2b x_1 x_2 + c x_2^2\right) e_2.$$

Thus $Q(x) = x$ gives

$$\cos^2 \beta + b \sin^2 \beta = \cos \beta$$

$$2b \cos \beta \sin \beta + c \sin^2 \beta = \sin \beta.$$

This system in b and c has the unique solution

$$b = \frac{\cos \beta - \cos^2 \beta}{\sin^2 \beta} = \frac{\cos \beta}{1 + \cos \beta}$$

$$c = \frac{\sin^2 \beta - 2\cos^2 \beta + 2\cos^3 \beta}{\sin^3 \beta},$$

so the polynomial is uniquely determined and therefore extreme, provided the values of b and c we have just found give that its norm is one. We will see that the condition $\beta \leq 2\pi/3$ is enough to get that $\|P\| = 1$.

Now $P(e_1) = P(x_\beta) = 1$. On $[\beta, \pi]$ we have $|f(\alpha)| \le 1$ since f is decreasing there. Let us see what happens on $[0, \beta]$.

Let $x = (\cos\alpha, \sin\alpha)$ with $\alpha \in (0, \beta)$. For each such x there is a $t \in (0,1)$ such that

$$x = \frac{te_1 + (1-t)x_\beta}{\|te_1 + (1-t)x_\beta\|}.$$

For $t = 1/2$ we obtain the point $x_{1/2} = (\cos\beta/2, \sin\beta/2)$. Then

$$\begin{aligned} P(x) &= \frac{t^3 + 3t(1-t)\langle e_1, x_\beta\rangle + (1-t)^3}{\left(2t^2 + 2t + 1 + 2t(1-t)\langle e_1, x_\beta\rangle\right)^{3/2}} \\ &= \frac{3t^2 - 3t + 1 + 3t(1-t)\cos\beta}{\left(2t^2 - 2t + 1 + 2t(1-t)\cos\beta\right)^{3/2}} \\ &= \frac{u(t)}{v(t)^{3/2}}. \end{aligned}$$

Let us observe that $u(t) - u(1/2) = 3t^2 - 3t + 1 + 3t\cos\beta - 3t^2\cos\beta - 1/4 - 3/4\cos\beta = 3(1-\cos\beta)(t^2 - t + 1/4) \ge 0$ for all t. At the same time $v(t) - v(1/2) = 2(1-\cos\beta)(t^2 - t + 1/4)$ so $v(t)^{3/2} \ge v(1/2)^{3/2} \ge 0$ and consequently $1/v(t)^{3/2} \le 1/v(1/2)^{3/2}$. If $u(t) \ge 0$, then $P(x) \ge 0 \ge -1$. If $u(t) < 0$ then

$$\begin{aligned} P(x) &= \frac{u(t)}{v(t)^{3/2}} \ge \frac{u(t)}{v(1/2)^{3/2}} \ge \frac{u(1/2)}{v(1/2)^{3/2}} \\ &= P(x_{1/2}) = \frac{2 + 6\cos\beta}{(2 + 2\cos\beta)^{3/2}} \ge -1 \end{aligned}$$

for $\beta \in (0, 2\pi/3]$ by the proof of Proposition 4.

Thus for b and c above we obtain a norm one polynomial which is extreme.

(c) $m = -1$. Since $f(\alpha - \pi) = -f(\alpha)$ this case is equivalent to (b). The only difference is that in the expression for b and c the angle β can take any value in $[-2\pi/3, 0)$.

(d) $m = -1$ and $M = 1$. This case is implicitly contained in the two above. Obviously the polynomial P is extreme. As in (b) this yields $P(x_{1/2}) = -1$ which gives $\cos\beta = -1/2$, so $\beta = 2\pi/3$ and $P(x) = x_1^3 - 3x_1x_2^2$ with the associated function $f(\alpha) = \cos 3\alpha$ and this is the only norm one polynomial which takes the value 1 at three points which must be equally spaced.

The cases (b) and (c) yield the following.

PROPOSITION 6. *A three-homogeneous polynomial of unit norm on a two-dimensional real Hilbert space that takes the value one at more than one unit vector is extreme.*

Summarizing all the information contained in the analysis above we obtain a characterisation of the extreme polynomials.

PROPOSITION 7. *A three-homogeneous polynomial P of unit norm on a two-dimensional real Hilbert space H is extreme if and only if there exists an orthonormal basis $\{e_1, e_2\}$ of H relative to which P has one of the following forms.*

(i) $P(x) = x_1^3 + 3/2 x_1 x_2^2$.

(ii) $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$, *where*

$$b = \frac{\cos \beta}{1 + \cos \beta}$$

$$c = \frac{\sin^2 \beta - 2\cos^2 \beta + 2\cos^3 \beta}{\sin^3 \beta}$$

with $\beta \in (-2\pi/3, 2\pi/3) \setminus \{0\}$.

(iii) $P(x) = x_1^3 - 3x_1x_2^2$.

Let us note that for $\beta = \pi/2$ we obtain the polynomial $P(x) = x_1^3 + x_2^3$ and for $\beta = -\pi/2$ we obtain the polynomial $P(x) = x_1^3 - x_2^3$.

The coefficients b and c are continuous functions of β on $[-2\pi/3, 2\pi/3]$. This can be used to show that the set of extreme three-homogeneous polynomials is a continuous image of a torus. Indeed let us fix a basis $\{e_1, e_2\}$ and let P be an extreme polynomial. There exists a unit vector f_1 such that $P(f_1) = 1$. If we write $f_1 = \cos \alpha e_1 + \sin \alpha e_2$ and take $f_2 = \cos(\alpha + \pi/2)e_1 + \sin(\alpha + \pi/2)e_2$ with α in $[-\pi, \pi]$ then the expression of P in the basis $\{f_1, f_2\}$ is $P(y) = y_1^3 + 3by_1y_2^2 + cy_2^3$, where $y = y_1 f_1 + y_2 f_2$ and b and c are given by the expression above for a β in $[-2\pi/3, 2\pi/3]$. This is the unique polynomial that takes the value 1 at the points $\cos \alpha e_1 + \sin \alpha e_2$ and $\cos(\alpha + \beta)e_1 + \sin(\alpha + \beta)e_2$ if $\beta \neq 0$. If $\beta = 0$ then P takes the value 1 only at f_1. If $\beta = \pm 2\pi/3$ then $\cos(\alpha - \beta)e_1 + \sin(\alpha - \beta)e_2$ is the third point at which P is 1. The expression of the polynomial in the basis $\{e_1, e_2\}$ is given by the function

$$
\begin{aligned}
u(\alpha,\beta) = (x_1\cos\alpha + x_2\sin\alpha)^3 \\
&+ 3b(x_1\cos\alpha + x_2\sin\alpha)(-x_1\sin\alpha + x_2\cos\alpha)^2 \\
&+ c(-x_1\sin\alpha + x_2\cos\alpha)^3.
\end{aligned}
$$

Since $b(2\pi/3) = b(-2\pi/3)$ and $c(2\pi/3) = c(-2\pi/3)$ for all $\alpha \in [-\pi,\pi]$ the function u satisfies the relation $u(\alpha, 2\pi/3) = u(\alpha, -2\pi/3)$. Obviously $u(\pi,\beta) = u(-\pi,\beta)$ for all $\beta \in [-2\pi/3, 2\pi/3]$. The function u is continuous and the remarks above show that the image of $[-\pi,\pi] \times [-2\pi/3, 2\pi/3]$ under u is the set of extreme three-homogeneous polynomials. However, the function u is not injective. It is easily seen that $u(\alpha,\beta) = u(\alpha+\beta, -\beta)$ and is the unique polynomial that takes the value 1 at $\cos\alpha e_1 + \sin\alpha e_2$ and $\cos(\alpha+\beta)e_1 + \sin(\alpha+\beta)e_2$. Thus we can conclude that the set of extreme three-homogeneous polynomials is the continuous (but not homeomorphic) image of a torus and consequently is connected and compact.

Remark 8. In fact, if we fix an orientation for H the number of orthonormal bases relative to which a polynomial of unit norm can be written as $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$ is exactly the number of the points at which P is 1, so for each polynomial there exist one, two, or three such bases.

The conditions in (ii) are equivalent to the relation

$$
c^2(1-2b)^3 = (4b^3 - 3b + 1)^2,
$$

with $b \in [-1, 1/2]$. Indeed it is easy to see that once there exists $\beta \in [-2\pi/3, 2\pi/3]$ such that b and c are given by the expressions in (ii), they satisfy the equation above. Conversely, if b and c satisfy the relation, then writing $\cos\beta = b/(1-b)$ we obtain that b and c have the form in (ii). Let us note that if $b = -1$ then $c = 0$. Thus Proposition 7 can be reformulated as:

THEOREM 9. *A three-homogeneous polynomial P of unit norm on a two-dimensional real Hilbert space H is extreme if and only if for any orthonormal basis $\{e_1, e_2\}$ of H relative to which $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$, the coefficients b and c satisfy one of the following conditions.*

(i) $b = 1/2$ *and* $c = 0$.

(ii) $c^2(1-2b)^3 = (4b^3 - 3b + 1)^2$ *with* $b \in [-1, 1/2)$.

5. SMOOTH POINTS OF THE UNIT BALL OF $\mathscr{P}(^3H)$

Since we are working on a two-dimensional real Hilbert space and we are interested in three-homogeneous polynomials, Corollary 3 gives that a three-homogeneous polynomial of unit norm P is smooth if and only if $P(x) = 1$ at just one point x in B_H. Thus we have:

PROPOSITION 10. *Let H be a two-dimensional real Hilbert space. The unit sphere of $\mathscr{P}(^3H)$ is the union of the set of its extreme points and the set of its smooth points and these two sets have a nonvoid intersection.*

Proof. Let P be a norm one polynomial. If P takes the value one at just one point, then P is smooth. If P takes the value 1 at more than one point then P is extreme according to Proposition 6.

If we consider $P(x) = x_1^3 + 3/2x_1x_2^2$ this polynomial is an extreme polynomial that takes the value 1 at just one point, so it is both extreme and smooth. ∎

Remark 11. Choi and Kim [2] observed that a two-homogeneous polynomial on a two-dimensional Hilbert space is either extreme or smooth so the unit sphere of $\mathscr{P}(^2H)$ is the union of the set of its extreme points and the set of its smooth points. Although, as we have just proved, the result is true for three-homogeneous polynomials as well, this is somewhat fortuitous. Indeed if we increase either the dimension of the space or the degree of the polynomial, this is no longer valid. Consider for instance the two-homogeneous polynomial $P(x) = x_1^2 + x_2^2 + 1/2x_3^2$ on $\mathbf{R}^3$ with its Hilbert space structure. This is neither extreme [5, 8] nor smooth (Proposition 1). The same is true for the four-homogeneous polynomial $P(x) = x_1^4 + x_2^4$ on the Hilbert space $\mathbf{R}^2$. It is not smooth since $P(e_1) = P(e_2) = 1$ and it is not extreme since $P = (P_1 + P_2)/2$ with $P_1(x) = x_1^4 + 2x_1^2x_2^2 + x_2^4$ and $P_2(x) = x_1^4 - 2x_1^2x_2^2 + x_2^4$.

Since we have characterized the extreme polynomials by the number of the points where they take the value 1, the analysis we have done also gives a characterisation of smooth polynomials.

PROPOSITION 12. *A three-homogeneous polynomial P of unit norm on a two-dimensional Hilbert space H is a smooth point of $B_{\mathscr{P}(^3H)}$ if and only if for any orthonormal basis $\{e_1, e_2\}$ of H relative to which $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$ the coefficients b and c satisfy one of the following conditions.*

(i) $b = 1/2$ *and* $c = 0$.

(ii) $c^2(1-2b)^3 \neq (4b^3 - 3b + 1)^2$ *with* $b \in [-1, 1/2)$.

Proof. As has been shown, there exists a basis $\{e_1, e_2\}$ of H such that in that basis P can be written $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$. Once the polynomial P has this form, according to the analysis carried out in the process of describing the extreme polynomials, in the first situation the polynomial $P(x) = x_1^3 + 3/2x_1x_2^2$ is extreme but, according to the proof of Proposition 5, e_1 is the only point where P takes the value 1, so P is smooth. In the last situation the polynomial is not extreme (Theorem 9), so according to Proposition 10, it is necessarily smooth. ∎

6. THE GEOMETRY OF THE UNIT BALL OF $\otimes_{s,\pi}^3 H$

In [7] Ryan and Turett showed that whenever X is a finite-dimensional normed space the extreme points of the unit ball of $\otimes_{s,\pi}^n X$ are exactly the points x^n with x in the unit sphere S_H. Thus the extreme points of the unit ball of $\otimes_{s,\pi}^3 H$ are the vectors x^3 with x in S_H. Using the analysis done in order to find the three-homogeneous extreme polynomials we can give a characterisation of the smooth points of the unit ball of $\otimes_{s,\pi}^3 H$.

PROPOSITION 13. *Let H be a two-dimensional real Hilbert space. A unit vector u is a smooth point of the unit ball of $\otimes_{s,\pi}^3 H$ if and only if u has one of the following forms.*

(a) $u = \lambda x^3 + \mu y^3$ *with* $0 \leq \lambda,\ \mu \leq 1,\ \lambda + \mu = 1$ *and* $x, y \in S_H$, $\langle x, y \rangle \geq -1/2$.

(b) $u = \lambda x^3 + \mu y^3 + \nu z^3$ *with* $0 \leq \lambda, \mu, \nu \leq 1,\ \lambda + \mu + \nu = 1$ *and* $x, y, z \in S_H$, $\langle x, y \rangle = \langle y, z \rangle = \langle z, x \rangle = -1/2$.

Proof. Let u be an element of the unit ball of $\otimes_{s,\pi}^3 H$. It can be written as a finite convex combination of vectors of the form x^3 with $x \in S_H$. If $u = \sum \lambda_i x_i^3$ and P is a three-homogeneous polynomial of unit norm on H then $P(u) = 1$ if and only if $P(x_i) = 1$ for all x_i. But such a polynomial can be one at no more than three points. Thus if u is smooth, then the convex combination cannot contain more than three terms. According to Proposition 4 for any two unit vectors x and y such that $P(x) = P(y) = 1$ we have $\langle x, y \rangle \geq -1/2$.

If a smooth u is a convex combination of three unit vectors x, y, z then necessarily $\langle x, y \rangle = \langle y, z \rangle = \langle z, x \rangle = -1/2$. On the other hand if we have three unit vectors x, y, z such that $\langle x, y \rangle = \langle y, z \rangle = \langle z, x \rangle = -1/2$ there exists only one three-homogeneous polynomial of unit norm such that $P(x) = P(y) = P(z) = 1$. If we choose a basis for H such that $e_1 = x$ then $P(x) = x_1^3 - 3x_1x_2^2$. Thus $u = \lambda x^3 + \mu y^3 + \nu z^3$ with $0 \leq \lambda, \mu, \nu \leq 1$, $\lambda + \mu + \nu = 1$ is a smooth point.

If a smooth u is a convex combination of two unit vectors x and y then $\langle x, y \rangle \geq -1/2$. Conversely, for any two unit vectors x and y such that $\langle x, y \rangle \geq -1/2$ there is a unique three-homogeneous polynomial of unit norm such that $P(x) = P(y) = 1$. If $\langle x, y \rangle = \cos \beta$ and we choose a basis for H such that $e_1 = x$ then $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$ with b and c as in Proposition 7. Hence $u = \lambda x^3 + \mu y^3$ with $0 \leq \lambda, \mu \leq 1, \lambda + \mu = 1$ is a smooth point.

No x^3 with $x \in B_H$ is a smooth point since there are many three-homogeneous polynomials of unit norm such that $P(x) = 1$. ∎

ACKNOWLEDGMENTS

This article is part of the author's Ph.D. thesis written under the supervision of Dr. Raymond A. Ryan. The author acknowledges the financial support of a Forbairt Basic Research Grant and a Postgraduate Fellowship from the National University of Ireland, Galway. The author is grateful to Dr. Ryan for introducing him to this kind of problem and for all his support.

REFERENCES

1. C. Boyd and R. A. Ryan, "Geometric Theory of Spaces of Integral Polynomials and Symmetric Tensor Products," preprint, 1997.
2. Y. S. Choi and S. G. Kim, The unit ball of $\mathscr{P}(^2 l_2^2)$, *Arch. Math.* **71** (1998), 472–480.
3. S. Dineen, "Complex Analysis in Locally Convex Spaces," North-Holland Math. Stud., Vol. 57, North-Holland, Amsterdam/New York, 1981.
4. K. Floret, "Natural Norms on Symmetric Tensor Products of Normed Spaces," preprint, 1998.
5. B. C. Grecu, "Extreme Polynomials On Hilbert Spaces," preprint, 1999.
6. R. A. Ryan, "Application of Topological Tensor Products to Infinite Dimensional Holomorphy," doctoral thesis, Trinity College, Dublin, 1980.
7. R. A. Ryan and B. Turett, Geometry of spaces of polynomials, *J. Math. Anal. Appl.* **221** (1998), 698–711.
8. K. Sundaresan, Geometry of spaces of quadratic forms, *in* "Inner Product Spaces and Applications," Pitman Research Notes in Mathematics Series **376**, pp. 261–265, Longman, Harlow, 1997.