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PHYSICS LETTERS B
Physics Letters B 579 (2004) 13–24
www.elsevier.com/locate/physletb

Charged black holes in Gauss–Bonnet extended gravity

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Received 16 October 2003; accepted 27 October 2003

Editor: M. Cvetič

Abstract

Charged black holes in Gauss–Bonnet extended gravity are studied. The electromagnetic field is coupled non-minimally, as in $U(2,2)$ Chern–Simons theory. We find that the geometrical properties of the solution exhibit "phase transitions" as one varies the mass and charge. The full phase diagram for all values of the ADM mass and charge is displayed.

1. Introduction, main results and conclusions

In a five-dimensional Universe, the Gauss–Bonnet density $\sqrt{-g}(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho})$ cannot be omitted in the gravitational Lagrangian. This term is covariant, its associated Einstein tensor is conserved, and, despite being quadratic in the curvature tensor, yields second order field equations for the metric [1,2]. The most general action for Gravity in five dimensions is then

$$I[g_{\mu\nu}] = \int_{M_5} \sqrt{-g}\big[\alpha_0 + \alpha_1 R + \alpha_2\big(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\big)\big]. \tag{1}$$

The presence of this term of course changes the dynamical equations, and many aspects of general relativity have to revisited. This issue is particularly relevant in the context of brane worlds models, and many papers have recently been devoted to the subject [3].

The simplest problem that can be analyzed in a closed form is the spherically symmetric five-dimensional black hole spacetime,

$$ds^2 = -N^2\,dt^2 + \frac{dr^2}{f^2} + r^2\,d\Omega_3. \tag{2}$$

Although the equations can be solved for arbitrary values of the three couplings [4–6], we are interested in the role of the Gauss–Bonnet term and then we set, for simplicity, $\alpha_1 = 1$, $\alpha_0 = 0$ and $\alpha_2 > 0$. The equations of motion

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doi:10.1016/j.physletb.2003.10.091

yield [4][1]

$$N^2 = f^2 = 1 + \frac{1}{\alpha_2}\left(r^2 - \sqrt{r^4 + 4\alpha_2 M}\right), \tag{3}$$

where M is an integration constant, that will be seen to be the ADM mass.

We first note that for $M = 0$ the metric reduces to flat space, which is the stable background for this theory [4]. It also follows that for $r^4 \gg 4M\alpha_2$,

$$N^2 \simeq 1 - \frac{2M}{r^2} \tag{4}$$

showing that, asymptotically, (3) approaches the five-dimensional Schwarzschild metric. Incidentally, note that if $M \gg \alpha_2$ then $r^4 \gg 4M\alpha_2$ would hold all the way to the Schwarzschild horizon at $r_+ = \sqrt{2M}$. If $\alpha_2 = 0$, (4) becomes an exact solution.

The function (3) has some interesting properties. The value of N^2 at the singularity $r = 0$ is finite,

$$N^2(0) = 1 - \sqrt{\frac{4M}{\alpha_2}},$$

although the curvature is still singular.[2] The location of the horizon, $N^2(r_+) = 0$, is

$$r_+ = \sqrt{2M - \frac{\alpha_2}{2}}. \tag{5}$$

We see from this expression that the horizon exists only for $M > \alpha_2/4$. We thus find a mass gap separating flat space from the spectrum of black holes:

$$\begin{aligned} &M = 0, && \text{flat space,} \\ &0 < M \leqslant \alpha_2/4, && \text{naked singularities,} \\ &M > \alpha_2/4, && \text{black holes.} \end{aligned} \tag{6}$$

The mass gap appears in all odd-dimensional theories containing the highest Lovelock [1] term. In three dimensions, this term is just the Hilbert term and the mass gap is present [12]. In this case, however, the "naked singularities" have a sensible interpretation in terms of particles [7].

In this Letter we add electric charge q to this black hole and study the corresponding spectrum. We shall see that the solution has some peculiarities not present in usual charged black holes.

As a first surprise, the addition of charge does not imply the existence of two horizons. There are open regions in the plane $\{M, q\}$ having non-extremal black holes with *one* horizon. In order to find solutions with two horizons, q has to be bigger than a certain critical value, $q > q_c$. Fig. 1 gives a summary of the properties of various solutions obtained by varying the values of M and q.

In one region of the space of parameters, we find a set of non-extremal charged black holes having only one horizon. We call this region the "heavy branch" because it is defined by the condition $M > M_{\text{crit}}$. For masses within the range

$$M_{\text{crit}} \geqslant M > M_{\text{ext}}, \qquad q > q_c \tag{7}$$

[1] The coefficient α_2 appearing here differs from that in (1) by a numerical constant. Note that $\alpha_1 = 1$ implies, in five dimensions, [mass] = [length^2] = [α_2].

[2] Consider the metric of a cone, $ds^2 = \alpha\, dr^2 + r^2\, d\phi^2$ with $0 < \phi < 2\pi$. It is known that its curvature is concentrated at $r = 0$. Consider now $ds^2 = \alpha\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$. The scalar curvature is $R = (\alpha - 1)/(\alpha r^2)$ showing that the geometry is not locally flat.

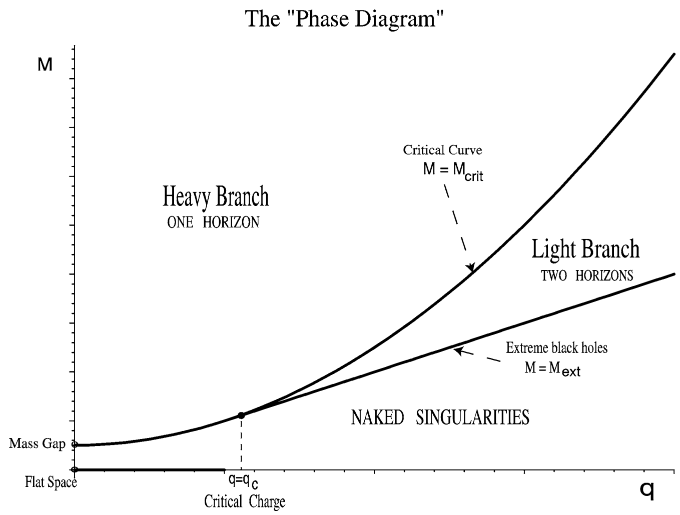

Fig. 1. The phase diagram. For each value of $q > q_c$ there exists two black holes phases. Flat space is located at $q = M = 0$, and it is disconnected from the black hole spectrum by a set of naked singularities.

we find the "light branch" with black holes with two horizons. At $M = M_{\text{ext}}$, we find extremal black holes with only one horizon (and zero Hawking temperature). Below the extreme value, the solution represents a naked singularity. There is also a "critical charge" below which the light branch ceases to exist. Flat space is located at $M = q = 0$, and thus the mass gap persists in the charged solution.

The terms "phase structure" and "phase transitions" are used here only in analogy with the statistical mechanics concept, without implying a direct connection. Of course, given the thermodynamical properties of black holes, this may turn out to be more than an analogy but we shall not study this issue here.

To avoid future confusions, we stress that the action considered in this Letter is not the usual minimally coupled Einstein–Maxwell system (plus a Gauss–Bonnet term). Those solutions were studied in [6] and do not exhibit this phase structure. Instead, we consider a five-dimensional Chern–Simons theory for the group $U(2,2)$ [8], which has a sensible interpretation as a gravitational plus electromagnetism theory, with a Gauss–Bonnet term. This interpretation, however, requires a symmetry breaking term because otherwise the equations of motion differ from the usual ones even asymptotically. This point was discussed in detail in [9].

The application of Chern–Simons theories to gravity has been discussed several times in the literature and we shall not repeat it here. The first constructions were reported in three dimensions in [10], and the same idea was then applied in [8] to five dimensions. See [11] for other aspects.

For the purposes of this Letter, we refer the reader to Ref. [9] were many details omitted here can be found. In particular, the asymptotic form of the charged black holes was already reported in that reference. The goal of the present Letter is to display the *exact* solution for an arbitrary mass M and electric charge q, and study the associated phase space.

2. The equations and their solution

2.1. The equations and spherically symmetric ansatz

We start by writing down the equations of motion associated to a Chern–Simons theory for the group $U(2,2)$. This group contains $SO(4,2)$ which would give pure gravity with a Gauss–Bonnet term. The extension to $U(2,2)$

incorporates an Abelian one-form, that we interpreted as an electromagnetic field, coupled non-minimally to gravity. See [9] for a detailed analysis, and other motivations to study this system.

A key issue in the analysis (and discussed in [9]) is the fact that the Chern–Simons equations of motion do not provide a sensible theory for the $SU(2,2)$ field. For example, there is no linearized theory and the metric is not asymptotically Schwarzschild. Given the strong topological roots of the Chern–Simons construction, it is still attractive as a field theory, and one would like to know if the equations can be "repaired" by some mechanism, hopefully within the same theory. Some progress in this direction was reported in [13].

Here we follow [9] in which a symmetry breaking term is added to the action. The Chern–Simons equations then becomes closer to the real world and one can start asking questions such as what is the structure of black holes, and what is the nature of the couplings between the gravitational and internal $U(N)$ gauge fields degrees of freedom. In this Letter we concentrate on the coupling between the gravitational and $U(1)$ field.

The symmetry breaking term added in [9] consists in a cosmological term (vacuum energy) and it is parameterised by a real number τ. Let e^a be the five-dimensional vielbein one-form, w^{ab} the spin connection, and A the Abelian one-form. The equations of motion following from the $U(2,2)$ Chern–Simons theory (including the symmetry breaking term proportional to τ) are,

$$\epsilon_{abcde}\big[\big(R^{ab}+e^a e^b\big)\wedge\big(R^{cd}+e^a e^b\big)-\tau^2 e^a\wedge e^b\wedge e^c\wedge e^d\big]=-4T_e\wedge F, \tag{8}$$

$$\epsilon_{abcde}\big(R^{ab}+e^a e^b\big)\wedge T^c=(R_{de}+e_d e_e)\wedge F, \tag{9}$$

$$\frac{1}{2}R^{ab}\wedge R_{ab}-d\big(e_a\wedge T^a\big)=F\wedge F, \tag{10}$$

where $R^{ab}=dw^{ab}+w^a{}_c w^{cb}$, $T^a=de^a+w^a{}_b e^b$ and $F=dA$. For $\tau=0$, these equations are equivalent to $\mathcal{F}\wedge\mathcal{F}=0$ with $\mathcal{F}\in U(2,2)$ which are the exact Chern–Simons equations.

Since we are interested in black hole solutions, we write the ansatz for the metric and gauge field with spherical symmetry

$$ds^2=-N(r)^2\,dt^2+f^2(r)\,dr^2+r^2\,d\Omega_3, \tag{11}$$

$$A=-\phi(r)\,dt, \tag{12}$$

where N, f and ϕ are functions to be determined.

As shown in [9], the dynamics of the Abelian form A is linked to the torsion tensor. If we assume from the very beginning that $T^a=0$, then the equations of motion for A do not give Maxwell's equations in any limit. On the contrary, letting T to be different from zero yields a system of equations that can be analyzed perturbatively and yields, to first order, Maxwell's theory for the potential $\phi(r)$. The relationship between the torsion T^a and the Maxwell field F is encoded in Eq. (9). To first order we ignore the right-hand side of (8) and find the gravitational background (AdS space) $R^{ab}=(\tau-1)e^a e^b$. Replacing in (9) it implies [9]

$$e_a\wedge T^a={}^*F, \tag{13}$$

where $*$ represents Hodge's dual. Replacing (13) into (10) one obtains the usual five-dimensional Maxwell–Chern–Simons system $d^*F=F\wedge F$. This is, in short, the mechanism that transforms A into a radiating field.

Incorporating the back reaction from the right-hand side of (8) produces corrections to (13), and the Maxwell equation. It is precisely the role of these corrections what we aim to investigate in this Letter.

Let us then assume that the torsion is not zero, and let $w^{ab}=w^{ab}(e)+\kappa^{ab}$ where $w^{ab}(e)$ is the solution to the equation $de^a+w^a{}_b(e)\wedge e^b=0$, and only depends on the metric. κ parameterizes a non-zero torsion by $T^a=\kappa^a{}_b\wedge e^b$. In order to prescribe a spherically symmetric ansatz for κ^{ab}, it is convenient to express all indices in the coordinate basis $\kappa_{\alpha\beta\mu}=e_{a\alpha}\kappa^{ab}{}_\mu\, e_{b\beta}$. The correct ansatz with spherical symmetry for this tensor follows by studying the equations of motion order by order starting from the AdS vacuum. The details were given in [9], and the result is,

$$\kappa_{\mu\nu\lambda} = \frac{\psi(r)}{2\sqrt{h}}\epsilon_{\mu\nu\lambda}{}^{\alpha\beta} z_{\alpha\beta} - z_{\mu\nu}U_\lambda + 2g_{\lambda[\mu}V_{\nu]}, \tag{14}$$

where $\epsilon_{\mu\nu\lambda}{}^{\alpha\beta}$ is the Levi-Civita tensor with $\epsilon^{\mathrm{tr}\theta_1\theta_2\theta_3} = 1$, $\sqrt{h} = r^3\sin^2\theta_1\sin\theta_2$, and $z = dt\wedge dr$, $U = \beta(r)\,dt$, $V = \alpha(r)\,dr$. The torsion then contributes with three more functions of r to be determined by the equations of motion, namely, $\alpha(r)$, $\beta(r)$ and $\psi(r)$. ψ is closely related to the electric field (see Eq. (13)), while α and β are auxiliary fields which will be eliminated algebraically from their own equations of motion.

We shall see that this ansatz does provide an exact solution to the full $U(2,2)$ system.

2.2. *The spherically symmetric reduced equations*

In this section we present some of the details in finding the equations and their solution. The reader interested only the final result can jump to the next section where the final form of the metric is displayed and its properties analyzed.

It is a direct but long calculation to replace in the equations of motion (8), (10) the ansatz with spherical symmetry shown in the last paragraph. The resulting equations turns out to be extremely complicated. Some simplification can be achieved by making field redefinitions which simplify the expressions for the curvature R^{ab} and torsion T^a. These field redefinitions involve all variables. We transform $\{\alpha,\beta,N,f,\psi,\phi\}\to\{\alpha_1,\beta_1,N_2,f_2,\psi_1,\phi_1\}$, according to

$$\begin{aligned}
&\alpha = \frac{\alpha_1-1}{r}, \qquad \beta = N\left(\frac{fN\beta_1}{g} + N' + N\alpha\right),\\
&f = \frac{\alpha_1}{f_1}, \qquad f_1 = \sqrt{1+r^2-\psi_1^2+f_2}, \qquad N = f_1N_1, \qquad N_1 = \exp\left(\int^r N_2\right),\\
&\phi = 4\int^r dr\,(N_1\psi_1\Phi_1), \qquad \psi = \frac{\psi_1}{r}.
\end{aligned} \tag{15}$$

Note that the horizon structure will be controlled by the zero'es of the function f_1.

Inserting the above ansatz into the equations of motion, we find two sets of equations which can be analyzed and solved separately.

2.2.1. *The α_1, ψ_1, f_2 system*

The first set of equations involves only the functions $\{\alpha_1,\psi_1,f_2\}$. The equations are the following (prime indicates radial derivative):

$$\left(4\psi_1^3 - 4r^2\psi_1 - 4\psi_1 - 2f_2\psi_1\right)\psi_1' - 2f_2r + 2rf_2\alpha_1 + 2\tau^2r^3\alpha_1 = 0, \tag{16}$$

$$\alpha_1f_2 - f_2 - 2r\psi_1\psi_1' = 0, \tag{17}$$

$$2\psi_1^2\psi_1' - (\psi_1f_2)' = 0. \tag{18}$$

We note that (17) and (18) can be easily solved. We find, respectively,

$$\alpha_1 = 1 + \frac{2r\psi_1\psi_1'}{f_2}, \tag{19}$$

$$f_2 = \frac{2}{3}\psi_1^2 + \frac{q_0}{\psi_1}, \tag{20}$$

where q_0 is an integration constant that will be related to the electric charge. It will be convenient in what follows to do yet one more redefinition,

$$\psi_1(r) = -\frac{q_0}{\Psi(r)}, \quad q_0 = \frac{\sqrt{6}q}{2}, \tag{21}$$

where $\Psi(r)$ is a new function of r. Replacing (20) into (19) we find the following expression for α_1,

$$\alpha_1 = \frac{(Pr)'}{P}, \tag{22}$$

where P is a short hand for

$$P = \frac{\Psi^3 - q^2}{\Psi^3}. \tag{23}$$

Finally, we replace (19) and (20) in (16) and obtain a closed equation for the function Ψ:

$$\left(6q^4\Psi^2 + 3q^2\Psi^6 - 6q^2r^4\Psi^4 - 6q^2\Psi^5 - 4q^6 + \Psi^9\right)\frac{d\Psi}{dr} + 2\tau^2 r^3\Psi^5\left(q^2 - \Psi^3\right) = 0. \tag{24}$$

This last equation is linear in Ψ'. The integral can be done explicitly and we find the implicit solution for $\Psi(r)$,

$$\begin{aligned}&\frac{\tau^2}{2}\left(\frac{\Psi^3 - q^2}{\Psi^3}\right)^4 r^4 - \frac{1}{2}\Psi^2 - 3\frac{q^2}{\Psi^2} + \frac{4}{7}\frac{q^6}{\Psi^7} - \frac{9}{2}\frac{q^6}{\Psi^8} + \frac{9}{10}\frac{q^8}{\Psi^{10}} + \frac{24}{11}\frac{q^8}{\Psi^{11}} + \frac{24}{5}\frac{q^4}{\Psi^5} - \frac{12}{13}\frac{q^{10}}{\Psi^{13}} \\ &\quad + \frac{1}{4}\frac{q^{12}}{\Psi^{16}} - \frac{3}{2}\frac{q^4}{\Psi^4} - \frac{3}{7}\frac{q^{10}}{\Psi^{14}} = -2M,\end{aligned} \tag{25}$$

where M is an integration constant that will be seen to be the ADM mass of the solution. This is an algebraic equation that should be inverted to find $\Psi(r)$.

2.2.2. The β_1, N_2, Φ system

We now proceed to find expressions for β_1, N_2 and Φ_1 in terms of Ψ. The three remaining equations are:

$$0 = f_2 r + \beta_1 f_2 + r^3, \tag{26}$$

$$\begin{aligned}0 &= \alpha_1 f_2 + \beta_1' f_2 + f_2'\beta_1 + N_2\beta_1 f_2 - 2\psi_1\psi_1'\beta_1 + 2r^2 + 4r\psi_1^2\Phi_1 \\ &\quad - 2r\psi_1\psi_1' - 2\alpha_1\beta_1 r + rf_2' + \alpha_1 r^2 + 2r\beta_1,\end{aligned} \tag{27}$$

$$\begin{aligned}0 &= 2\alpha_1\beta_1 - 2f_2\Phi_1 - 2N_2\psi_1^2 + 2N_2 + f_2' + 2N_2 f_2 - 2\psi_1\psi_1' + 2r\beta_1' \\ &\quad + 2r\alpha_1 + 2r + 2rN_2\beta_1 + 2r^2N_2.\end{aligned} \tag{28}$$

This set of equations can also be solved in a closed form. We shall not go into the details on how to find the solution, we only quote the result. Using the value of α_1 found in the previous paragraph, we find that Eqs. (26)–(28) are solved by:

$$\beta_1 = -\frac{r(f_2 + r^2)}{f_2}, \tag{29}$$

$$N_2 = \frac{8\psi_1\psi_1'}{f_2}, \tag{30}$$

$$\Phi_1 = \frac{1}{2}\frac{(3f_2^2 - 2r^4)\psi_1'}{f_2^2\psi_1}. \tag{31}$$

Note that making the redefinition (21), and using the solution (20) for f_2 we have

$$N_1 = e^{\int N_2} = \left(\frac{\Psi^3 - q^2}{\Psi^3}\right)^4.$$

This completely solves the problem. All functions are known in terms of r and Ψ, and Ψ is known in terms of r by (25).

3. Charged black holes and a phase transition

3.1. The metric

Let us summarize the results of the analysis of the equations of motion. The metric ansatz was

$$ds^2 = -N(r)^2\, dt^2 + f^2(r)\, dr^2 + r^2\, d\Omega_3. \tag{32}$$

The functions N and f are fixed by the equations of motion as (prime denotes radial derivative)

$$N^2 = P^8 f_1^2, \tag{33}$$

$$f^2 = \left[\frac{(Pr)'}{P}\right]^2 \frac{1}{f_1^2}, \tag{34}$$

where

$$f_1^2 = 1 + r^2 - \frac{1}{2}\frac{q^2}{\Psi^2} - \Psi, \tag{35}$$

and P is given in (23). Finally, Ψ is a function of r defined by the algebraic equation (25).

In some applications it may be convenient to define a new radial coordinate, as suggested by Eqs. (25) and (34),

$$\rho = Pr. \tag{36}$$

In terms of this new coordinate the metric takes a simple form,

$$ds^2 = -P^8 f_1^2\, dt^2 + \frac{1}{P^2}\left(\frac{d\rho^2}{f_1^2} + \rho^2\, d\Omega_3^2\right) \tag{37}$$

and the relation (25) becomes

$$\tau^2\rho^4 + 4M = \Psi^2 + 6\frac{q^2}{\Psi^2} - \frac{8}{7}\frac{q^6}{\Psi^7} + 9\frac{q^6}{\Psi^8} - \frac{9}{5}\frac{q^8}{\Psi^{10}} - \frac{48}{11}\frac{q^8}{\Psi^{11}}\frac{48}{5}\frac{q^4}{\Psi^5} + \frac{24}{13}\frac{q^{10}}{\Psi^{13}} - \frac{1}{2}\frac{q^{12}}{\Psi^{16}} + 3\frac{q^4}{\Psi^4} + \frac{6}{7}\frac{q^{10}}{\Psi^{14}}. \tag{38}$$

From now we shall only consider the case $\tau = 1$. This is only for simplicity in some calculations, but it does not affect the main conclusions.

3.2. Known limits of the solution. Reissner–Nordstrom solution

Since the exact solution displayed in the previous paragraph is rather complicated, as a first check we analyze how this solution reduces to the known ones, in various limits. We first study the uncharged solution, found in [4] and discussed in the introduction. Then we show how in the limit of small charges and large radial coordinate r, we recover the usual Reissner–Nordstrom spacetime.

Consider first the uncharged solution with $q = 0$. In this case, $P = 1$ (hence $r = \rho$), we can solve Ψ explicitly, $\Psi = \sqrt{r^4 + 4M}$, and obtain for f_1 the closed expression,

$$f_1^2 = 1 + r^2 - \sqrt{r^4 + 4M}, \tag{39}$$

representing the uncharged solution [4], described in the introduction, with $\alpha_2 = 1$.

Consider now the charge as a small parameter, and seek for a perturbative solution in q^2 to the Eq. (25). Let

$$\Psi(r) = \sqrt{r^4 + 4M} + q^2 h_1(r). \tag{40}$$

Replacing in (25) and keeping only the linear terms in q^2 we find for the first order perturbation

$$h_1(r) = -\frac{2r^4 + 3\sqrt{r^4 + 4M}}{(r^4 + 4M)^2}. \tag{41}$$

Replacing in (20), and taking the limit $r^4 \gg 4M$, we find

$$N^2 = \frac{1}{f^2} = 1 - \frac{2M}{r^2} + \frac{3}{2}\frac{q^2}{r^4} + \mathcal{O}\left(\frac{1}{r^6}\right) \tag{42}$$

coinciding exactly with the Reissner–Nordstrom spacetime. We can also see that the parameter that we called q is in fact the electric charge, up to a normalization.

3.3. The origin, the Kasner singularity, and physical range of radial coordinate

The metric (37) has a curvature singularity at the origin $\rho = 0$. There is another singularity at the point where P (P was defined in (23)) vanishes,

$$\Psi^3 - q^2 = 0. \tag{43}$$

At this singularity, proper times are shrink to zero while spacelike separations are stretch to infinity. The volume element, however, remains finite, $\det g \sim P^8 \times (P^{-2})^4 = 1$. For this reason, we call this point the "Kasner singularity".

Both singularities can be shown to be physical in the sense that the components of the curvature tensor in an inertial frame diverge. It is then important to ask whether these singularities are protected by horizons.

The analysis of existence of horizons is greatly simplified by noticing that the function Ψ can be used as a radial coordinate. Although the algebraic relation (38) between ρ^4 and Ψ is quite untractable and attempts to invert it explicitly are hopeless, we can in fact show that in the domain of interest, it is an invertible function.

We first note that the derivative of (38) can be factorized in the form,

$$\frac{d\rho^4}{d\Psi} = 2\frac{(\Psi^3 + 2q^2 - \sqrt{6}q\Psi)(\Psi^3 + 2q^2 + \sqrt{6}q\Psi)(\Psi^3 - q^2)^4}{\Psi^{17}}. \tag{44}$$

Since $\Psi^3 = q^2$ is a curvature singularity, we do not need to worry about non-invertibility at that point. We need to focus on either $\Psi^3 > q^2$ or $\Psi^3 < q^2$. Since, asymptotically, $\Psi \simeq r^2$ is a large positive number, the physical domain of the function Ψ is

$$q^{2/3} < \Psi < +\infty, \tag{45}$$

and we explore invertibility of (38) on this domain.

All factors in (44) are positive definite in the physical domain except for the first one. We would then like to know if the solutions Ψ_c to the equation ${\Psi_c}^3 + 2q^2 - \sqrt{6}q\Psi_c = 0$ lie in the physical range of the function Ψ or not. Let us first note this equation has positive solutions only if $q < q_c := (2/3)^{3/2}$. Thus, our first conclusion is that the relation (38) is invertible for $q > q_c$.

Recall now that the metric has two singularities, $\Psi^3 = q^2$, and $\rho = 0$. Let us call Ψ_0 the particular value of Ψ such that $\rho(\Psi_0) = 0$. It turns out (this is most easily done by a graphical analysis) that if $q < q_c$ the three numbers Ψ_c, Ψ_0 and $q^{2/3}$ are ordered according to

$$q^{2/3} < \Psi_c < \Psi_0. \tag{46}$$

This means that the non-invertible point Ψ_c is beyond the origin $\rho = 0$ and thus, it does not affect the physical domain.

3.4. Horizon structure

Given the form (37) of the metric it is clear that horizons will arise whenever the function f_1^2 vanishes. As mentioned before, the relation between ρ and Ψ is invertible and we can study f_1^2 as a function of Ψ. We write here the explicit form of f_1^2 in terms of Ψ, the ADM mass M and the charge q,

$$\begin{aligned} f_1^2(\Psi) = {} & 1 - \frac{1}{2}\frac{q^2}{\Psi^2} - \Psi + \frac{\Psi^6}{(\Psi^3 - q^2)^2} \\ & \times \sqrt{\Psi^2 - \frac{1}{2}\frac{q^{12}}{\Psi^{16}} + \frac{6q^2}{\Psi^2} + \frac{24}{13}\frac{q^{10}}{\Psi^{13}} - \frac{9}{5}\frac{q^8}{\Psi^{10}} - \frac{48}{11}\frac{q^8}{\Psi^{11}} - \frac{8}{7}\frac{q^6}{\Psi^7} + \frac{6}{7}\frac{q^{10}}{\Psi^{14}} + \frac{9q^6}{\Psi^8} + \frac{3q^4}{\Psi^4} - \frac{48}{5}\frac{q^4}{\Psi^5} - 4M}. \end{aligned} \tag{47}$$

In Fig. 2 we have plotted $f_1(\Psi)^2$ in the domain $q^{2/3} < \Psi < \infty$, for $q = 6$, and five different values of the mass M. (The picture is actually generic for all values $q > q_c$.) Let us analyze each curve separately.

3.4.1. Light black holes, $M < M_{\text{crit}}$, and extreme black holes

The lightest case, corresponding to $M_{\text{ext}} = 7.057\ldots$, represents the extreme black hole. It touches the horizontal line once, and its derivative is zero there too. The masses $M_{\text{ext}}(q)$ are defined by the equations

$$f_1^2 = 0, \qquad \frac{df_1^2}{d\Psi} = 0. \tag{48}$$

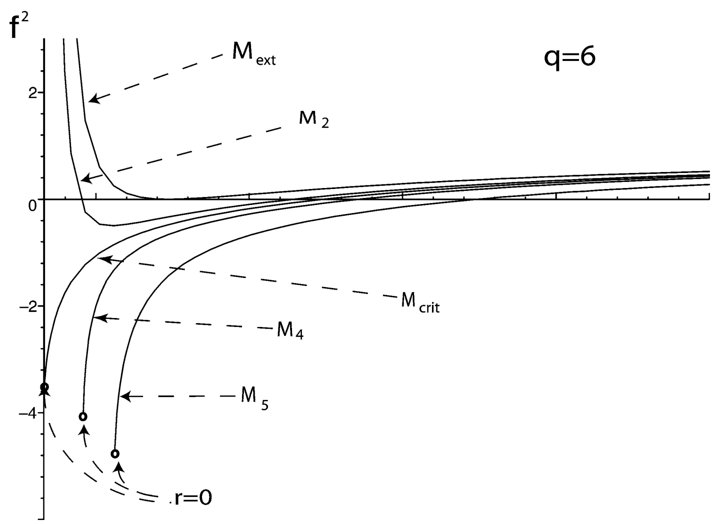

Fig. 2. The function $f_1^2(\Psi)$ in the domain $q^{2/3} < \Psi < \infty$ for $q = 6$ and five different values of M.

These equations can be solved numerically and we have found a *linear* relation

$$M_{\mathrm{ext}}(q) \simeq -0.30914\ldots + (1.2247\ldots)q. \tag{49}$$

(The linear approximation is better for charges $q > 2$.) This result is remarkable because Eq. (48) defining $M_{\mathrm{ext}}(q)$ form a extremely non-linear system. Note also that the slope of the curve approaches $1.2247\ldots = \sqrt{3/2}$, which is precisely the value obtained by the asymptotic solution (valid for $q/r^2 \ll 1$),

$$f_1^2 \simeq 1 - \frac{2M}{r^2} + \frac{2}{3}\frac{q^2}{r^4} = \left(1 - \sqrt{\frac{2}{3}}\frac{q}{r^2}\right)^2, \quad M = \sqrt{\frac{3}{2}}q.$$

A word of caution is in order here. This analysis does not imply that the function $M_{\mathrm{ext}}(q)$ is exactly linear, for all values of q. We only claim that the linear relation is a good approximation for that curve.

Let us now we move to the curve $M_2 = 7.8$. This looks very much like a standard charged black hole. f_1^2 intersects the horizontal line twice, and thus there are two horizons.

The black holes discussed so far have one or two horizons, and f_1^2 diverges as one approaches $\Psi \to q^{2/3}$; in these cases, the Kasner singularity at $\psi^3 = q^2$ is met *before* the origin $\rho = 0$.

3.4.2. The critical mass $M = M_{\mathrm{crit}}$

If we carry on making the black hole heavier, we reach the curve (for $q = 6$) $M_{\mathrm{crit}} = 8.113\ldots$ where something new happens (we give a close expression for M_{crit} below). This curve intersects the horizontal line only once, and thus it has only one horizon. Also, at the origin, f_1^2 has a finite value.

To have a better understanding of this case, consider the function f_1^2 displayed in (47). For generic values of M there is a explicit singularity at $\Psi^3 = q^2$. However, if M is fine-tuned such that the numerator (square root) vanishes at that point, the pole is cancelled. In fact, one observes that the zero in the square root is stronger than the zero in the denominator, and that whole term vanishes at $\Psi^3 = q^2$. The value of f_1^2 at that point is then,

$$f_1^2\big|_{\Psi^3=q^2} = 1 - \frac{3}{2}q^{2/3}, \tag{50}$$

which is in fact finite. Since M enters linearly in the square root, the value of M, called M_{crit}, such that the square root vanishes at $\Psi^3 = q^2$ can be calculated directly,

$$M_{\mathrm{crit}} = \frac{3^9}{8\cdot 13!!}q^{2/3}\bigl(33q^{2/3} + 26\bigr), \tag{51}$$

$(13!! \equiv 13 * 11 * 9 \cdot 3 * 1)$.

Finally, recall that the square root is nothing but ρ^2 (see Eq. (38)). This means that, by definition, at the critical mass M_{crit}, the origin $\rho = 0$ and the Kasner singularity $\Psi^3 = q^2$ coincide.

3.4.3. Heavy black holes: $M > M_{\mathrm{crit}}$

Let us now increase the value of M above M_{crit}. We find the curves $M_4 = 8.5$ and $M_5 = 10$. These curves intersect the horizontal line only once. The associated black holes then have only one horizon, despite being charged.

In this class of solutions (with $M > M_{\mathrm{crit}}$) the origin $\rho = 0$ is met *before* the Kasner singularity. This is the reason that the curve stops before reaching $\Psi^3 = q^2$. At $\rho = 0$, f_1^2 has a finite value (just like the uncharged black hole discussed in the introduction).

3.4.4. The critical charge

We have seen that the spectrum of black holes is separated into two branches, the heavy branch with $M > M_{\mathrm{crit}}$ and the light branch with $M < M_{\mathrm{crit}}$. The interphase is defined by the critical curve $M = M_{\mathrm{crit}}$ displayed in (51)

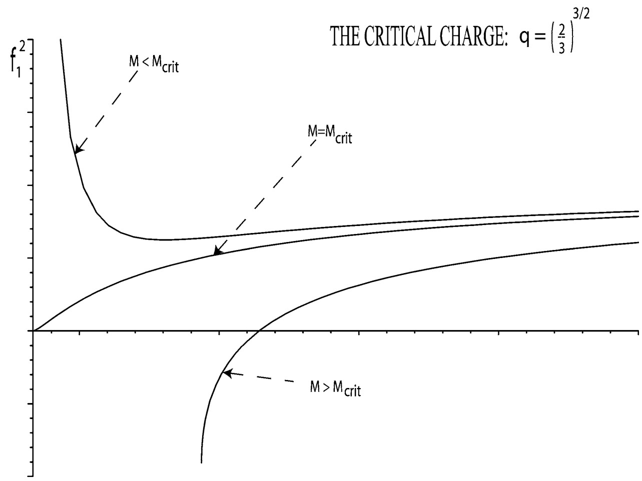

Fig. 3. The function f_1^2 as a function of Ψ for $q = (2/3)^{3/2}$.

which depends on the charge q. We shall now see that there exists a particular value of q, namely,

$$q_c = \left(\frac{2}{3}\right)^{3/2}, \tag{52}$$

for which the light branch produces only naked singularities.

In fact, going back to Eq. (50) we note that for $q = q_c$, the value of f_1^2 at $\Psi^3 = q^2$ is zero. We have plot in Fig. 3 the function f_1^2 for $q = q_c$ and three different masses, $M < M_{\text{crit}}$, $M = M_{\text{crit}}$ and $M > M_{\text{crit}}$. We observe that the curve $M < M_{\text{crit}}$ cannot intersect the horizontal line. The light branch thus gives rise only to naked singularities.

For charges $q < q_c$ the critical curve is pulled upwards. The light branch will carry on producing naked singularities, while the heavy branch will have both, black holes and naked singularities depending on M.

These results are summarized in the "phase diagram" displayed in the introduction, Fig. 1, showing the various black holes types for all values of M and q. The most important aspect of that diagram, and of this Letter, is the existence of two types of black holes for charges $q > q_c$, which are continuously connected by varying the parameters M and q.

3.5. The Coulomb potential

So far we have only analyzed the properties of the metric. To compute the value of the electrostatic potential we first go back to the redefinitions (15), and recall that the functions N_1, ψ_1 and Ψ_1 are known in terms of Ψ, which is algebraically related to the radial coordinate. The full expression for the potential is not very illuminating so we do not display it here. We only quote the result in the asymptotic limit $r \to \infty$

$$\phi(r) \simeq -3\sqrt{6}\frac{q}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{53}$$

showing as stated above that the electromagnetic potential is asymptotically controlled by Maxwell equations.

Acknowledgements

The author would like to thank Guido Tarrach for useful conversations, and Kayll Lake for introducing him into GRTensor II (http://grtensor.org/), which was very helpful in various calculations. This work was partially supported by grant No. 1000744 (FONDECYT, Chile), and from Ecos(France)–Conicyt(Chile) grant No. C01E05.

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