Linear Algebra and its Applications 430 (2009) 2659–2680

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Magic square spectra

Peter Loly [a,*], Ian Cameron [a], Walter Trump [b], Daniel Schindel [c]

[a] Department of Physics and Astronomy, The University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2
[b] Gymnasium Stein, Faber-Castell-Allee 10, D-90547 Stein, Germany
[c] Department of Physics and Astronomy, University of Western Ontario, London, Ontario, Canada N6A 3K7

A R T I C L E I N F O

Article history:
Received 26 December 2007
Accepted 30 October 2008
Available online 19 December 2008

Submitted by R.A. Brualdi

AMS classification:
00A08
05B20
11C20
15A03
15A18
15A36

Keywords:
Magic squares
Vector spaces
Characteristic equations
Eigenvalues
Singularities
Diagonability
Frenicle and Schroeppel complete sets

A B S T R A C T

To study the eigenvalues of low order singular and non-singular magic squares we begin with some aspects of general square matrices. Additional properties follow for general semimagic squares (same row and column sums), with further properties for general magic squares (semimagic with same diagonal sums). Parameterizations of general magic squares for low orders are examined, including factorization of the linesum eigenvalue from the characteristic polynomial.
For nth order natural magic squares with matrix elements $1, \ldots, n^2$ we find examples of some remarkably singular cases. All cases of the regular (or associative, or symmetric) type (antipodal pair sum of $1 + n^2$) with $n - 1$ zero eigenvalues have been found in the only complete sets of these squares (in fourth and fifth order). Both the Jordan form and singular value decomposition (SVD) have been useful in this study which examines examples up to eighth order.
In fourth order these give examples illustrating a theorem by Mattingly that even order regular magic squares have a zero eigenvalue with odd algebraic multiplicity, m. We find eight cases with $m = 3$ which have a nondiagonal Jordan form. The regular group of 48 squares is completed by 40 squares with $m = 1$, which are diagonable. A surprise finding is that the eigenvalues of 16 fourth order pandiagonal magic squares alternate between $m = 1$, diagonable, and $m = 3$, non-diagonable, on rotation by $\pi/2$. Two eighth order natural magic squares, one regular and the other pandiagonal, are also examined, found to have $m = 5$, and to be diagonable.
Mattingly also proved that odd order regular magic squares have a zero eigenvalue with even multiplicity, $m = 0, 2, 4, \ldots$. Analyzing

* Corresponding author.
E-mail address: loly@cc.umanitoba.ca (P. Loly).

doi:10.1016/j.laa.2008.10.032

results for natural fifth order magic squares from exact backtracking calculations we find 652 with $m = 2$, and four with $m = 4$. There are also 20,604 singular seventh order natural ultramagic (simultaneously regular and pandiagonal) squares with $m = 2$, demonstrating that the co-existence of regularity and pandiagonality permits singularity. The singular odd order examples studied are all non-diagonable.

1. Introduction

1.1. Types of magic squares

Here we consider the general case of a square array, A, with real elements a_{rs}, deferring to a later section (6.11.2) some rarer types which make more sense once we discuss magic squares composed of sets of sequential natural numbers (see e.g. [31]).

- **Semi-magic squares**: when all row and column sums are constrained to have the same magic sum, $S(A)$:

$$\sum_{r=1}^{n} a_{rs} = \sum_{r=1}^{n} a_{sr} = S(A) \quad \text{for } s = 1, 2, \ldots, n, \tag{1}$$

  the resultant matrices describe semimagic (or doubly affine) squares.
- **Magic squares**: if in addition to the above, both the principal diagonal and the dexter diagonal [11] also sum to $S(A)$:

$$\sum_{r=1}^{n} a_{rr} = \sum_{r=1}^{n} a_{r,n-r+1} = S(A) \quad \text{for } s = 1, 2, \ldots, n, \tag{2}$$

  we have a general magic square.
- **A pandiagonal magic square** has all the broken (or co-)diagonals (n consecutive elements parallel to the main diagonals under tiling, as indicated below for the set sloping down to the right) with the same magic sum, $S(A)$:

$$\begin{array}{|c|c|c|c|}\hline \spadesuit & \heartsuit & \clubsuit & \diamondsuit \\ \hline \diamondsuit & \spadesuit & \heartsuit & \clubsuit \\ \hline \clubsuit & \diamondsuit & \spadesuit & \heartsuit \\ \hline \heartsuit & \clubsuit & \diamondsuit & \spadesuit \\ \hline \end{array} \tag{3}$$

  It may help to tile a copy of a magic square to an edge of itself to see the continuity of n element lines, or even to wrap the square onto a torus (see e.g. [26]) to join all opposite edges for the same effect (periodic boundary conditions).
- **Regular (or associative, or symmetric) magic squares** exist if all pairs of elements which are antipodal to each other have the same pair sum:

$$a_{ij} + a_{n-i+1,n-j+1} = \text{constant}; \quad i, j = 1, \ldots, n \tag{4}$$

  For odd n the centre element, which can be seen as pairing with itself, must be half of this *constant*.
- **Ultramagic squares** have both the regular and pandiagonal properties.

1.2. Magic squares considered as matrices

McClintock [50] evidently reckoned that if magic squares were worthy of study by "an Euler or a Cayley" then they have legitimacy. Magic squares considered as matrices have a rich connection with linear algebra. As far as we have been able to determine, Fox [23] in 1956 was the first to use the term 'magic square matrix' in the context used here. His context was the inverse when the matrix was non-singular. Studies fall into several categories according to the nature of their matrix elements: (a) general results for fields, e.g., rational, real, complex numbers, (b) general results for rings, e.g., the integers (which have no multiplicative inverse), and (c) the fascinating cases of sequential natural numbers. In order to clarify what is already known from a considerable but widely spread literature we first review the more general case of magic squares composed of real entries, before examining the **natural** (or **classical**) cases composed of equally spaced sequential integer elements, the nth order with entries consisting of the first n^2 natural numbers, which often possess strikingly simple properties. A recent review of scientific aspects of magic squares has been given by Loly [42]. Here the focus is with the mathematics. In much of the literature normal is used in place of **natural**, but since **normal** matrices have another meaning in the literature on linear algebra, we have chosen to use **natural**, while **classical** could include closely related magic squares comprised of $0, \ldots, (n^2 - 1)$, which are sometimes convenient mathematically.

The present work began by noting a strikingly simple eigenvector associated with the magic sum (the "magic eigenvalue" or linesum) in experimental computations on natural magic squares from third order to more than 20th order. Initially Mathematica® was used in this study, which then continued using Maple® for a study of the complete set of fourth order magic squares, before changing focus to find examples of singular magic squares which illustrate two theorems of Mattingly [48]. However many of the results can be obtained through simple analysis of characteristic equations. The examples studied include degenerate (multiple) zero eigenvalues which can render the matrix non-diagonable (e.g. [14,62,15,76]). Eventually the need for a proper introduction led to a careful review of earlier work on magic squares for orders $n = 3, 4$, and 5, a number of new results, and some insight into higher order magic squares through examples up to eighth order. Singular value decomposition (SVD) and the Jordan form of matrices have been helpful in discussing the singular cases.

In studying the eigenproperties of natural magic squares, which may be singular or not, this work sheds light on the determinants of natural magic squares when they do not vanish. Of course, historically magic squares and determinants predate the 19th century invention of matrices.

2. Preliminary observations

A few properties of general square matrices are worth noting at the start, before proceeding with those additional properties possessed by general semimagic squares, and finally general magic squares. From any of the standard texts on linear algebra or matrix computations (e.g. [33,51]), and when A is an $n \times n$ matrix we list the following:

(a) There are two conventions for the characteristic polynomial, (i) $p(\lambda) = \det(A - \lambda I)$, with a term $(-1)^n\lambda^n$, or (ii) $P(\lambda) = \det(\lambda I - A)$, with a term $+\lambda^n$ (see Horn and Johnson [33]). Then $P(\lambda) = (-1)^n p(\lambda)$. The characteristic equation is obtained by setting each equal to zero and thus the sign of λ^n may be taken to be $+1$ in both cases. N.B. In Mathematica® the `CharacteristicPolynomial` function corresponds to (i), while in Maple® the `charpoly` function corresponds to (ii).

(b) If λ_i are the eigenvalues of A, the roots of the characteristic equation, $P(\lambda) = 0$, may be written as a product of terms equal to a polynomial:

$$\prod_i^n (\lambda - \lambda_i) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n = 0, \tag{5}$$

where we followed Meyer's [51] notation for the remaining coefficients of the characteristic equation.

In standard notation, e.g., Meyer [51], one can write the coefficients of (5) as

$$c_k = (-1)^k s_k, \tag{6}$$

where

$$s_k = \sum(\text{all } k \times k \text{ principal minors}). \tag{7}$$

The trace of A is then

$$s_1 = \sum_{i=1}^{n} \lambda_i = -c_1, \tag{8}$$

the sum of all 2×2 principal minors of A is

$$s_2 = \sum_{i \neq j} \lambda_i \lambda_j = c_2, \tag{9}$$

and the determinant of A:

$$s_n = \prod_{i=1}^{n} \lambda_i = (-1)^n c_n. \tag{10}$$

(c) A and its transpose A^T have the same characteristic polynomial, the same eigenvalues, with the same multiplicities, since $\det(\lambda I - A^T) = \det(\lambda I - A)$.
(d) If A has n distinct eigenvalues then it is diagonalizable (semisimple). A matrix with some multiple eigenvalues may not be diagonizable.
(e) The determinant of A is the product of its eigenvalues (having in mind its algebraic multiplicities) and is equal to $p(0)$, the constant term in $p(\lambda)$.
(f) The trace of A is the sum of its eigenvalues (having in mind its algebraic multiplicities) and except for possibly for a sign, is equal to the coefficient of λ^{n-1} in the polynomial $p(\lambda)$.
(g) For A the following conditions are equivalent: (i) A is singular (non-invertible). (ii) The determinant of A is zero. (iii) $\lambda = 0$ is an eigenvalue of A. If the matrix is singular, there may still be useful information in the other coefficients of (5).

Hereafter we will use x in place of λ, noting that some authors use z, e.g., Amir-Moez and Fredricks [2].

2.1. Rotations and reflections

Prompted by Hruska's [35] study which used two orientations of the third order natural magic square, Loly [43] began a detailed examination of the effect of rotations, as well as reflections, on both the determinant and the characteristic equation for general square matrices. The dihedral group D_4 [36] transforms a plane square into itself under rotations by units of $\pi/2$ radians (a quarter turn, or 90°), and by reflections about the horizontal and vertical medians. Reflections about the diagonals can be obtained by a combination of group elements, and constitute transposes about both diagonals. We will call the four front views and the four back views of a square matrix the eight **phases** (or variants) of the matrix. They will be shown in an explicit numerical example later in Section 4.2. There are generally two distinct diagonals for these eight phases which can be divided into two interlaced sets, one for each diagonal (see later in Section 4.2 for an explicit third order example where they are labelled (a) and (b)). Recently Chu [14] and Styan [62] have called the four matrices associated with one diagonal "sweet" and those associated with the other diagonal "sour".

Finding curious effects for fourth and fifth order magic squares, Loly [43] realized that these issues were best approached by first considering the situation for general square matrices. Loly's [43] results include: (i) for non-singular matrices the sign of the determinant of a general square matrix changes under a rotation of $\pi/2$ for $n \bmod 4 > 1$ (or for $n = 2 + 4r$ and $n = 3 + 4r$, $r = 0, 1, 2, \ldots$), i.e., for $n = 2, 3; 6, 7; \ldots$, but not for $n = 4, 5; 8, 9; \ldots$, and, (ii) there are just two characteristic equations, one for each diagonal, with potentially two sets of eigenproperties for the eight phases. This behaviour of the determinant and eigenproperties for the eight phases of a general square matrix under D_4 does not appear to be well known.

We shall see later in Section 6 that even when the determinant vanishes in singular cases there are differences when one looks at the eigenproperties of the sweet and sour phases, except in extreme singular cases.

2.2. Semimagic squares

Beyond $S(A)$ in (1) we may identify further properties. The magic sum (1) does not change under rotations by $\pi/2$ or by reflection about the horizontal or vertical medians, and the uniform (n-agonal) right and left eigenvectors $(1, 1, \ldots, 1)$ and $(1, 1, \ldots, 1)^T$, have a common eigenvalue, say λ_1, equal to the magic sum (e.g., see [64]). In 1984 Amir-Moéz and Fredricks [2] proved that the remaining roots:

$$\lambda_2, \ldots, \lambda_n, \tag{11}$$

which they called **complementary characteristic roots**, are unchanged by addition of complex factors of E_n, the $n \times n$ matrix of all ones. Also, for semimagic squares with non-negative elements, Khan [37] proved that λ_1 is not less than the absolute value of any other eigenvalue, and since Perron's theorem (see e.g. [33]) states that a matrix with positive entries has a unique eigenvalue of maximum modulus, then λ_1 must be the largest eigenvalue for a semimagic square with positive entries.

3. General magic squares

The trace of magic squares is obviously invariant under rotation or reflection because each diagonal has the same sum. Amir-Moéz and Fredricks [2] then proved a powerful factorization theorem for the characteristic polynomial:

$$(x - \lambda_1)q(x), \tag{12}$$

where

$$q(x) = x^{n-1} + a_{n-3}x^{n-3} + \cdots + a_0. \tag{13}$$

We note the absence of an x^{n-2} term in (13). This follows since one of the eigenvalues of a magic square is the magic sum from 2.2, and the trace is the sum of the eigenvalues, so that the other eigenvalues must sum to zero (see e.g. [64,29,75]). It is also easy to show that in (13), the coefficient $a_{n-3} = c_2$, the sum of the 2×2 principal minors in (9) (see also e.g. [76]).

The result in (13) has been overlooked in much of the subsequent literature, perhaps because Amir-Moéz and Fredricks [2] only applied it to the invertibility of the general third order magic square. However it elegantly describes some results discussed later which we found in examples of low order magic squares. From (13), using (5) and (6), it also follows that:

$$c_n = -\lambda_1 a_0, \tag{14}$$

from which the determinant can be obtained via (10).

3.1. Parameterization of general magic squares

The general nth order magic square can be parametrized in terms of $n(n-2)$ variables (see e.g. [68,13,74,16,49]). This is the number of matrix elements (n^2), less twice the number of rows (columns). Moreover if the line constant is fixed, then there is one less independent variable.

We continue by reviewing aspects of general third and fourth order magic squares in order to set the scene for an appreciation later of the unique role of integer magic squares.

3.2. Parameterization of general third order magic squares

The earliest reference we have been able to find is due to Lucas [46] in 1891, and the essence of that result is repeated in many places, often without prior referencing (see e.g. [40,39,13]).

First we add a common average value, c, to each cell of Lucas's parameterization [46] to give the general expression of a magic square of order three:

$$\text{Lucas} = \begin{bmatrix} c+u & c+v-u & c-v \\ c-u-v & c & c+u+v \\ c+v & c+u-v & c-u \end{bmatrix}, \tag{15}$$

where the sum of all rows columns and the main diagonals is now $3c$. It is also a regular magic square with antipodal pairs summing to $2c$. The characteristic equation is then:

$$x^3 - 3cx^2 + 3(v^2 - u^2)x - 9c(v^2 - u^2) = 0. \tag{16}$$

3.2.1. Factoring out the magic eigenvalue

Relatively little attention has been paid to the simple factorizing out of the magic sum eigenvalue, $3c$, from the characteristic equation (16). We provide that now, finding a simple quadratic for the complementary eigenvalues:

$$(x - 3c)(x^2 - 3u^2 + 3v^2) = 0. \tag{17}$$

The determinant is $9c(v^2 - u^2)$, showing that the matrix is singular if $u = \pm v$, or if the centre value c vanishes. The eigenvalues are $3c$, and a signed pair $\pm\sqrt{3}\sqrt{u^2 - v^2}$, which are either real or imaginary. The sum of this pair of eigenvalues vanishes as expected and the absence of the linear term in the second bracket is also in accord with (13).

Interchanging u and v, which is a reflection about the middle row, changes the signs of the coefficients of x and x^0 in the characteristic equation.

These eigenvalues have been given in many places, but not quite as simply, although several are particularly instructive for the present study. Wardlaw [75] posed the third order parameterization as a problem for readers of the *American Mathematical Monthly* in May 1991 to determine whether the matrix corresponding to 3×3 magic squares is singular or not, for in the latter case it has a non-vanishing determinant and is therefore invertible. That journal reported receiving 57 solutions, printing one by Hartman [29] in December 1992.

3.2.2. The basic rotation

Rotate (15) by $\pi/2$ clockwise:

$$\begin{bmatrix} c+v & c-v-u & c+u \\ c+u-v & c & c+v-u \\ c-u & c+u+v & c-v \end{bmatrix}. \tag{18}$$

Now the characteristic equation is

$$x^3 - 3cx^2 + 3(u^2 - v^2)x - 9c(u^2 - v^2) = 0, \tag{19}$$

where the linear and constant terms have changed sign: the determinant has changed sign, $-9c(v^2 - u^2)$; eigenvalues are $3c$, as before, but with a signed pair $\pm\sqrt{3}\sqrt{v^2 - u^2}$, so that if those eigenvalues were previously real (imaginary), they are now imaginary (real). Now simultaneous transformations $\{v \to u, u \to (-)v\}$ restore the matrix to our Lucas form in (15). Note that Dernham [17] stated the result for the determinant of this magic square changing sign on reflection, but erred in stating that rotation did not change the sign.

3.3. Parameterization for fourth order magic squares

So far we have only looked in detail only at the third order case, so it would be cautionary not to assume that everything is similar for higher orders. The simple alternation between real and imaginary eigenvalues for $n = 3$ appears to only be true for special cases in higher orders, and then only for singular squares. The effect of rotations discussed in Section 2.1 might suggest that fourth order magic squares are simpler under rotation, than are the third order variety, because fourth order determinants do not change sign under rotation or reflection, but we shall see later that this is definitely not the case because of changes in other coefficients of the characteristic polynomials. In 1910 Bergholt [7] found a parameterization for the fourth order general magic square, which is also found in Kraitchik [40], Ball and Coxeter [3] and Descombes [18]:

$$\begin{bmatrix} A-a & C+a+c & B+b-c & D-b \\ D+a-d & B & C & A-a+d \\ C-b+d & A & D & B+b-d \\ B+b & D-a-c & A-b+c & C+a \end{bmatrix}. \tag{20}$$

Factoring out the magic sum eigenvalue, $A+B+C+D$ from the characteristic polynomial leaves behind a complicated cubic equation with no quadratic term in accord with (13):

$$x^3 + \beta x + \gamma = 0. \tag{21}$$

If the remaining eigenvalues are called p, q, r, then by forming:

$$(x-p)(x-q)(x-r) = x^3 - (p+q+r)x^2 + (pq+qr+rp)x - pqr = 0, \tag{22}$$

we see that the sum $p+q+r$ vanishes because the trace is the magic sum, in accord with the theorem of Amir-Moez and Fredricks [2]. This will be useful later in analyzing fourth order results.

3.4. Parameterization for fifth order magic squares

The parameterization of the general fifth order magic square (see [13,18]), after factoring out the magic eigenvalue, leaves a reduced form of quartic, namely one without a cubic term. For the regular case, after factoring out the magic eigenvalue, one is left with an even more reduced form of quartic, namely one with no cubic and linear terms:

$$x^4 + \beta x^2 + \gamma = 0, \tag{23}$$

which being even in x, means that the solutions for x are simply two signed pairs of quadratic solutions, which is consistent with Mattingly's [48] theorem with $m = 0$.

Chernick [13] outlined a procedure for carrying out these parameterizations for arbitrary order.

3.5. Powers of magic squares

van den Essen [22] showed that odd powers of third order magic squares were also magic via an analysis which recognizes the important role of semimagic squares.

Thompson [64] studied odd powers of magic squares for third, fourth and fifth orders and in the process focussed on the invertibility of the squares by analyzing their eigenvalue structure. Both Thompson [64] and van den Essen [22] showed that odd powers of 3×3 magic squares are themselves magic, and Thompson [64] also shows that pandiagonal magic squares in fourth and fifth orders have this property. Excluding the line sum eigenvalue, Thompson [64] conjectured that these results follow from the symmetrical placement of the remaining eigenvalues about the origin, i.e., as signed pairs. Thompson's [64] analysis is useful in the present paper when we come to examine specific examples provided by complete sets of natural magic squares.

Chu [14] and Styan et al. [62], and their collaborators [15], have recently studied odd powers of magic squares, obtaining several interesting theorems. Trenkler [66] discussed the eigenvalues of the general third order magic square by studying the eigenvalues of its square and found a signed pair of eigenvalues in addition to the line sum eigenvalue. Gauthier [28] also looked at powers and polynomials of the general third order magic square.

4. Natural magic squares

Natural semimagic and magic squares using the equally spaced set $1, \ldots, n^2$ have a magic sum given by

$$S_n = \frac{n}{2}(n^2 + 1). \tag{24}$$

We note that these natural squares are by definition not symmetric. That all the s_k in (7) are now integers means that the value of the determinant is a positive or negative integer, unless it vanishes.

4.1. Integer versus real squares

The recreational mathematics literature is dominated by construction and counting, so it is not surprising to find there an emphasis on squares with the natural numbers for their matrix elements. It is clear that natural magic squares can be scaled by real, complex or rational factors to produce corresponding behaviour (properties) for an infinite subset of general magic squares. The reverse situation, from general squares with real elements, cannot generally scale to an integer square. Brock [9] has an insightful discussion of the rarity of singular matrices which is broader than the present concern with magic squares, and Hetzel et al. [32] have discussed the related issue of when integer matrices are diagonalizable.

It is worth noting that there is a close relationship between pandiagonal and regular magic squares which causes some confusion in the extant literature, e.g., neither natural regular nor natural pandiagonal squares exist for singly even order, i.e., $n = 6, 10, \ldots$ (see e.g. [58,73,56]), although they may exist in the general case. Clearly the finite sets of squares found under the restriction to natural squares offer a special opportunity.

We note that the physical moment of inertia of magic squares is constant for all natural magic squares of a given order [41]. This concept can be extended to general semimagic squares, but it does not then give an invariant for a given order because their elements are not constrained to a particular equally spaced set of values.

Non-classical magic squares use a non-sequential set of integers, while in this paper all the magic squares studied use the natural sequence $1, \ldots, n^2$.

4.2. Lo Shu – the unique third order natural magic square

The original ancient Chinese orientation of the Lo Shu (or Lo-shu, Loshu, Luoshu, ...) is

$$L = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix} \tag{25}$$

It has the following pairs of eigenvectors and eigenvalues, beginning with the linesum:

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \leftrightarrow 15, \quad \begin{Bmatrix} -\frac{2}{7}i\sqrt{6} - \frac{5}{7} \\ \frac{2}{7}i\sqrt{6} - \frac{2}{7} \\ 1 \end{Bmatrix} \leftrightarrow -2i\sqrt{6}, \quad \begin{Bmatrix} \frac{2}{7}i\sqrt{6} - \frac{5}{7} \\ -\frac{2}{7}i\sqrt{6} - \frac{2}{7} \\ 1 \end{Bmatrix} \leftrightarrow 2i\sqrt{6}. \tag{26}$$

For L the determinant has the value 360, the characteristic polynomial is $x^3 - 15x^2 + 24x - 360 = (x - 15)(x^2 + 24)$. From (5) $c_1 = -15, c_2 = 24, c_3 = -360$. For $n = 3$ the determinant divided by n and by the line sum λ_1 is also an integer (see [17]). Also, since L and its transpose do not commute, this magic square is not a **normal** matrix, i.e., one which commutes with its conjugate transpose (see e.g. [5]).

Other authors have used a different orientation (or phase), e.g., Hruska [35] studied two of those phases. Beginning with the Lo Shu phase, we rotate with operator R successively clockwise by $\pi/2$ in Table 1 before taking the transpose of the original to display its reverse side and then listing its rotations.

Table 1 Rotation of the Lo Shu magic square (R is the basic rotation by $\pi/2$ and $L^{\perp}$ is a transpose about the dexter diagonal), $c_1 = -15$, (a) and (b) are the alternating eigenvalue sets, (c) is Hruska's [35] A, (d) is in Frenicle's [24] standard form [6], (e) is the horizontal flip of L, and (f) is Hruska's [35] B and the vertical flip of L.

Phase	Face	Top row	c_2	c_3	Eigenvalues
L (25)	1	4, 9, 2	+24	−360	(a) $15, \pm 2i\sqrt{6}$
RL	1	8, 3, 4	−24	360	(b) $15, \pm 2\sqrt{6}$
R^2L (c)	1	6, 1, 8	+24	−360	(a)
R^3L (d)	1	2, 7, 6	−24	360	(b)
L^T	2	4, 3, 8	+24	−360	(a)
RL^T (e)	2	2, 9, 4	−24	360	(b)
$R^2L^T \equiv L^{\perp}$	2	6, 7, 2	+24	−360	(a)
R^3L^T (f)	2	8, 1, 6	−24	360	(b)

In Table 1, two groups of four phases whose members alternate share one of two characteristic polynomials. From (5) the signs of c_2 and c_3 coefficients alternate but c_1 is always -15, and from (6) the trace is always $s_1 = +15$. Why this happens is now clear in general from Loly [43].

Hereafter we only need consider one of each set, for which we choose a pair related by the basic rotation. It may be of interest to note that the physicist (Loly) used rotations, while the mathematician and statistician Styan [62] used the reversal (flip) permutation operator J of all ones along the dexter diagonal. In computer codes the J operator is often convenient, because of the absence of a built-in rotation operator in some software packages.

In the case of a general third order magic square, both c_2 and c_3 either alternate in sign or vanish as shown earlier in (16). Later we show more complicated results on rotation for fourth order magic squares. Despite these rotational and flip effects it has been common to count just one of the eight phases as distinct in most studies.

Anticipating additional information needed for higher order magic squares, especially for singular squares, we introduce these aspects in this simplest case. First the Jordan form (see e.g. [5]), which consists of Jordan blocks on the diagonal. The Lo Shu shows the simplest form with first order (diagonal) 1-by-1 Jordan blocks:

$$JF(L) = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 2i\sqrt{6} & 0 \\ 0 & 0 & -2i\sqrt{6} \end{bmatrix} \tag{27}$$

showing the non-degenerate eigenvalues in a diagonalized matrix.

Second, singular value decomposition (SVD) (see [34,21]), in which the singular values are the square roots of the eigenvalues of the two Gramian products of the transpose of a matrix with itself (see e.g. [25]). For the Lo Shu these Gramian products are

$$L^TL = \begin{bmatrix} 89 & 59 & 77 \\ 59 & 107 & 59 \\ 77 & 59 & 89 \end{bmatrix} \quad \text{and} \quad LL^T = \begin{bmatrix} 101 & 71 & 53 \\ 71 & 83 & 71 \\ 53 & 71 & 101 \end{bmatrix}. \tag{28}$$

In general these Gramians are symmetric matrices, but in the Lo Shu case they are also bisymmetric. They both have the same characteristic polynomials and thus the same eigenvalues, 225, 48, 12, the square roots of which give the singular values: $15, 4\sqrt{3}, 2\sqrt{3}$. These singular values are the same for all phases, as shown by Loly [43] for general square matrices.

5. Complete sets of natural magic squares

Central to the present contribution is the availability of complete finite sets of natural magic squares for order 4, numbering 880 distinct squares, and also for order 5. Those of order four were first enumerated by Frénicle de Bessy [24] (see also [6]) in 1693, and have been classified in various ways, with the count definitively analyzed by Ollerenshaw and Bondi [55] (see also [8]). In 1973 Schroeppel (see [26,4]) used an early PDP-10 computer to count the fifth order set, finding 275, 305, 224 distinct squares by an exhaustive backtracking procedure. We (see [73]) also used a similar backtracking approach for fifth order, and for the ultramagic subset of seventh order.

However, apart from a study of the determinants for the whole 880 by Trigg [70] and Brown [10], there has until recently been no thorough eigenvalue study, and much less for fifth order. It is well within the scope of a summer project for undergraduates to analyze these datasets. While our work in fourth order began with a dataset assembled by Heinz [30] in terms of Dudeney's [20] classification, the present authors have also used versions of a backtracking approach begun by one of us [73] for that order, for which the resulting dataset could be reordered into any desired sequence. Dudeney [20] identified 12 Groups based on patterns of element pairs which add to half the magic sum of 34. We summarize some aspects of these groups later in Tables 3 and 8.

5.1. The MATLAB® algorithms

In 1993 Moler [53] noted that natural magic squares generated by standard algorithms encoded in the MATLAB® `magic(n)` function were all singular in even order, but nonsingular in odd order. Shortly after Moler's observations Kirkland and Neumann [38] examined the doubly-even natural magic squares produced by the MATLAB® algorithm, finding for all orders, $4k, k = 1, 2, \ldots$, a rank 3 square and a general formula for their eigenvalues and singular value decomposition. In view of the elegant results from that study the fourth order square produced by MATLAB® is worth further discussion since it provides valuable guidance in interpreting results for other magic squares:

$$\mathtt{magic}(4) = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}. \tag{29}$$

The determinant of $\mathtt{magic}(4)$ vanishes and its characteristic polynomial may be written as $x(x^3 - 34x^2 - 80x + 2720) = x(x-34)(x^2-80)$, so that the eigenvalues are $0, 34, \pm 4\sqrt{5}$. This magic square has rank 3, with singular values $34, 8\sqrt{5}, 2\sqrt{5}, 0$. The Jordan form for this case is diagonal.

This magic square is closely related to one made famous by Albrecht Dürer in his 1514 engraving Melencholia I, differing only by the exchange of the middle columns to reveal that date.

For order $n = 4k$ Kirkland and Neumann [38] gave the 3 non-zero eigenvalues as $S_n, \pm\frac{n}{2}\sqrt{\frac{n(n^2-1)}{3}}$, and their three non-zero singular values as $S_n, \frac{n}{2}\sqrt{\frac{n^2(n^2-1)}{3}}$ and $\frac{n}{2}\sqrt{\frac{(n^2-1)}{3}}$, i.e., rank 3, which agree for $n = 4$ with the computed values above, while giving insight into a general form for one square in each higher order $(n = 8, 12, 16, \ldots)$.

Regular (or associative, or symmetric) natural magic squares satisfy the antipodal constraint:

$$a_{ij} + a_{n-i+1,n-j+1} = n^2 + 1; \quad i,j = 1, \ldots, n. \tag{30}$$

The odd and doubly-even algorithms in MATLAB® produce regular magic squares, while the singly even algorithm does not.

5.2. Mattingly's theorems

More recently Mattingly [48] shed considerable light on Moler's [53] observations by analyzing regular, but not necessarily natural, magic squares with the help of Perron's theorem, and through the use of deflation techniques to produce a skew-centrosymmetric matrix where the magic constant eigenvalue is replaced by a zero eigenvalue. In essence, Mattingly proved that regular magic squares have an odd multiplicity of non-zero eigenvalues, meaning that, e.g., in fourth order there may in principle be either one or three zero eigenvalues, while in fifth order there may be either none, two or four zero eigenvalues.

It is worth noting that Mattingly's [48] analysis of regular magic squares depends largely on the semimagic property and is also applicable to squares with real or complex elements.

We have identified all singular regular natural magic squares in orders four and five, and discuss these in the light of Mattingly's work [48]. In both cases we find maximally singular cases with just one non-zero eigenvalue. Our new results come from explicit studies of the only complete sets of natural magic squares, those of order four and five, and we have extracted the regular subset from each of those sets.

Mattingly's theorems are now illustrated separately for even and odd orders in the light of our new results.

6. Even order natural magic squares

Mattingly [48] showed that regular natural magic squares of even order are singular by proving that they have zero eigenvalues of odd multiplicity, $m = 1, 3, 5, \ldots, (n-1)$, but gave only one example of order four (4-by-4) with a single zero eigenvalue, $m = 1$.

6.1. Example of a 4-by-4 regular magic square with one zero eigenvalue

Mattingly's [48] fourth order example is the transpose about the dexter diagonal of Dürer's famous magic square which is listed as Dudeney [20] Group III, Frénicle index 175:

$$F175 = \begin{bmatrix} 1 & 12 & 8 & 13 \\ 14 & 7 & 11 & 2 \\ 15 & 6 & 10 & 3 \\ 4 & 9 & 5 & 16 \end{bmatrix}. \tag{31}$$

$F175$ has the following pairs of eigenvectors and eigenvalues:

$$\begin{Bmatrix} -1 \\ -3 \\ 3 \\ 1 \end{Bmatrix} \leftrightarrow 0, \quad \begin{Bmatrix} 7 \\ -3 \\ -5 \\ 1 \end{Bmatrix} \leftrightarrow -8, \quad \begin{Bmatrix} \frac{1}{7} \\ -\frac{5}{7} \\ -\frac{3}{7} \\ 1 \end{Bmatrix} \leftrightarrow 8, \quad \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \leftrightarrow 34.$$

The determinant vanishes and the characteristic polynomial factors as follows: $x^4 - 34x^3 - 64x^2 + 2176x = x(x+8)(x-8)(x-34)$. It is worth noting that all these eigenvalues are integers, something of a rarity (see [47]). The Jordan form is diagonal, and the singular values are $34, 8\sqrt{5}, 2\sqrt{5}, 0$ for rank 3. The trace of all fourth order natural magic squares is always $s_1 = 34 = S_4$.

Trenkler [67] has recently completed an interesting study of all Dürer-like magic squares.

6.2. Example of a 4-by-4 regular magic square with three zero eigenvalues

One of these squares is Dudeney Group III, Frénicle index 790:

$$F790 = \begin{bmatrix} 5 & 4 & 16 & 9 \\ 11 & 14 & 2 & 7 \\ 10 & 15 & 3 & 6 \\ 8 & 1 & 13 & 12 \end{bmatrix}. \tag{32}$$

Maple® gives the eigenvectors: $\begin{Bmatrix} -1 \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{Bmatrix} \leftrightarrow 0, \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \leftrightarrow 34$, with the characteristic equation $x^3(x-34) = 0$, showing the magic constant eigenvalue, $S_4 = 34$, and the triple degeneracy of the zero eigenvalue. $F790$ has rank 3 and the same singular values as $F175$. There are seven more magic squares in this group with the same spectra.

Going beyond Mattingly's examples, Chu [14] and Styan et al [62] noted that the eight natural regular squares with three zero eigenvalues are non-diagonable. We demonstrate this for $F790$ using the Jordan form which has these eigenvalues on the diagonal, but is not diagonal, being composed of the linesum eigenvalue and a 3-by-3 Jordan block:

Table 2
Regular types in Group III compared. (The prefix R denotes a rotation of $\pi/2$.)

Singular square ($c_4 = 0$)	c_2	c_3	Eigenvalues
$F790$, $m = 3$, all phases	0	0	34, 0, 0, 0
$F175$, $m = 1$ (sweet)	−64	+2176	$34, \pm 8, 0$
$RF175$, $m = 1$ (sour)	+64	−2176	$34, \pm 8i, 0$

$$JF(F790) = \begin{bmatrix} 34 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

This nondiagonal Jordan form is an example of an upper (or lower) triangular canonical form (see e.g. [52]).

Table 2 summarizes the effect of rotation on these two examples.

Note the sign alternation for c_2 and c_3 on rotation for $F175$ and $RF175$. Square $F790$ with just one non-vanishing eigenvalue has the same characteristic equation for all phases, while $F175$ is similar to the pattern of the third order square, with the fourth eigenvalue vanishing.

6.3. Parameterization of regular $n = 4$ natural magic squares

There are fewer independent variables when other constraints are applied, e.g., the pandiagonal, or the regular condition. The regular fourth order magic squares, which are of prime importance for the present study, have five independent variables and the magic eigenvalue factorizes out as expected [45]. If we fix the linesum at 34 and the antipodal pair sum at 17, the value for natural $n = 4$ magic squares, this set may be parameterized as follows:

$$\begin{bmatrix} 17-b & a+b+c-17 & b-a+d & 34-d-c-b \\ 17-c & 17-a & a+c-d & d \\ 17-d & d-a-c+17 & a & c \\ b+c+d-17 & a-b-d+17 & 34-a-b-c & b \end{bmatrix}. \tag{34}$$

This matrix has rank 3, with a characteristic polynomial which factorizes as

$$x(x-34)(x^2 + 1156 + 4d^2 - 68b + 4db + 4c^2 - 4da + 4cb - 136c + 4ca - 68d) \tag{35}$$

showing that all regular fourth order magic squares are singular, a specific instance of Mattingly's theorem for even n [48].

The Jordan form of (34) is diagonal:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 34 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & -\delta \end{bmatrix}, \tag{36}$$

where $\delta = 2\sqrt{da - ac - c^2 + 34c - bc - d^2 - bd + 17b + 17d - 289}$.

With the values for $a = 3, b = 12, c = 6, d = 7$ corresponding to $F790$ the sum of the constants in the second bracket vanishes of (35), as expected. For the values in $F790$ the δ's in (36) vanish, appearing to leave a diagonal Jordan form, in contrast to the specific result earlier that $F790$ is not diagonable. Perhaps this is a result of the presence of three zero eigenvalues not being anticipated in the general parameterization above in (34).

6.4. Change of multiplicity on rotation in Group I

Up to this point we focused on the regular Group III because of Mattingly's theorems [48]; however it seemed appropriate to check some other fourth order magic squares. Since the pandiagonal magic squares have a central place in the annals of magic square studies, it seemed appropriate to examine these first. From Dudeney's Group I we present square $F109$ which on rotation alternates between $m = 3$ and $m = 1$:

$$F109 = \begin{bmatrix} 1 & 8 & 11 & 14 \\ 15 & 10 & 5 & 4 \\ 6 & 3 & 16 & 9 \\ 12 & 13 & 2 & 7 \end{bmatrix}. \tag{37}$$

The eigenvalues are 34, 8, 0, −8, and the characteristic polynomial: $x^4 - 34x^3 - 64x^2 + 2176x = x(x - 34)(x - 8)(x + 8)$. $F109$ has a diagonal Jordan form and rank 3 from singular values $34, 4\sqrt{17}, 2\sqrt{17}, 0$. The Gramian matrices for Groups I and II are symmetric, unlike the regular squares of Group III which are bisymmetric.

However if $F109$ is rotated by $\pi/2$ to $RF109$, or flipped, it has different eigenproperties. Now the eigenvalues are 34, 0, 0, 0, i.e., triply degenerate zero eigenvalues, with the characteristic polynomial: $x^4 - 34x^3$, clearly different from the characteristic polynomial for $F109$. $RF109$ is non-diagonable with the same Jordan form as $F790$ in (33).

Thompson [64] also focussed on this pandiagonal Group I, with a particular interest for when powers of the squares remained magic.

6.5. Dudeney's Group II magic squares

Group II exhibits a new feature called **semi-bent diagonals**. For even order any combination of half diagonals from the corners which is not straight is called a bent-diagonal, as are those shifted left–right or up–down, including those wrapped over the edges. Following a particular symbol in the diagram below shows the bent diagonals which are staggered to the right:

♠	♥	♣	◇
◇	♠	♥	♣
◇	♠	♥	♣
♠	♥	♣	◇

(38)

For orders $n = 8$ and $n = 16$ Franklin (see [61]) introduced squares with bent diagonal sums equal to the row and column sums, which themselves are comprised of equal sum half rows and columns. Franklin's squares are semimagic because they do not have magic main diagonals, and are further discussed in Section 6.11.2.

The Group II magic squares have alternating semi-bent diagonals with the magic sum, interleaved with ones which do not have that sum. In Dudeney's Groups II through VI-P, shown in the next section, there is a similar semi-pandiagonal alternation of magic and non-magic pandiagonals.

The alternation of the number of zero eigenvalues on rotation shown in the previous section for some Group I magic squares does not occur for the regular set, Group III, but does occur for some in Group II, the semi-pandiagonal semi-bent set. Loly and Tromp [45] observed that Groups I and II have "global" constraints, i.e., surviving under tiling, whereas the regular constraint in Group III, having centred antipodal pairs is "local".

6.6. Multiplicities of the singular fourth order squares

We have analyzed the complete set of the 880 order four natural magic squares, where there are 640 singular magic squares. In Group III of this set there are 48 regular squares, 40 with $m = 1$ (including one from the MATLAB® algorithm), which are highlighted in Table 3. A summary for these 640 natural fourth order magic squares according to Dudeney's [20] Groups and their multiplicity of zero eigenvalues (m) which takes into account the effects of rotation (or flipping) are shown in Table 3.

Recently Vehkalahti [76] has used Mustonen's Survo [54] graphics package to animate the 640 singular magic squares in order to rapidly compare the squares in a given Group.

Table 3 Census of singular fourth order magic squares by Dudeney [20] Group and multiplicity (m) [30]. The last column shows that eight squares in Groups I and II oscillate between $m = 1$ and $m = 3$ on change of phase by rotation or flipping. The regular Group III is shown in bold, and VI-S (simple) has no pandiagonal or bent features.

Group	Type	Number	$m = 1$	$m = 3$	$1 \leftarrow m \rightarrow 3$
I	Pandiagonal	48	32	0	16
II	Semi-pandiagonal, semi-bent	48	32	0	16
III	**Regular**, semi-pandiagonal	**48**	**40**	**8**	**0**
IV	Semi-pandiagonal	96	96	0	0
V	Semi-pandiagonal	96	96	0	0
VI-P	Semi-pandiagonal	96	96	0	0
VI-S	Simple	208	208	0	0

6.7. Some general observations concerning the singular cases

Thompson [64] approaches these singular cases in a more general way. Using the trace property he gives three cases to consider: (i) gives the complementary eigenvalues as $0, \pm\lambda$, which would accommodate our $m = 3$ result in (6.2) if $\lambda = 0$, and the $m = 1$ case otherwise, while (ii) gives $\lambda, \omega\lambda, \omega^2\lambda$, where ω is the complex cube root of unity, as well as the possibility of a non-singular result if $\lambda \neq 0$ to describe an invertible square. In cases (i) and (ii) these non-zero eigenvalues are symmetrical about the origin. Clearly the $m = 3$ results for fourth order natural regular squares are covered if $\lambda = 0$ in either of these cases! Thompson's case (iii) is relevant for the non-singular examples in Section 6.9 where the characteristic eigenvalues are not symmetrical about the origin.

6.8. Singular values for Groups I, II and III

A further breakdown follows from the analysis of the singular values, especially for those cases exhibiting three vanishing eigenvalues, in part because of the invariance of the SVD results under rotation or flipping.

For Groups I, II, and III there are just three sets of singular values which are common to each Group (see Table 4).

The SVD results do not differentiate between the 16 squares alternating between $m = 1$ and $m = 3$ in each of Groups I and II, and the unchanging $m = 3$ pattern for eight squares in Group III because the SVD values are invariant under rotation (see [43]).

Set α has already been shown above in connection with the MATLAB® magic(4) and Kirkland and Neumann's study [38], while set β appeared for $F109$ in (37). The expressions in brackets follow from Kirkland and Neumann [38] for α, while those for β and γ were first inspired from the decimal expressions by those for α, and then confirmed by Gramian analysis. For Group III we construct Table 5 to show which eigenvalue (EV) cases are found for each of the three SVD.

While all three Groups have $m = 3$ cases, only in Group III do their eigenvalues not change in magnitude under rotation, because in those eight cases the value of m does not change. This Group also differs from the other two in that the magnitude of the eigenvalues values of none of the other members changes magnitude under rotation, changing only between real and imaginary. Rather than list all 48 members we display only the 16 members with the SVD sets α and β, noting that there are also a further 16 members in Table 5 with the SVD sets α and γ, and 16 members with the SVD sets β and γ.

Table 4
The three SVD sets for Groups I, II and III (the fourth singular value vanishes for singular squares).

SVD set	First	Second	Third
α	34	$17.88854382\left(8\sqrt{5}\right)$	$4.472135955\left(2\sqrt{5}\right)$
β	34	$16.49242250\left(4\sqrt{17}\right)$	$8.246211251\left(2\sqrt{17}\right)$
γ	34	$16.12451550\left(2\sqrt{5}\sqrt{13}\right)$	$8.944271910\left(4\sqrt{5}\right)$

Table 5
Eigenvalues found for each SVD set in Group III. (The $m = 3$ cases are shown in bold.)

SVD set	EV cases (in addition all have 0, 34)
α	$\mathbf{0}, \mathbf{0}$; ± 8; $\pm 8i$; $\pm 4\sqrt{5}$; $\pm 4i\sqrt{5}$; $\pm 4\sqrt{3}$; $\pm 4i\sqrt{3}$;
β	$\mathbf{0}, \mathbf{0}$; ± 8; $\pm 8i$; $\pm 2\sqrt{34}$; $\pm 2i\sqrt{34}$; $\pm 2\sqrt{30}$; $\pm 2i\sqrt{30}$;
γ	$\pm 4\sqrt{5}$; $\pm 4i\sqrt{5}$; $\pm 4\sqrt{3}$; $\pm 4i\sqrt{3}$; $\pm 2\sqrt{34}$; $\pm 2i\sqrt{34}$; $\pm 2\sqrt{30}$; $\pm 2i\sqrt{30}$

Table 6
α and β SVD sets for Group III with Dudeney indices. (All have additional eigenvalues 0, 34.)

EVs	EVs rotated	SVD set	Frenicle indices
0, 0	0, 0	α	290, 360, 790, 803
0, 0	0, 0	β	299, 377, 489, 535
± 8	$\pm 8i$	α	113, 175, 835, 850
± 8	$\pm 8i$	β	122, 185, 637, 695

Table 7
Non-singular examples compared (type: simple).

Square	c_2	c_3	c_4	Eigenvalues
Group 11, $F181$	-40	$+1552$	-6528	$34, -8, 4 \pm 2\sqrt{2}i$
$RF181$	$+8$	-80	-6528	$34, -5.30783, 2.65391 \pm 5.3972i$
Group 7, $F268$	-110	$+3964$	-7616	$34, -11.3873, 9.26392, 2.1234$
$RF268$	-142	$+5052$	-7616	$34, -12.6382, 11.0315, 1.60667$

Note that the $m = 3$ cases with three vanishing eigenvalues occur only for SVD sets α and β, and from Tables 5 and 6 that each eigenvalue set appears only in two SVD sets.

6.9. Two non-singular examples

All the non-singular fourth order magic squares have rank 4. We have selected the Dudeney Group XI, Frénicle index 181 square for illustration of another eigenvalue pattern:

$$F181 = \begin{bmatrix} 1 & 12 & 13 & 8 \\ 16 & 9 & 4 & 5 \\ 2 & 7 & 14 & 11 \\ 15 & 6 & 3 & 10 \end{bmatrix}, \tag{39}$$

with characteristic polynomial $x^4 - 34x^3 - 40x^2 + 1552x - 6528$, which factorizes as $(x - 34)(x + 8)(x^2 - 8x + 24)$. Note that factoring out $(x + 8)$ after $(x - 34)$ leaves the full form of a quadratic which may have complex roots, and not just the real or imaginary roots from the simpler quadratics which are missing the linear term, because the complementary characteristic roots add to zero.

The magic square with Frénicle index 268 in Dudeney Group VII provides another interesting case:

$$F268 = \begin{bmatrix} 2 & 5 & 16 & 11 \\ 8 & 12 & 1 & 13 \\ 9 & 7 & 14 & 4 \\ 15 & 10 & 3 & 6 \end{bmatrix}, \tag{40}$$

with characteristic polynomial: $x^4 - 34x^3 - 110x^2 + 3964x - 7616$, which factorizes as $(x - 34)(x^3 - 110x + 224)$. Note again the 'reduced' form of a cubic equation.

Maple® solutions for both cases are not simple and are best rendered numerically. They are given in the Table 7 for each matrix and its rotation by $\pi/2$.

These squares exhibit the first complex eigenvalues in this study, and by contrast with order 3, have no sign change on rotation for the determinant. The complementary eigenvalues $(\lambda_2, \lambda_3, \lambda_4)$ do sum to zero as expected. Note the change of magnitude of c_2, c_3 and the eigenvalues on rotation in both cases. These are associated with quite different characteristic polynomials.

Table 8
Number of distinct SVD sets by Dudeney Group.

Groups	Type	Population	Common SVD sets
I,II,III	See Table 3	48 each	3
IV,V,VI-P	Semi-pandiagonal	96 each	10
VI-S	Simple	208	26
VII,VIII,IX,X	Simple	56 each	22
XI,XII	Simple	8 each	2

Again, though not singular, singular value analysis is helpful here because it is invariant under rotation or flipping, simplifying the non-singular results just presented. The singular values for $F181$ are 34, 17.44170811, 5.656854249, 1.945974860, which are the square roots of the Gramian eigenvalues $1156, 32, 154 \pm 2\sqrt{5641}$, while for $F268$ the singular values are 3415, 15.64649515, 9.642766781, 1.484667786, with no simple surd form.

A further observation from Table 7 is that the determinant, c_4, divided by the product of the line sum and the order $(n = 4)$ is an integer. Trigg [69] posed this as a question, first for $n = 3$, then for other orders [71], and finally Trigg [70] proved this for all fourth order natural magic squares.

6.10. SVD summary for the 880 fourth order magic squares

Nothing particularly notable is found for the eigenvalues in other Groups except for the distinct SVD sets associated with each Group summarized as shown in Table 8.

It is worth noting the similarities in Table 8 for two singular Groupings I, II, III, and IV, V, VI-P, as well as the merit of the distinction (see e.g. [30]) between VI-P and VI-S which differ markedly in their singular values. There are also two Groupings for the non-singular Groups VII to XII.

6.11. Other even orders

6.11.1. Sixth order natural regular magic squares

There are no natural regular or pandiagonal magic squares of order six [73], and since Trump [73] (see also Pickover [57]) estimates the number of natural eighth order regular magic squares at $2.5228(14) \times 10^{27}$ out of a total population of $5.2210(70) \times 10^{54}$, a complete study is out of the question. We note that Mattingly [48] showed a natural, but not regular, sixth order square with eigenvalues $111, 0, \pm 27, \pm 4\sqrt{6}$.

6.11.2. Eighth order – regular natural magic squares and Franklin squares

Some examples of regular magic squares are available from the archive of a useful web site by Suzuki [63], from which we have chosen one by Woodruff in 1916:

$$\begin{bmatrix} 1 & 32 & 34 & 63 & 37 & 60 & 6 & 27 \\ 48 & 49 & 15 & 18 & 12 & 21 & 43 & 54 \\ 19 & 14 & 52 & 45 & 55 & 42 & 24 & 9 \\ 62 & 35 & 29 & 4 & 26 & 7 & 57 & 40 \\ 25 & 8 & 58 & 39 & 61 & 36 & 30 & 3 \\ 56 & 41 & 23 & 10 & 20 & 13 & 51 & 46 \\ 11 & 22 & 44 & 53 & 47 & 50 & 16 & 17 \\ 38 & 59 & 5 & 28 & 2 & 31 & 33 & 64 \end{bmatrix}. \tag{41}$$

The eigenvalues are $0(\text{five}), 260, \pm 4\sqrt{546}$, with characteristic polynomial: $x^8 - 260x^7 - 8736x^6 + 2271360x^5 = x^5(x - 260)(x^2 - 8736)$, again in agreement with Section 3 [2]. The Jordan form is diagonal, so we see here a fivefold degenerate zero eigenvalue case which is diagonable. These five eigenvectors are

$$\left\{\begin{array}{lllll} \frac{1}{11} & -\frac{5}{33} & -\frac{28}{33} & \frac{28}{33} & \frac{31}{33} \\ 0 & -1 & 0 & -1 & -1 \\ -\frac{12}{11} & \frac{5}{33} & -\frac{5}{33} & -\frac{28}{33} & -\frac{31}{33} \\ 0 & 0 & 0 & 0 & 1 \\ 1 & ,0 & ,0 & ,0 & ,0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right\} \leftrightarrow 0.$$

On rotation the eigenvalues alternate with $0, 260, \pm 4i\sqrt{546}$. Singular values: $260, 4\sqrt{1041}, 72, 0, 0, 0, 0, 0$ (rank 3).

Another regular eighth order square worth noting here is generated by MATLAB®'s `magic(8)` function:

$$\begin{bmatrix} 64 & 9 & 17 & 40 & 32 & 41 & 49 & 8 \\ 2 & 55 & 47 & 26 & 34 & 23 & 15 & 58 \\ 3 & 54 & 46 & 27 & 35 & 22 & 14 & 59 \\ 61 & 12 & 20 & 37 & 29 & 44 & 52 & 5 \\ 60 & 13 & 21 & 36 & 28 & 45 & 53 & 4 \\ 6 & 51 & 43 & 30 & 38 & 19 & 11 & 62 \\ 7 & 50 & 42 & 31 & 39 & 18 & 10 & 63 \\ 57 & 16 & 24 & 33 & 25 & 48 & 56 & 1 \end{bmatrix} \tag{42}$$

with eigenvalues: $0, 0, 0, 0, 0, 260, \pm 8\sqrt{42}$, and characteristic polynomial: $x^8 - 260x^7 - 2688x^6 + 698880x^5 = x^5(x - 260)(x^2 - 2688)$.

The Jordan form is also diagonal, so we see here another fivefold degenerate zero eigenvalue situation which is diagonable. On rotation the last pair of eigenvalues become $-8i\sqrt{42}, 8i\sqrt{42}$. Non-zero SVD values: $260, 32\sqrt{21}, 4\sqrt{21}$.

At eighth order there are two important complete subsets of magic squares, albeit not regular ones, which should be mentioned.

- **Most-perfect pandiagonal natural magic squares**, originally called **complete magic squares** by McClintock [50] (see also [56]) exist for all doubly even orders and have two additional properties, (a) that all 2×2 subsquares have the same sum, including those that run over the edges when tiled or when wrapped over a torus (see [26]), and (b) that each integer is complementary to the one distant from it $n/2$ places along the same pandiagonal, which precludes them from being regular. Ollerenshaw and Brée [56] were able to give a combinatorial formula for most-perfect pandiagonal magic squares for doubly even orders (at fourth order they are the pandiagonal set of Dudeney's Group I).
  However they only have three non-zero eigenvalues $(s_8, \pm\eta)$ which makes that subset of 368,640 order eight squares rather interesting.
- **Franklin squares** have all bent diagonals with the magic sum, all half rows and columns with half that sum, and in addition all the 2-by-2 subsquare sums are the same. Schindel et al. [61] modified the backtracking approach for their count of eighth order natural Franklin squares. Also all 368,640 eighth order natural pandiagonal Franklin squares which they found have exactly three non-zero eigenvalues, i.e., again the same pattern of eigenvalues as found by Kirkland and Neumann [38] mentioned earlier.

7. Odd order regular natural magic squares

Mattingly [48] also left as an open question whether odd order regular magic squares could be singular, proving that the multiplicity of a zero eigenvalue must be even, $m = 0, 2, \ldots, (n - 1)$, but giving no examples. The unique order three natural magic square is regular but not singular, so it has no zero eigenvalues. Moler's MATLAB® magic squares [53] for odd n have $m = 0$, are also non-singular.

7.1. Examples of 5-by-5 regular natural magic squares with even multiplicity of zero eigenvalues

Schindel tailored a backtracking program, based on ideas outlined by Trump [73], from which we found 48,544 order five regular magic squares, of which there are 656 singular cases, with 4 squares having four zero eigenvalues. We exhibit one of the 652 squares with $m = 2$

$$\begin{bmatrix} 15 & 12 & 21 & 10 & 7 \\ 2 & 6 & 17 & 18 & 22 \\ 25 & 23 & 13 & 3 & 1 \\ 4 & 8 & 9 & 20 & 24 \\ 19 & 16 & 5 & 14 & 11 \end{bmatrix}. \tag{43}$$

This has the factorized characteristic equation: $x^2(x-65)(x^2-340)=0$, where $S_5 = 65$. The eigenvalues are $0, 0, 65, \pm 2\sqrt{85}$, and on matrix rotation the last pair become $\pm 2i\sqrt{85}$. The rank is 4 but the Jordan form is not diagonal, having a 2-by-2 nondiagonal block. The squares of the singular values are $4225 (= 65^2), 550 \pm 2\sqrt{71705}, 200, 0$. These results are reminiscent of the effect of rotation for the third order magic square, but with the added pair of zero eigenvalues.

In addition there are four extraordinary cases with $m = 4$, i.e., with just a single non-zero eigenvalue corresponding to the magic constant. One of these is

$$\begin{bmatrix} 2 & 11 & 21 & 23 & 8 \\ 16 & 14 & 7 & 6 & 22 \\ 25 & 17 & 13 & 9 & 1 \\ 4 & 20 & 19 & 12 & 10 \\ 18 & 3 & 5 & 15 & 24 \end{bmatrix} \tag{44}$$

with the characteristic equation: $x^4(x-65)=0$, four degenerate zero eigenvalues, and again rank 4. These eigenvalues are invariant under matrix rotation. Again the Jordan form is not diagonal, now having a 4-by-4 nondiagonal block. The squares of the singular values are $4225, 700, 300 \pm 60\sqrt{5}, 0$.

Thompson [64] has shown that general fifth order pandiagonal magic squares have eigenvalues $S, \pm\eta_1, \pm\eta_2$, and so may be singular or not. In any case they provide support for Mattingly's theorem in a wider context than just the regular squares for which it was derived.

7.2. Natural ultramagic squares in fifth order

Suzuki [63] listed the 16 fifth order ultramagic squares, none of which are singular, e.g.,

$$\begin{bmatrix} 1 & 15 & 22 & 18 & 9 \\ 23 & 19 & 6 & 5 & 12 \\ 10 & 2 & 13 & 24 & 16 \\ 14 & 21 & 20 & 7 & 3 \\ 17 & 8 & 4 & 11 & 25 \end{bmatrix}. \tag{45}$$

Characteristic polynomial: $x^5 - 65x^4 - 250x^3 + 16250x^2 + 12245x - 795925$, which factorizes to $(x - 65)(x^4 - 250x^2 + 12245)$. This is an invertible magic square and a good example of Thompson's [64] pandiagonal criterion. Using $a = \sqrt{125 - 26\sqrt{5}}, b = \sqrt{26\sqrt{5} + 125}$, the changes on rotation are easily compared in compact tabular form given in Table 9.

Note again the signed pairs. As expected from Section 2.1, the determinant $(s_5 = -c_5)$ has not changed, but it is perhaps worth noting the lack of change in $c_4 = s_4$, which involves the sum of all possible quadruple products of the eigenvalues. The squares of the singular values are $4225, 325 \pm 142\sqrt{5}, 325 \pm 122\sqrt{5}$. Finally, multiplying all the eigenvalues in this example shows that its determinant is divisible by both $n = 5$ and then by the linesum, $\lambda_1 = 65$.

Table 9 Characteristic polynomial coefficients for the fifth order ultramagic square in (45), where columns c_0 (always 1) and c_1 (always −65) are omitted.

Coefficient	c_2	c_3	c_4	c_5	Eigenvalues
Original (45)	−250	+16250	+12245	−795925	$65, \pm a, \pm b$
$\pi/2$ rotation	+250	−16250	+12245	−795925	$65, \pm ia, \pm ib$

7.3. Natural ultramagic squares in seventh order

One of us, Trump [72], has used an exhaustive backtracking method to show that there are $1.125151(51) \times 10^{18}$ seventh order regular squares out of a total population of $3.79809(50) \times 10^{34}$. We have now been able to find many singular regular magic squares in order seven from the exact census of the seventh order ultramagic squares, numbering exactly 20,190,684. Trump [72], made extensive use of transformations to reduce computation time, and Francis Gaspalou [27] has recently found a further transformation reduction. One example is

$$\begin{bmatrix} 35 & 48 & 3 & 1 & 6 & 40 & 42 \\ 19 & 34 & 28 & 21 & 20 & 46 & 7 \\ 11 & 26 & 38 & 13 & 45 & 33 & 9 \\ 18 & 36 & 27 & 25 & 23 & 14 & 32 \\ 41 & 17 & 5 & 37 & 12 & 24 & 39 \\ 43 & 4 & 30 & 29 & 22 & 16 & 31 \\ 8 & 10 & 44 & 49 & 47 & 2 & 15 \end{bmatrix}. \tag{46}$$

The characteristic polynomial factorizes to $x^2(x-175)(x^2-9)(x^2+231)$, showing the $S_7 = 175$ line sum eigenvalue. The full set of eigenvalues contains two signed pairs: $175, \pm 3, \pm i\sqrt{231}, 0, 0$, and is invariant on rotation. With two zero eigenvalues this demonstrates that regular magic squares, when also pandiagonal, can be singular, although that is not the case with any of the 16 ultramagic squares in fifth order [72], none of which are singular. This magic square has singular values: 175, 74.36923250, 53.96991351, 28.03076252, 20.79580951, 11.75909658, 0, for rank 6. In this case the eigenvalues of the Gramian matrices are themselves very complicated expressions. The Jordan form is not diagonal, having one 2-by-2 nondiagonal block.

We find that 20,604 of these seventh order ultramagic squares are singular, all with a pair of zero eigenvalues. Moreover, Trump finds less than 0.06% singular regular magic squares in random sets in seventh order where the estimated population is so large that it is unlikely that the complete set can be studied.

8. Conclusion

The present study made use of the availability of the only complete sets of natural magic squares which occur for orders four and five, together with the seventh order ultramagics, to illustrate Mattingly's recent theorems [48]. It is clear from the present work that odd order regular magic squares can be singular, while the MATLAB® algorithm only gives a non-singular case. Also, for eighth order we have shown that there are many highly singular magic squares.

The more constraints, e.g., regular, pandiagonal, ultramagic, complete, or Franklin, possessed by a magic square, then the smaller the number of independent parameters needed to describe a general magic square (the dimension of its vector space). The occurrence of singular natural magic squares appears to grow with increasing order, except when some constraints cannot be satisfied, e.g., the non-existence of regular and pandiagonal cases for singly even orders. It is not clear at present if the percentage of singular magic squares also grows.

A number of issues are left for future investigations:

- The graphs for characteristic equations are another way of looking at the eigenvalue structure of magic squares and make a good project for low orders. De Alba [1] has recently studied the roots of cubic polynomials in terms of the Perron–Frobenius theorem, including a detailed graphical analysis.
- Trenkler [65,67] has advanced studies of the Moore–Penrose inverse of singular magic squares and some of the singular squares studied here may be worthy examples for such studies.
- Beyond the scope of the present paper Chan and Loly [12] have been able to compound [12] regular magic squares, and with the help of Rempel et al. [59] to produce larger regular magic squares of composite orders $3 \times 3 = 9, 3 \times 4 = 12 = 4 \times 3, 4 \times 4 = 16$, etc. The preservation of the regular property on compounding is readily understood (see [59]). Furthermore compounding reg-

ular magic squares always produces singular magic squares, even if the constituent squares are not themselves singular, e.g., the non-singular regular 3-by-3 magic square when compounded to a 9-by-9 has five non-zero eigenvalues. This will be examined in a future report with Rogers and Styan [60]. Moreover, if one or both of the initial squares are non-diagonable, so too are the compounded squares.

- Another look at the non-magic pandiagonal squares of Loly and Steeds [44] in the light of some aspects of the present study is probably worthwhile, especially since those squares only have two non-zero eigenvalues of the form $S_n/2 \pm d_n, n = 2^p, p = 1, 2, 3, \ldots$, whose sum is the magic constant, and where d_n is a complicated function of n.
- A reasonable conjecture suggested by this work is that the rank of natural magic squares be greater than or equal to three. A proof has since been provided by Drury [19].
- Finally, it would seem to be useful to put together a complete database of all of the characteristics, including SVDs, Gramians and Jordan form, that we have discussed for the whole set of 880 fourth order magic squares. This would be useful preparation for a complete study of the fifth order squares.

Acknowledgments

We thank George Styan, especially for help in understanding the issue of non-diagonable matrices, Harvey Heinz for providing a datafile of the 4-by-4's and Bruce Mattingly for much encouragement, as well as Götz Trenkler, Kimmo Vehkalahti, Ka Lok Chu, Karl Gustafson, John Tromp, David Lavis, Matt Rempel, Adam Rogers, Jason Fiege, Marcus Steeds, Joe Williams, Garth Irwin, Saleem Ahmed and Frank Hruska for many useful discussions or communications. D. Schindel acknowledges summer scholarship support from the Natural Sciences and Engineering Research Council and from the Faculty of Science at the University of Manitoba; P. Loly acknowledges a Post-Secondary Research Grant from the Winnipeg Foundation, and benefitted from a visit to the Strens–Guy Archive at the University of Calgary. Helpful guidance from the referees resulted in improvements in the final version.

References

[1] L.M. De Alba, Cubic polynomials, their roots and the Perron–Frobenius theorem, Int. J. Math. Ed. Sci. Technol., 33 (2002) 96–111.
[2] A.R. Amir-Moéz, G.A. Fredricks, Characteristic polynomials of magic squares, Math. Mag. 57 (1984) 220–221.
[3] W.W.R. Ball, H.S.M. Coxeter, Mathematical Recreations & Essays, University of Toronto Press, 12th ed., 1939, 1974.
[4] M. Beeler, Appendix 5: The Order 5 Magic Squares, 1973, pp. 1–16 (this is a report of R. Schroeppel's work with the enumeration of order 5 magic squares mentioned by Gardner [26]).
[5] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, second ed., Wiley, InterScience, 2002.
[6] W.H. Benson, O. Jacoby, New Recreations with Magic Squares, Dover, 1976 (see Appendix).
[7] E. Bergholt, The magic square of sixteen cells. A new and completely general formula, Nature 83 (1910) 368–369.
[8] E.R. Berlekamp, J.H. Conway, R.K. Guy, More on magic squares, in: Winning Ways: For Your Mathematical Plays, Games in Particular, vol. 2, Academic Press, London, 1982, pp. 778–783 (reprinted in Winning Ways: For Your Mathematical Plays, Second Revised Edition, vol. 4, AK Peters, 2004.
[9] K.G. Brock, How rare are singular matrices?, The Math. Gazette 89 (2005) 378–384.
[10] K.S. Brown, The Determinants of 4×4 Magic Squares. <http://www.mathpages.com/home/kmath310.htm>.
[11] A.C. Cayley, Note on magic squares, Messenger Math. VI (1877) 168.
[12] W. Chan, P.D. Loly, Iterative compounding of square matrices to generate large-order magic squares, Math. Today 38 (2002) 113–118.
[13] J. Chernick, Solution of the general magic square, Am. Math. Month. 45 (1938) 172–175.
[14] K.L. Chu, Some comments on magic square matrices with three nonzero eigenvalues, Invited talk (abstract #W-8, joint work with G.P.H. Styan) at the 16th International Workshop on Matrices and Statistics, Windsor, Ontario, Canada, June 2007.
[15] K.L. Chu, G.P.H. Styan, G. Trenkler, K. Vehkalahti, Some comments on magic squares, with special emphasis on magic matrices having 3 nonzero eigenvalues, in preparation (paper based on the talks at the 16th International Workshop on Matrices and Statistics, Windsor, Ontario, Canada, June 2007 by Chu [14], Styan[62] and Vehkalahti [73]).
[16] J. Conway, Some mathematical Magic, Quantum 1 (March/April 1991) 28–29, 45 and 61.
[17] M. Dernham, Magic square as a determinant: solution II, Am. Math. Month 56 (1949) 33–34 (problem E813 proposed by Trigg [16]).
[18] R. Descombes, Les Carrés Magiques, Vuibert, Paris, 2000.
[19] S.W. Drury, There are no magic squares of rank 2, August 2007, personal communication.

[20] H.E. Dudeney, Amusements in Mathematics, Reprint Edition, Dover, New York, 1970 (unabridged and unaltered republication of the work originally published by Thomas Nelson, London, 1917, and reprinted by Dover, New York, 1958).
[21] L. Eldén, Matrix Methods in Data Mining and Pattern Recognition, SIAM, 2007.
[22] A. van den Essen, Magic squares and linear algebra, Am. Math. Month. 97 (1990) 60–62.
[23] C. Fox, Magic matrices, Math. Gazette 40 (1956) 209–211.
[24] Frénicle de Bessy, Des Quarrez ou Tables Magiques, Mémoires de l'Académie Royale des Sciences depuis 1666 jusqu'a 1699, Tome V Paris, 1729, p. 209.
[25] F.R. Gantmacher, The Theory of Matrices, Chelsea Publishing Company, 1977.
[26] M. Gardner, Mathematical Games, Scientific American 234 (January 1976), pp. 118–122.
[27] F. Gaspalou, Ultramagic squares of order 7 – improvements. <http://www.trump.de/magic-squares/ultramagic-7/improvements.htm>.
[28] N. Gauthier, Singular matrices applied to 3×3 magic squares, Math. Gazette 81 (1997) 225–230.
[29] J. Hartman, Nonsingular magic matrices: solution to elementary problem E3440, proposed by Wardlaw [75], Am. Math. Month. 99 (1992) 966–967.
[30] H. Heinz, Order-4 Magic Squares. <http://www.geocities.com/~harveyh/order4list.htm>.
[31] H.D. Heinz, J.R. Hendricks, Magic squares lexicon: illustrated, 2000. ISBN:0-9687985-0-0, See HDH's: <http://www.geocities.com/~harveyh/>.
[32] A.J. Hetzel, J.S. Liew, K.E. Morrison, The probability that a matrix of integers is diagonalizable, Am. Math. Month. 114 (2007) 491–499.
[33] R.A. Horn, C.A. Johnson, Matrix Analysis, Cambridge University Press, UK, 1985
[34] R.A. Horn, C.A. Johnson, Topics in Matrix Analysis, Cambridge University Press, UK, 1991
[35] F. Hruska, Magic squares, matrices, planes and angles, J. Recreat. Math. 23 (1991) 183–189.
[36] B.L. Johnston, Fred Richman, Numbers and Symmetry, CRC Press, 1997
[37] N.A. Khan, Characteristic roots of semi-magic matrices, Am. Math. Month. 64 (1957) 261–263.
[38] S.J. Kirkland, M. Neumann, Group inverses of *M*-matrices associated with nonnegative matrices having few eigenvalues, Linear Algebra Appl. 220 (1995) 181–213.
[39] M. Kraitchik, La Mathématique des Jeux, Stevens, Brussels, 1930 (revised and translated into English as [40]).
[40] M. Kraitchik, Mathematical Recreations, second revised ed., Dover, New York, 2006 (Unabridged republication of the 1953 Dover edition of the work originally published by W.W. Norton, New York, 1942. First published in French as [39]).
[41] P.D. Loly, The invariance of the moment of inertia of magic squares, Math. Gazette 88 (2004) 151–153.
[42] P.D. Loly, Franklin squares – a chapter in the scientific studies of magical squares, Complex Syst. 17 (2007) 143–161.
[43] P.D. Loly, The characteristic equation under rotation, 2008, in preparation.
[44] P.D. Loly, M. Steeds, A new class of pandiagonal squares, Int. J. Math. Ed. Sci. Technol. 36 (2005) 375–388.
[45] P.D. Loly, J. Tromp, Vector spaces of magical squares, 2008, in preparation.
[46] E. Lucas, Les carrés magiques: sur le carré de 3 et sur les carrés a deux degrés, Les Tablettes du Chercheur: Journal de Jeux d'Ésprit et de Combinaisons, No. 5, page 7, 1 March 1891. Reprinted in "Récréations mathématiques" by E. Lucas, vol. 4, 1891, pp. 224–228. Full-text pdf files of all 4 volumes of "Récréations mathématiques" are available at the American Libraries Internet Archive. <http://www.archive.org/details/recretionmatedou04lucarich>.
[47] G. Martin, E.B. Wong, Almost all integer matrices have no integer eigenvalues, <arXiV:0712.3060v1 [math.NT]> (18.12.07).
[48] R.B. Mattingly, Even order regular magic squares are singular, Am. Math. Month. 107 (2000) 777–782. (The Mathematical Association of America, Washington, DC. See also his personal webpage: http://web.cortland.edu/mattinglyb/research.htm: "I receive a couple of emails each year about this paper, which contains a conjecture that odd order regular magic squares are nonsingular. However, Peter Loly and his colleagues have discovered that this is false!")
[49] F. Mayoral, Semi-magic squares and their orthogonal complements, Math. Gazette 80 (1996) 308–313.
[50] E. McClintock, On the most perfect forms of magic squares, with methods for their production, Am. J. Math. 19 (1897) 99–120.
[51] C. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM (Society for Industrial and Applied Mathematics, Philadelphia), 2000
[52] L. Mirsky, An Introduction to Linear Algebra, Oxford at the Clarendon Press, 1955
[53] C. Moler, MATLAB's magical mystery tour, The MathWorks Newslett. 7 (1993) 8–9.
[54] S. Mustonen, SURVO MM: computing environment for creative processing of text and numerical data, 2001. <http://www.survo.fi/mm/english.html>.
[55] K. Ollerenshaw, H. Bondi, Magic squares of order four, Phil. Trans. Royal Soc. A 306 (1982) 443–532.
[56] K. Ollerenshaw, D.S. Brée, Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration, The Institute of Mathematics and its Applications, Southend-on-Sea, UK, 1998
[57] C.A. Pickover, The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures across Dimensions, Second printing and first paperback printing, Princeton University Press, 2003.
[58] C. Planck, Pandiagonal magic squares of orders 6 and 10 with minimal numbers, Monist 29 (1919) 307–316.
[59] M. Rempel, W. Chan, P.D. Loly, More on Compounding Magic Squares, 2008, in preparation.
[60] A. Rogers, P.D. Loly, G.P.H. Styan, Some comments on composite magic matrices using Kronecker products, 2008, in preparation.
[61] D.G. Schindel, M. Rempel, P. Loly, Enumerating the bent diagonal squares of Dr Benjamin Franklin FRS, Proceedings of the Royal Society A: Physical, Mathematical and Engineering 462 (2006) 2271–2279.
[62] G.P.H. Styan, An illustrated (philatelic) introduction to magic matrices and statistics, with special emphasis on magic matrices having 3 nonzero eigenvalues, Invited talk (joint work with K.L. Chu and G. Trenkler) at the Fifth Annual Meeting of the Statistical Society of Canada, St. Johns, Newfoundland, Canada, June 2007.
[63] M. Suzuki, Magic Squares. <http://mathforum.org/te/exchange/hosted/suzuki/MagicSquare.html> (frozen at 2001).
[64] A.C. Thompson, Odd magic powers, Am. Math. Month. 101 (1994) 339–342.
[65] G. Trenkler, Singular magic squares, Int. J. Math. Ed. Sci. Technol. 25 (1994) 595–597.
[66] G. Trenkler, On the eigenvalues of 3×3 magic squares, Int. J. Math. Ed. Sci. Technol. 30 (1999) 307–308.

[67] G. Trenkler, On the Moore–Penrose inverse of magic squares, Invited talk (abstract #W-41) at the 16th International Workshop on Matrices and Statistics, Windsor, Ontario, Canada, June 2007.
[68] J.E. Trevor, Determinants whose arrays are magic squares, Am. Math. Month. 31 (1924) 216–222.
[69] C.W. Trigg, Magic square as a determinant, Am. Math. Month. 55 (1948) 248
[70] C.W. Trigg, Determinants of fourth order magic squares, Am. Math. Month. 55 (1948) 558–561.
[71] C.W. Trigg, Problem E 813: remarks by the proposer, Am. Math. Month. 56 (1949) 34–37.
[72] W. Trump, Ultramagic squares of order 7. <http://www.trump.de/magic-squares/ultramagic-7>.
[73] W. Trump, Notes on magic squares and cubes. <www.trump.de/magic-squares>.
[74] J.E. Ward III, Vector spaces of magic squares, Math. Mag. 53 (1980) 151–153.
[75] W.P. Wardlaw, [Elementary Problem] E3440, Am. Math. Month. 98 (1991) 437 (Solution by Hartman [29]).
[76] K. Vehkalahti, Some comments on magic squares and Survo puzzles, Invited talk (abstract #W-43) at the 16th International Workshop on Matrices and Statistics, Windsor, Ontario, Canada, June 2007.