Representation of solutions of discrete delayed system $x(k + 1) = Ax(k) + Bx(k - m) + f(k)$ with commutative matrices

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Abstract

In the investigation performed we give, on half-infinity discrete intervals, formulas for solution of initial problem of linear discrete systems $x(k + 1) = Ax(k) + Bx(k - m) + f(k)$ with constant square matrices $A, B$ such that $AB = BA$, $\det A \neq 0$ and with a vector function $f(k)$. Corresponding representations are obtained with the aid of so-called discrete matrix delayed exponential, which permits to represent solutions in a matrix form similarly as for ordinary differential systems with constant matrices, or as well as for differential systems with constant matrices and constant delay.

Keywords: Discrete system; Initial problem; Discrete matrix delayed exponential

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1. Introduction

For given integers $s, q, s < q$, we set $\mathbb{Z}_q^s := \{s, s + 1, \ldots, q\}$. Possibility $s = -\infty$ or $q = \infty$ is admitted, too. The subject of our investigation is linear discrete system

$$x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad (1.1)$$

where $m \geq 1$ is a fixed integer, $k \in \mathbb{Z}_\infty^0$, $A = (a_{ij})$, det $A \neq 0$ and $B = (b_{ij})$ are constant $n \times n$ matrices admitting commutative property

$$AB = BA, \quad (1.2)$$

$f : \mathbb{Z}_\infty^0 \to \mathbb{R}^n$, $x : \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$. Following the terminology (used, e.g., in [1,2]), we refer (1.1) as delayed discrete system if $m \geq 1$ and as nondelayed discrete system if $m = 0$. Together with Eq. (1.1) we consider initial (Cauchy) problem

$$x(k) = \varphi(k) \quad (1.3)$$

with given $\varphi : \mathbb{Z}_{-m}^0 \to \mathbb{R}^n$.

The existence and uniqueness of solution of initial problem (1.1), (1.3) on $\mathbb{Z}_{-m}^\infty$ is obvious. We recall that solution $x : \mathbb{Z}_{-m}^\infty \to \mathbb{R}^n$ of initial Cauchy problem (1.1), (1.3) is defined as an infinite sequence

$$\{x(-m) = \varphi(-m), x(-m + 1) = \varphi(-m + 1), \ldots, x(0) = \varphi(0), x(1), x(2), \ldots, x(k), \ldots\}$$

such that for any $k \in \mathbb{Z}_0^\infty$ equality (1.1) holds.

1.1. An equivalent form of system (1.1)

We consider on $\mathbb{Z}_0^\infty$ nonhomogeneous system (1.1) together with initial data (1.3). We suppose, moreover, that $A$ is nonsingular. Substituting in (1.1)

$$x(k) = A^ky(k) \quad (1.4)$$

with $k \in \mathbb{Z}_{-m}^\infty$, we get

$$y(k + 1) = y(k) + B_1y(k - m) + A^{-k-1}f(k)$$

with $B_1 = A^{-k-1}BA^{-m}$. Due to property (1.2), we obtain $B_1 = A^{-1}B^{-m}A^{-m}$ and matrix $B_1$ becomes a constant matrix. Using difference operator, we write equivalent form to (1.1) as

$$\Delta y(k) = B_1y(k - m) + A^{-k-1}f(k), \quad k \in \mathbb{Z}_0^\infty. \quad (1.5)$$

Corresponding equivalent initial data with respect to (1.5) are

$$y(k) = A^{-k}\varphi(k), \quad k \in \mathbb{Z}_{-m}^0. \quad (1.6)$$

1.2. Formulation of the problem considered

Let us compare homogenous initial problem corresponding to (1.5), (1.6), i.e., the problem
\[ \Delta y(k) = B_1 y(k - m), \quad k \in \mathbb{Z}_0^\infty, \]
\[ y(k) = A^{-k} \varphi(k), \quad k \in \mathbb{Z}_{-m}^0, \]

with an analogous one (in a sense) for ordinary differential system with one delay
\[ \dot{z}(t) = A z(t - \tau), \quad t \in (0, \infty), \quad (1.7) \]
\[ z(t) = \varphi(t), \quad t \in [-\tau, 0]. \quad (1.8) \]

We suppose that the matrix \( A \) is a \( n \times n \) constant matrix, \( t \in [0, \infty) \), \( \tau > 0 \), \( z: [-\tau, \infty) \to \mathbb{R}^n \), and \( \varphi \) is a continuously differentiable initial function on \( [-\tau, 0] \).

In [9] is proved that solution of the initial Cauchy problem (1.7), (1.8) can be represented on interval \( [-\tau, \infty) \) in the form
\[ z(t) = e^{A t \tau} \varphi(-\tau) + \int_{-\tau}^{0} e^{A (t - s - \tau)} \varphi'(s) \, ds, \quad (1.9) \]

where \( e^{A t} \) is so-called delayed exponential of matrix \( A \), defined as
\[
e^{A t} := \begin{cases} \Theta & \text{if } -\infty < t < -\tau, \\ I & \text{if } -\tau \leq t < 0, \\ I + \frac{1}{1!} A t & \text{if } 0 \leq t < \tau, \\ I + \frac{1}{1!} A t + \frac{1}{2!} A^2 (t - \tau)^2 & \text{if } \tau \leq t < 2 \tau, \\ \vdots & \\ I + \frac{1}{1!} A t + \frac{1}{2!} A^2 (t - \tau)^2 + \cdots + \frac{1}{k!} A^k [t - (k - 1) \tau]^k & \text{if } (k - 1) \tau \leq t < k \tau, \\ \cdots, & \\
\end{cases}
\]
\( \Theta \) is null \( n \times n \) matrix and \( I \) is unit \( n \times n \) matrix. Discussing the meaning of formula (1.9), we conclude that the delayed exponential is an useful formalizing tool and formula (1.9) formalizes explicit computation of solution of initial problem for systems with delay.

**Problem under consideration.** In this contribution we use a discrete analogy of delayed exponential for representing of solution of problem (1.1), (1.3) by formulas which are discrete counterpart of formulas for continuous case like formula (1.9).

2. **Discrete matrix delayed exponential and its basic property**

For \( n \times n \) constant matrix \( B \) we define a discrete matrix function \( \exp_m (Bk) \) called the discrete matrix delayed exponential:
We explain the sense of such definition of discrete matrix delayed exponential. It permits to formalize computation of solution of initial problem and admits (similarly as fundamental matrix in theory of ordinary differential equations) to write its solution in a compact form.

2.1. Fundamental property of the discrete matrix delayed exponential

The main property of discrete matrix delayed exponential is given in the following result.

**Theorem 2.1.** Let $B$ be a constant $n \times n$ matrix. Then for $k \in \mathbb{Z}_{\infty}$,

$$
\Delta e^{Bk}_m = Be^{B(k-m)}_m. \quad (2.1)
$$

The proof of Theorem 2.1 will be given in the next part. Now we use immediately result given by formula (2.1). We consider an initial Cauchy problem for homogeneous linear matrix equation:

$$
X(k+1) = AX(k) + BX(k-m), \quad k \in \mathbb{Z}_{\infty}, \quad (2.2)
$$

$$
X(k) = A^k, \quad k \in \mathbb{Z}_0^\infty \quad (2.3)
$$

with $n \times n$ matrices $A$ and $B$, satisfying conditions formulated in Introduction. Here $X: \mathbb{Z}_m^\infty \to \mathbb{R}^{n \times n}$ is an unknown matrix.

**Theorem 2.2.** The matrix

$$
X = X_0(k) := A^k e^{B_1k}_m, \quad k \in \mathbb{Z}_m^\infty \quad (2.4)
$$

solves the problem (2.2), (2.3).

**Proof.** We put

$$
X(k) = X_0(k), \quad k \in \mathbb{Z}_m^\infty,
$$
in matrix equation (2.2). Then
\[ A^{k+1}e_{m}^{B_{1}(k+1)} = A^{k+1}e_{m}^{B_{1}k} + B A^{k-m}e_{m}^{B_{1}(k-m)} \]
and
\[ \Delta e_{m}^{B_{1}k} = B_{1}e_{m}^{B_{1}(k-m)}, \quad k \in \mathbb{Z}_{-\infty}^{\infty}. \]
This equality is valid, due to formula (2.1), for every \( k \in \mathbb{Z}_{-\infty}^{\infty}. \)

\[ \Delta e_{m}^{B_{1}k} = B_{1}e_{m}^{B_{1}(k-m)}, \quad k \in \mathbb{Z}_{-\infty}^{\infty}. \]

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This equality is valid, due to formula (2.1), for every \( k \in \mathbb{Z}_{-\infty}^{\infty}. \)

\[ \Delta e_{m}^{B_{1}k} = B_{1}e_{m}^{B_{1}(k-m)}, \quad k \in \mathbb{Z}_{-\infty}^{\infty}. \]

2.1.1. Proof of Theorem 2.1

Let a matrix \( B \) and a positive integer \( m \) be fixed. For \( k \in \mathbb{Z}_{-(m+1)}^{\infty} \) is formula (2.1)
obvious. If
\[ (\ell - 1)(m + 1) + 1 \leq k \leq \ell(m + 1) \quad \text{and} \quad \ell = 0, 1, 2, \ldots, \]
then (in accordance with definition of \( e_{m}^{B_{1}k} \)) since \( \Delta I = \Theta \) relation
\[ \Delta e_{m}^{B_{1}k} = \Delta \left[ I + \sum_{j=1}^{\ell} B_{j}^{j} \frac{(k - (j - 1)m)!}{j! (k - (j - 1)m - j)!} \right] \]
\[ = \Delta \left[ \sum_{j=1}^{\ell} B_{j}^{j} \frac{(k - (j - 1)m)!}{j! (k - (j - 1)m - j)!} \right] \]
holds. Here and throughout the paper we adopt customary notations \( \sum_{i=k+s}^{k} \circ(i) = 0 \) and \( \prod_{i=k+s}^{k} \circ(i) = 1 \), where \( k \) is an integer, \( s \) is a positive integer and “\( \circ \)” denotes the function considered independently on the fact if it is for indicated arguments defined or not. For the proof of (2.5) seems to be reasonable to divide values of \( k \) into two groups. The first group contains values \( k \) and \( k + 1 \) between numbers \( (\ell - 1)(m + 1) + 1 \) and \( \ell(m + 1) \), the second group contains such values \( k \) and \( k + 1 \) that do no satisfy this condition, i.e., \( k = \ell(m + 1) \) and \( k + 1 = \ell(m + 1) + 1 \). The reason for it is the definition of increment itself since
\[ \Delta e_{m}^{B_{1}k} = e_{m}^{B_{1}(k+1)} - e_{m}^{B_{1}k}. \]
In the following we put \( \ell > 0 \) since the case \( \ell = 0 \) is trivial.

*The case \( (\ell - 1)(m + 1) + 1 \leq k < k + 1 \leq \ell(m + 1) \).* In this case we have
\[ k - m \in \mathbb{Z}_{(\ell-1)(m+1)}^{(\ell-2)(m+1)+1} \]
and, by definition,
\[ e_{m}^{B_{1}(k-m)} = I + \sum_{j=1}^{\ell-1} B_{j}^{j} \frac{(k - m - (j - 1)m)!}{j! (k - m - (j - 1)m - j)!}. \]
In this case we prove the formula (2.1), i.e., we prove that
\[ \Delta e_{m}^{B_{1}k} = B_{m}^{B_{1}(k-m)} = B_{m} \left[ I + \sum_{j=1}^{\ell-1} B_{j}^{j} \frac{(k - m - (j - 1)m)!}{j! (k - m - (j - 1)m - j)!} \right]. \]
With the aid of (2.5) and (2.6) we get
\[
\Delta^{B_k} e_m^\ell = e_m^{\ell(k+1)} - e_m^{B_k} \\
= \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(k+1 - (j-1)m)!}{(k+1 - (j-1)m - j)!} - \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(k-(j-1)m)!}{(k-(j-1)m - j)!} \\
= \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(k-(j-1)m)!}{(k+1 - (j-1)m - j)!} \left[ \frac{(k+1 - (j-1)m)!}{(k+1 - (j-1)m - j)!} - \frac{(k-(j-1)m)!}{(k-(j-1)m - j)!} \right] \\
= \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(k-(j-1)m)!}{(k+1 - (j-1)m - j)!} \cdot \frac{(k+1 - (j-1)m)!}{(k+1 - (j-1)m - j)!} \\
= B \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(k-(j-1)m)!}{(j-1)! (k+1 - (j-1)m - j)!} \\
= B \left[ I + \sum_{j=2}^\ell \frac{B_j}{j!} \frac{(k-(j-1)m)!}{(j-1)! (k+1 - (j-1)m - j)!} \right].
\]

Now we change the index of summation \(j\) by \(j+1\). Then
\[
\Delta^{B_k} e_m^\ell = B \left[ I + \sum_{j=1}^{\ell-1} \frac{B_j}{j!} \frac{(k-jm)!}{(k+1-jm - (j+1))!} \right] \\
= B \left[ I + \sum_{j=1}^{\ell-1} \frac{B_j}{j!} \frac{(k-m - (j-1)m)!}{(k-m - (j-1)m - j)!} \right].
\]

At the end we use relation (2.7). Formula (2.8) is proved.

**The case** \(k = \ell(m+1), k+1 = \ell(m+1) + 1\). In accordance with the definition we write
\[
e_m^{B_k} = e_m^{\ell(m+1)} = I + \sum_{j=1}^\ell \frac{B_j}{j!} \frac{(\ell(m+1) - (j-1)m)!}{(\ell(m+1) - (j-1)m - j)!} \\
\text{and}
\]
\[
e_m^{B_{\ell(k+1)}} = e_m^{\ell(m+1)+1} = I + \sum_{j=1}^{\ell+1} \frac{B_j}{j!} \frac{(\ell(m+1) + 1 - (j-1)m)!}{(\ell(m+1) + 1 - (j-1)m - j)!}.
\]

Since
\[
k - m = \ell(m+1) - m \in \mathcal{E}_{(\ell-1)(m+1)+1}^{\ell(m+1)},
\]
the discrete matrix delayed exponential \( \exp_m(B(k - m)) \) is defined as

\[
e^B_{m(k-m)} = I + \sum_{j=1}^{\ell} \frac{B^j}{j!} \frac{(k - m - (j - 1)m)!}{(k - m - (j - 1)m - j)!}.
\]

(2.9)

Now we consider

\[
\Delta e^B_{m(k+1)} = e^B_{m(k+1)} - e^B_{m(k)} = e^B_{m(\ell(m+1)+1)} - e^B_{m(\ell+1)}
\]

\[
\Delta e^B_{m(k+1)} = \sum_{j=1}^{\ell} \frac{B^j}{j!} \frac{(\ell(m+1) + 1 - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} - \frac{(\ell(m+1) - (j - 1)m)!}{(\ell(m+1) - (j - 1)m - j)!}
\]

\[
= \sum_{j=1}^{\ell} \frac{B^j}{j!} \left[ \frac{(\ell(m+1) + 1 - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} - \frac{(\ell(m+1) - (j - 1)m)!}{(\ell(m+1) - (j - 1)m - j)!} \right]
\]

\[
= \sum_{j=1}^{\ell} \frac{B^j}{j!} \left[ \frac{(\ell(m+1) + 1 - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} - \frac{(\ell(m+1) - (j - 1)m)!}{(\ell(m+1) - (j - 1)m - j)!} \right]
\]

\[
= \sum_{j=1}^{\ell} \frac{B^j}{j!} \frac{(\ell(m+1) + 1 - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} + B^{\ell+1}
\]

\[
= \sum_{j=1}^{\ell} \frac{B^j}{j!} \frac{(\ell(m+1) + 1 - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} + B^{\ell+1}
\]

\[
= B \sum_{j=1}^{\ell-1} \frac{B^{j-1}}{j!} \frac{(\ell(m+1) - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} + B^{\ell+1}
\]

\[
= B + B \sum_{j=2}^{\ell} \frac{B^{j-1}}{j!} \frac{(\ell(m+1) - (j - 1)m)!}{(\ell(m+1) + 1 - (j - 1)m - j)!} + B^{\ell+1}.
\]

We change the summation index \( j \) by \( j + 1 \). Then

\[
\Delta e^B_{m(k+1)} = B \left[ I + \sum_{j=1}^{\ell-1} \frac{B^j}{j!} \frac{(\ell(m+1) - jm)!}{(\ell(m+1) + 1 - jm - (j + 1))!} + B^{\ell} \right].
\]
Since in the case considered $k = \ell(m + 1)$, we get

$$
\Delta e^{Bk}_m = B \left[ I + \sum_{j=1}^{\ell-1} \frac{B_j}{j!} \frac{(k - m - (j - 1)m)!}{(k - m - (j - 1)m - j)!} + \frac{B^\ell}{\ell!} \frac{(k - m - (\ell - 1)m)!}{(k - m - (\ell - 1)m - \ell)!} \right].
$$

Finally with the aid of definition (2.9) we obtain

$$
\Delta e^{Bk}_m = B \left[ I + \sum_{j=1}^{\ell} \frac{B_j}{j!} \frac{(k - m - (j - 1)m)!}{(k - m - (j - 1)m - j)!} \right] = Be^{B(k-m)}_m.
$$

Relation (2.5) is proved.

3. Matrix solution of problem (1.1), (1.3)

At this part we give a matrix form of solution of the homogeneous and nonhomogeneous initial problem (1.1), (1.3). We use the matrix function $X_0(k)$ defined as a discrete matrix delayed exponential by formula (2.4).

3.1. Homogeneous initial problem

Consider homogeneous initial problem (1.1), (1.3), i.e., the problem

$$
x(k + 1) = Ax(k) + Bx(k - m), \quad k \in \mathbb{Z}_0^\infty, \quad x(k) = \varphi(k), \quad k \in \mathbb{Z}_{-m}^0.
$$

Theorem 3.1. Let $A$, $B$ be a constant $n \times n$ matrices, $AB = BA$ and $\det A \neq 0$. Then solution of the problem (3.1), (3.2) can be expressed as

$$
x(k) = X_0(k)A^{-m}\varphi(-m) + A^m \sum_{j=-m+1}^{0} X_0(k - m - j)\left[ \varphi(j) - A\varphi(j - 1) \right],
$$

where $k \in \mathbb{Z}_0^\infty$.

Proof. We put $x(k) = A^ky(k)$, $k \in \mathbb{Z}_0^\infty$. Then the problem (3.1), (3.2) turns into (see formulas (1.4)–(1.6))

$$
\Delta y(k) = B_1y(k - m), \quad k \in \mathbb{Z}_0^\infty,
$$

$$
y(k) = A^{-k}\varphi(k), \quad k \in \mathbb{Z}_{-m}^0.
$$

We will try to find solution of the problem (3.4), (3.5) for $k \in \mathbb{Z}_{-m}^\infty$ in the form

$$
y(k) = e^{B_1k}C + \sum_{j=-m+1}^{0} e^{B_1(k-m-j)}\Delta\pi(j - 1),
$$
where $C$ is an unknown constant vector and $\pi : \mathbb{Z}_0^0 \rightarrow \mathbb{R}^n$ is a discrete function. At first we show that expression (3.6) is a solution of homogeneous system (3.4) for arbitrary $C$ and $\pi$ and for $k \in \mathbb{Z}_0^\infty$. We compute $\Delta y(k), k \in \mathbb{Z}_-^\infty$. Due to linearity, we get

$$\Delta y(k) = \Delta \left[ e_m^{B_1 k} C + \sum_{j=-m+1}^{0} e_m^{B_1 (k-m-j)} \Delta \pi(j-1) \right]$$

$$= \Delta e_m^{B_1 k} C + \sum_{j=-m+1}^{0} \Delta \left[ e_m^{B_1 (k-m-j)} \Delta \pi(j-1) \right]$$

$$= \Delta \left[ e_m^{B_1 k} C \right] + \sum_{j=-m+1}^{0} \Delta \left[ e_m^{B_1 (k-m-j)} \right] \Delta \pi(j-1).$$

We apply formula (2.1) relative to the increments of discrete exponential:

$$\Delta y(k) = B_1 e_m^{B_1 (k-m)} C + \sum_{j=-m+1}^{0} B_1 e_m^{B_1 (k-2m-j)} \Delta \pi(j-1)$$

$$= B_1 \left[ e_m^{B_1 (k-m)} C + \sum_{j=-m+1}^{0} e_m^{B_1 (k-2m-j)} \Delta \pi(j-1) \right]. \quad (3.7)$$

Consequently,

$$\Delta y(k) = B_1 y(k-m)$$

for $k \in \mathbb{Z}_0^\infty$ and expression (3.6) really solves homogeneous system (3.4) for arbitrary $C$ and $\pi$.

Now we try to fix $C$ and $\pi$ in order to satisfy initial condition (3.5) for $k \in \mathbb{Z}_0^\infty$. We use the representation of increment (3.7) and we put it into system (3.4). Easy simplification leads to relation

$$e_m^{B_1 (k-m)} C + \sum_{j=-m+1}^{0} e_m^{B_1 (k-2m-j)} \Delta \pi(j-1) = y(k-m), \quad k \in \mathbb{Z}_0^m.$$ 

We choose the vector $C$ and function $\pi$ in such manner that initial conditions (3.5) will hold. We change the index $k$ by index $k + m$. Then the last relation can be written as

$$e_m^{B_1 (k-m)} C + \sum_{j=-m+1}^{0} e_m^{B_1 (k-2m-j)} \Delta \pi(j-1) = y(k) = A^{-k} \varphi(k),$$

where $k \in \mathbb{Z}_0^m$. Moreover, let us rewrite the last formula again. We get

$$e_m^{B_1 k} C + \sum_{j=-m+1}^{k} e_m^{B_1 (k-m-j)} \Delta \pi(j-1) + \sum_{j=k+1}^{0} e_m^{B_1 (k-m-j)} \Delta \pi(j-1) = A^{-k} \varphi(k).$$

(3.8)
Consider the first sum. If integer $j > k$ then, by definition, the first sum equals to zero. Therefore we consider only the case $j \leq k$. In this case we have

$$k - m - j \geq k - m - k = -m$$

and, moreover since $j \geq -m + 1$ and $k \leq 0$,

$$k - m - j \leq -m - j \leq -m + m - 1 = -1.$$

Due to definition of discrete matrix delayed exponential

$$e_m^{B_1(k-m-j)} \equiv I$$

and the first sum is equivalent to

$$\sum_{j=-m+1, j \leq k}^k e_m^{B_1(k-m-j)} \Delta \pi(j - 1) = \sum_{j=-m+1, j \leq k}^k \Delta \pi(j - 1) = \pi(k) - \pi(-m).$$

Now we consider the second sum. If integer $j > 0$ then, by definition, the second sum equals to zero. This holds for $k = 0$, i.e., it is enough to consider $k \in \mathbb{Z}_{-m}^{-1}$ only. Since $j \geq k + 1$ then

$$k - m - j \leq k - m - k - 1 = -m - 1 < -m$$

and due to definition of discrete matrix delayed exponential

$$e_m^{B_1(k-m-j)} \equiv \Theta.$$

Finally since $e_m^{B_1 k} \equiv I$ if $k \in \mathbb{Z}_{-m}^{0}$, relation (3.8) becomes

$$C + \pi(k) - \pi(-m) = A^{-k} \varphi(k)$$

and one can put

$$\pi(k) := A^{-k} \varphi(k), \quad k = -m, -m + 1, \ldots, 0 \quad \text{and} \quad C := \pi(-m) = A^{m} \varphi(-m).$$

In order to get formula (3.3) it remains to put $C$ and $\pi$ into (3.6). We take into account that from definition of discrete exponential it follows that matrices $A$ and discrete exponential commute. Now

$$x(k) = A^k y(k) = A^k \left[ e_m^{B_1 k} A^{-m} \varphi(-m) + \sum_{j=-m+1}^0 e_m^{B_1(k-m-j)} \Delta A^{-(j-1)} \varphi(j - 1) \right]$$

$$= X_0(k) A^{-m} \varphi(-m) + \sum_{j=-m+1}^0 A^{m+j} X_0(k - m - j)$$

$$\times \left[ A^{-j} \varphi(j) - A^{-(j-1)} \varphi(j - 1) \right]$$

$$= X_0(k) A^{-m} \varphi(-m) + A^m \sum_{j=-m+1}^0 X_0(k - m - j) \left[ \varphi(j) - A \varphi(j - 1) \right].$$
3.2. Nonhomogeneous initial problem

We consider nonhomogeneous initial Cauchy problem (1.1), (1.3), i.e., the problem
\[
x(k + 1) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}^\infty,
x(k) = \varphi(k), \quad k \in \mathbb{Z}^0_{-m}.
\] (3.9)
We get solution of this problem, in accordance with theory of linear equations, as the sum of solution of adjoint homogeneous problem (3.1), (3.2) (satisfying the same initial data) and a particular solution of (3.9) being zero on initial interval. Therefore we will try to find such a particular solution. We give some auxiliary material. In the following definition and theorem we suppose that all expressions are well defined.

Definition 3.2. Let a discrete function \( F(k, n) \) of two discrete variables be given. We define a partial difference operator \( \Delta_k \) as
\[
\Delta_k F(k, n) := F(k + 1, n) - F(k, n).
\]

Lemma 3.3. Let a discrete function \( F(k, n) \) of two discrete variables be given. Then
\[
\Delta_k \left[ \sum_{j=1}^{k} F(k, j) \right] = F(k + 1, k + 1) + \sum_{j=1}^{k} \Delta_k F(k, j).
\] (3.10)

Proof. Considering the left-hand side of (3.10), we get
\[
\begin{align*}
\Delta_k \left[ \sum_{j=1}^{k} F(k, j) \right] &= \sum_{j=1}^{k+1} F(k + 1, j) - \sum_{j=1}^{k} F(k, j) \\
&= F(k + 1, k + 1) + \sum_{j=1}^{k} F(k + 1, j) - \sum_{j=1}^{k} F(k, j) \\
&= F(k + 1, k + 1) + \sum_{j=1}^{k} \Delta_k F(k, j).
\end{align*}
\]
Now we find solution \( x = x_p(k), k \in \mathbb{Z}^\infty_{-m} \) of the problem
\[
x(k) = Ax(k) + Bx(k - m) + f(k), \quad k \in \mathbb{Z}^\infty_0,
x(k) = 0, \quad k \in \mathbb{Z}^0_{-m}.
\] (3.11) (3.12)

Theorem 3.4. Let \( A, B \) be a constant \( n \times n \) matrices, \( AB = BA \) and \( \det A \neq 0 \). Then solution \( x = x_p(k) \) of the initial Cauchy problem (3.11), (3.12) can be represented on \( \mathbb{Z}^\infty_{-m} \) in the form
\[
x_p(k) = A^m \sum_{j=1}^{k} X_0(k - m - j) f(j - 1).
\]
Proof. We put \( x(k) = A^k y(k), \ k \in \mathbb{Z}^\infty_{-m} \). Then the problem (3.11), (3.12) turns into (see formulas (1.4)–(1.6))

\[
\Delta y(k) = B_1 y(k - m) + A^{-k-1} f(k), \quad k \in \mathbb{Z}^\infty_0,
\]

(3.13)

\[
y(k) = 0, \quad k \in \mathbb{Z}^0_{-m}.
\]

(3.14)

We will try to find particular solution \( y_p(k) \) of the problem (3.13), (3.14) on \( \mathbb{Z}^\infty_1 \), employing the idea of method of variation of arbitrary constant (see, e.g., [6–8]), in the form

\[
y_p(k) = \sum_{j=1}^k e_m B_1^{(k-m-j)} \omega(j),
\]

(3.15)

where \( \omega : \mathbb{Z}^\infty_1 \rightarrow \mathbb{R}^n \) is a discrete function. We put \( y_p(k) \) into (3.13). Then

\[
\Delta \left[ \sum_{j=1}^k e_m B_1^{(k-m-j)} \omega(j) \right] = B_1 \left[ \sum_{j=1}^{k-m} e_m B_1^{(k-2m-j)} \omega(j) \right] + A^{-k-1} f(k).
\]

(3.16)

Considering \( \Delta \) as \( \Delta_k \), we obtain with the aid of formula (3.10),

\[
e_m B_1^{((k+1)-m-(k+1))} \omega(k + 1) + \sum_{j=1}^k \Delta [e_m B_1^{(k-m-j)} \omega(j)]
\]

\[
= B_1 \left[ \sum_{j=1}^{k-m} e_m B_1^{(k-2m-j)} \omega(j) \right] + A^{-k-1} f(k).
\]

Using the formula (2.1), we have

\[
\Delta e_m B_1^{(k-m-j)} = B_1 e_m B_1^{(k-2m-j)}
\]

and the last relation becomes

\[
e_m B_1^{(-m)} \omega(k + 1) + B_1 \sum_{j=1}^k e_m B_1^{(k-2m-j)} \omega(j)
\]

\[
= B_1 \sum_{j=1}^{k-m} e_m B_1^{(k-2m-j)} \omega(j) + A^{-k-1} f(k).
\]

(3.16)

Since \( e_m B_1^{(-m)} \equiv I \) and

\[
\sum_{j=1}^k e_m B_1^{(k-2m-j)} \omega(j) = \sum_{j=1}^{k-m} e_m B_1^{(k-2m-j)} \omega(j) + \sum_{j=1}^{k-m+1} e_m B_1^{(k-2m-j)} \omega(j),
\]

where due to the definition of discrete matrix delayed exponential

\[
e_m B_1^{(k-2m-j)} \equiv \Theta \quad \text{if} \ j \in \mathbb{Z}^k_{k-m+1},
\]
the relation (3.16) turns into
\[
\omega(k + 1) + B_1 \sum_{j=1}^{k-m} e_m B_1(k-2m-j) \omega(j) = B_1 \sum_{j=1}^{k-m} e_m B_1(k-2m-j) \omega(j) + A^{-k-1} f(k).
\]
Both sides will be equivalent if we define
\[
\omega(k) := A^{-k} f(k-1), \quad k \in \mathbb{Z}_1^\infty,
\]
and put this function into (3.15). This ends the proof since
\[
x_p(k) = A^k y_p(k) = \sum_{j=1}^{k} A^{m+j} A^{-m-j} e_m B_1(k-m-j) A^{-j} f(j-1)
\]
\[
= \sum_{j=1}^{k} A^m X_0(k-m-j) f(j-1).
\]

Collecting results of Theorems 3.1 and 3.4, we get immediately

**Theorem 3.5.** Solution \( x = x(k) \) of the initial Cauchy problem (1.1), (1.3) can be on \( \mathbb{Z}_m^\infty \) represented in the form
\[
x(k) = X_0(k) A^{-m} \varphi(-m) + A^m \sum_{j=-m}^{0} X_0(k-m-j) \left[ \varphi(j) - A \varphi(j-1) \right]
\]
\[+ A^m \sum_{j=1}^{k} X_0(k-m-j) f(j-1).
\]

**Remark 3.6.** Note that results obtained can be directly used to investigation such asymptotic problems as boundedness or convergence of solutions (with using of different methods such problems at recent were investigated, e.g., in [3–5]).

**References**