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# A noncrossing basis for noncommutative invariants of $SL(2, \mathbb{C})$

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ABSTRACT

Noncommutative invariant theory is a generalization of the classical invariant theory of the action of $SL(2, \mathbb{C})$ on binary forms. The dimensions of the spaces of invariant noncommutative polynomials coincide with the numbers of certain noncrossing partitions. We give an elementary combinatorial explanation of this fact by constructing a noncrossing basis of the homogeneous components. Using the theory of free stochastic measures this provides a combinatorial proof of the Molien–Weyl formula in this setting.



Invariant theory has played a major role in 19th century mathematics. It has seen a revival in the last decades and one of the recent generalizations is noncommutative invariant theory. The study of noncommutative invariants of $SL(n, \mathbb{C})$ has been initiated by Almkvist, Dicks, Formanek and Kharchenko [6,5,2], see [1] for a survey. An approach using Young tableaux was realized by Teranishi [15] and the symbolic method was adapted from the classical to the noncommutative setting by Tambour [14]. The latter provides the ground on which we establish a natural basis of the noncommutative invariants which is in bijection with certain noncrossing partitions. It arose after computer experiments and subsequent consulting of Sloane's database [12]. This bijection is applied to provide a combinatorial proof of the Molien–Weyl integral formula for the Hilbert–Poincaré series in this setting, using free cumulants and free stochastic measures.

This note is organized as follows. In Section 1 we give a short survey of invariant theory and the statement of the problem. In Section 2 we review a few facts from free probability theory and noncrossing partitions. In Section 3 we explain the symbolic method and construct the noncrossing basis announced in the title. In Section 4 we review the necessary combinatorial aspects of free

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stochastic measures and conclude by a proof of the Molien–Weyl formula using the newly found noncrossing basis.

## 1. An outline of invariant theory

### 1.1. Introduction

Let $X$ be a set and $G$ a group acting on $X$ from the left. Consider a class $\mathcal{A}$ of functions $f : X \to Y$, usually an algebra or at least a vector space, on which the induced action of $G$

$$(g \cdot f)(x) = f\bigl(g^{-1}x\bigr) \tag{1.1}$$

makes sense. The objects of *invariant theory* are the fixed point sets

$$\mathcal{A}^G = \bigl\{f \in \mathcal{A}\colon\ (g \cdot f) = f\ \forall g \in G\bigr\}$$

of such actions.

**Example 1.1.** A favorite example is provided by quadratic polynomials and the group $G$ of translations of the real axis $\mathbb{R}$:

$$g_s : x \mapsto x + s.$$

Denote by $X = \mathbb{R}_2[x] = \{a_0 + a_1x + a_2x^2\colon\ a_0, a_1, a_2 \in \mathbb{R}\}$ the space of polynomials of degree 2 with the action of $G$

$$(g_s \cdot p)(x) = p(x - s) = a_0 - a_1s + a_2s^2 + (a_1 - 2a_2s)x + a_2x^2.$$

Now one may ask which properties of a polynomial $p = a_0 + a_1x + a_2x^2$ do not change under translation. One significant parameter is the number of distinct real roots of a quadratic polynomial, and the three possibilities are distinguished by the sign of the *discriminant*

$$\Delta = a_1^2 - 4a_0a_2$$

and the latter is indeed invariant under the action of $G$. Moreover, it is in some sense the only invariant of $G$: if $\mathcal{A} = \mathbf{P}(\mathbb{R}_2[x])$ is the algebra of polynomials over $\mathbb{R}_2[x]$ (i.e., the polynomials in the coefficients $a_0$, $a_1$, $a_2$) then $\mathcal{A}^G$ is the subalgebra generated by $\Delta$ and $\Delta$ is the only “simple” invariant.

Returning to the general case, if $\mathcal{A}$ is graded

$$\mathcal{A} = \bigoplus_{n\geqslant 0} \mathcal{A}_n$$

with $\dim \mathcal{A}_n < \infty$ then one is interested in the dimensions $d_n = \dim \mathcal{A}_n^G$. These are collected in the *Hilbert–Poincaré series*

$$H\bigl(\mathcal{A}^G; z\bigr) = \sum_{n=0}^{\infty} d_n z^n.$$

### 1.2. Notation

Before proceeding to the invariants of interest let us fix some notation. We are going to consider matrix groups with their actions on certain vector spaces. Let $V$ be a (complex) vector space. As usual $V^*$ denotes the space of linear functionals $v^* : V \to \mathbb{C}$ and there is a natural dual pairing $\langle v^*, w\rangle = v^*(w)$ on $V^* \times V$.

The standard action of $G$ on $V = \mathbb{C}^2$ induces a dual action on $V^*$ via (1.1), namely

$$\langle g \cdot v^*, v\rangle = \langle v^*, g^{-1} \cdot v\rangle$$

i.e., by the invariance requirement

$$\langle g \cdot v^*, g \cdot v\rangle = \langle v^*, v\rangle.$$

Next we induce the action on $V^{*m} \times V^n$ by setting

$$g \cdot \left(v_1^*, v_2^*, \ldots, v_m^*, w_1, w_2, \ldots, w_n\right) = \left(g \cdot v_1^*, g \cdot v_2^*, \ldots, g \cdot v_m^*, g \cdot w_1, g \cdot w_2, \ldots, g \cdot w_n\right).$$

Then for example, on $V^* \times V$, the map

$$\begin{aligned} &f : V^* \times V \to \mathbb{C}, \\ &\left(v^*, w\right) \mapsto \langle v^*, w\rangle \end{aligned} \tag{1.2}$$

is invariant under the action and similarly for any $m \in \mathbb{N}$ the map

$$\begin{aligned} &V^{*m} \times V^m \to \mathbb{C}, \\ &\left(v_1^*, v_2^*, \ldots, v_m^*, w_1, w_2, \ldots, w_m\right) \mapsto \langle v_1^*, w_1\rangle\langle v_2^*, w_2\rangle \cdots \langle v_m^*, w_m\rangle. \end{aligned}$$

In fact, for $V = \mathbb{C}^n$ these are the only multilinear functions which are invariant under the canonical action of $GL(n, \mathbb{C})$, but for $SL(n, \mathbb{C})$ there are more as we shall see below.

The space of $d$-linear functionals $f : V^d \to \mathbb{C}$ will be identified with the $d$-fold tensor product $T^d(V^*) = V^* \otimes \cdots \otimes V^*$. We denote by $S^d(V^*)$ the subspace of symmetric $d$-linear forms, i.e., the $d$-linear functionals which are invariant under permutation of the arguments. This space can be identified with the space of $d$-homogeneous polynomials on $V$ as follows. First note that a symmetric $d$-linear form $f$ is completely determined by the values of the $d$-homogeneous map $\tilde{f}(v) = f(v, v, \ldots, v)$, because the other values can be obtained by *polarization*:

$$f(v_1, v_2, \ldots, v_d) = \sum_{I \subseteq \{1,\ldots,d\}} (-1)^{d-|I|} \tilde{f}\left(\sum_{i \in I} v_i\right).$$

Now if we choose a basis $e_1, \ldots, e_n$ of $V$ and denote by $x_1, \ldots, x_n$ the dual basis of $V^*$, then $S^d(V^*)$ is the linear span of the monomials $x_{k_1,\ldots,k_n} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ with $\sum k_i = d$ where

$$\tilde{x}_{k_1,\ldots,k_n}(v) = \langle x_1, v\rangle^{k_1} \cdots \langle x_n, v\rangle^{k_n}.$$

### 1.3. *Classical invariant theory*

In the present paper we are interested in certain invariants of $G = SL(2, \mathbb{C})$, which acts on $V = \mathbb{C}^2$ by left multiplication. *Classical invariant theory* is interested in the invariants of the space $R_d$ of $d$-homogeneous polynomials on $V$, which are called *binary forms* of degree $d$. Denoting the standard basis vectors of the dual space $V^*$ by $X$ and $Y$, this space can be written as

$$R_d = \left\{ \sum_{k=0}^{d} \binom{d}{k} \xi_k X^k Y^{d-k} \colon \xi_k \in \mathbb{C} \right\}$$

and it is isomorphic to $S^d(V^*)$, the $d$-fold symmetric tensor product of $V^*$. The object of classical invariant theory is the polynomials in the coefficients $\xi_0, \xi_1, \ldots, \xi_d$ which are invariant under the action of $SL(2, \mathbb{C})$, i.e., the space

$$\bigoplus_{m \geqslant 0} S^m\left(R_d^*\right)^{SL(2,\mathbb{C})}.$$

Similarly, *noncommutative invariant theory* is interested in the invariant noncommutative polynomials, i.e., the invariant elements of the full tensor algebra

$$\bigoplus_{m\geqslant 0} T^m\bigl(R_d^*\bigr).$$

Indeed the $m$-fold tensor product $T^m(R_d^*)$ can be identified with the space of $m$-linear forms on $R_d$ as follows: Denote by $a_0, a_1, \dots, a_d$ the canonical basis of $R_d^*$, i.e.,

$$\left\langle a_k, \sum_{j=0}^{d} \binom{d}{j} \xi_j X^j Y^{d-j} \right\rangle = \xi_k,$$

then the space of $m$-linear forms $T^m(R_d^*)$ is spanned by the noncommuting monomials

$$\begin{aligned} a_{k_1} a_{k_2} \cdots a_{k_m} &\left( \sum_{j=0}^{d} \binom{d}{j} \xi_{1j} X^j Y^{d-j}, \sum_{j=0}^{d} \binom{d}{j} \xi_{2j} X^j Y^{d-j}, \dots, \sum_{j=0}^{d} \binom{d}{j} \xi_{mj} X^j Y^{d-j} \right) \\ &= \xi_{1k_1} \xi_{2k_2} \cdots \xi_{mk_m} \end{aligned} \tag{1.3}$$

and we want to determine the space $T^m(R_d^*)^G$ of noncommutative polynomials which are invariant under the action of $G = SL(2, \mathbb{C})$ on $R_d$.

### 1.4. *The fundamental theorems*

Let us now take a closer look at the actions of $G = SL(2, \mathbb{C})$ on $V = \mathbb{C}^2$ and its dual. There are more invariant functions than for $GL(2, \mathbb{C})$. Denoting the standard basis vectors of $V$ by $e_1$ and $e_2$ and decomposing $v_i = \eta_{i1} e_1 + \eta_{i2} e_2$ we can define another invariant function, namely the *bracket*

$$\begin{aligned} &V \times V \to \mathbb{C}, \\ &(v_1, v_2) \mapsto [v_1\ v_2] := \det \begin{bmatrix} \eta_{11} & \eta_{21} \\ \eta_{12} & \eta_{22} \end{bmatrix}; \end{aligned}$$

this function is indeed invariant, because

$$\begin{aligned} [g \cdot v_1\ g \cdot v_2] &= \det\left( g \cdot \begin{bmatrix} \eta_{11} & \eta_{21} \\ \eta_{12} & \eta_{22} \end{bmatrix} \right) \\ &= \det g\ \det \begin{bmatrix} \eta_{11} & \eta_{21} \\ \eta_{12} & \eta_{22} \end{bmatrix} \\ &= [v_1\ v_2]. \end{aligned}$$

Similarly one can define a determinant on $V^* \times V^*$. The first fundamental theorem states that these together with (1.2) are *all* the invariant functions.

**Theorem 1.2** *(First fundamental theorem). Every $SL(2, \mathbb{C})$-invariant multilinear function $f : V^{*m} \times V^n \to \mathbb{C}$ is a linear combination of products of the functions*

$$\langle v^*, w \rangle, \quad \bigl[v_1^*\ v_2^*\bigr], \quad [w_1\ w_2]. \tag{1.4}$$

The functions (1.4) are not independent from each other, they satisfy certain relations, called *syzygies*:

$$[v_1\ v_2] = -[v_2\ v_1], \tag{1.5}$$

$$[v_1\ v_2][v_3\ v_4] = [v_1\ v_3][v_2\ v_4] - [v_1\ v_4][v_2\ v_3]. \tag{1.6}$$

The identity (1.6) is called *Plücker relation*. The second fundamental theorem states that these are the only relations.

**Theorem 1.3** *(Second fundamental theorem). The algebra of invariant polynomials on $V^{*m} \times V^n$ is isomorphic to the free algebra generated by the functions* (1.4) *modulo the relations* (1.5) *and* (1.6) (*and their analogs on* $V^* \times V^*$).

We have thus a complete classification of the invariant functions on $V^{*m} \times V^n$ and the so-called *symbolic method* provides a means to reduce other spaces to this one.

## 2. Free probability

Free probability was invented by Voiculescu [17] as a means to study the von Neumann algebras of free groups, see [16]. For our purpose the combinatorial approach of R. Speicher is appropriate, see the lectures [10] for information beyond the following short survey.

The basic notion of free probability is a *noncommutative probability space* $(\mathcal{A}, \varphi)$ which consists of a unital $C^*$-algebra $\mathcal{A}$ and a faithful state $\varphi$ (i.e., a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ with the properties $\varphi(I) = 1$ and $\varphi(X^*X) \geqslant 0$; faithfulness means that $\varphi(X^*X) = 0$ if and only if $X = 0$). The elements of $\mathcal{A}$ are called *noncommutative random variables*. This definition follows the general strategy of noncommutative geometry to replace commutative algebras of functions by more general noncommutative ones. In this case the commutative von Neumann algebra $L^\infty(\Omega, \mathcal{F}, \mu)$ of bounded measurable functions associated to a probability space $(\Omega, \mathcal{F}, \mu)$ provides the motivating example. We call *distribution* of a noncommutative random variable $X$ the collection of its moments

$$\varphi\big(X^{k_1} X^{*l_1} X^{k_2} X^{*l_2} \cdots X^{k_m} X^{*l_m}\big).$$

When considering a bounded self-adjoint random variable $X$, the sequence of moments $\varphi(X^k)$, $k = 1, 2, \ldots$ uniquely determines a probability measure $\mu_X$ on the spectrum of $X$, which is called the (*spectral*) *distribution* of $X$ and satisfies

$$\varphi\big(X^k\big) = \int t^k \, d\mu_X(t)$$

for all $k \in \mathbb{N}$. There are various notions of noncommutative independence, and *free independence* or *freeness* is the most successful so far.

**Definition 2.1.** Given a noncommutative probability space $(\mathcal{A}, \varphi)$, the subalgebras $\mathcal{A}_i \subseteq \mathcal{A}$ are called *free* if

$$\varphi(X_1 X_2 \cdots X_n) = 0$$

whenever $X_j \in \mathcal{A}_{i_j}$ with $\varphi(X_j) = 0$ and $i_j \neq i_{j+1}$ for $j = 1, 2, \ldots, n-1$.

Free probability shares a lot of features from classical probability. There is for example a central limit theorem which can be formulated exactly like the classical one and the limit distribution is Wigner's semicircle law

$$d\mu(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \, dt.$$

The *free convolution* of two measures $\mu$ and $\nu$, denoted $\mu \boxplus \nu$, which is the distribution of the sum of two free random variables $X$ and $Y$ with spectral distributions $\mu_X = \mu$ and $\mu_Y = \nu$. This operation is well defined because it can be shown that the distribution of $X + Y$ only depends on the distributions $\mu_X$ and $\nu_Y$ and not on the particular realizations of $X$ and $Y$.

Correspondingly, a probability measure $\mu$ is called *free infinite divisible* if for every $n$ there exists a measure $\mu_n$, such that $\mu = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n$ ($n$-fold convolution). A random variable $X$ is called free infinite divisible if its spectral distribution $\mu_X$ has this property.

To compute the free convolution, the rôle of the *characteristic function* of a random variable is played by Voiculescu's *R-transform*, but for our purposes we chose Speicher's cumulant approach to freeness.


**Fig. 1.** The partitions $\{\{1,5,6\},\{2,3\},\{4\}\}$ and $\{\{1,3,4\},\{2,5,6\}\}$.

### *2.1. Noncrossing partitions*

**Definition 2.2.** Denote by $\Pi_n$ the set of partitions $\pi = \{B_1, B_2, \dots, B_k\}$ of the set $[n] = \{1, 2, \dots, n\}$. Equivalently, a partition $\pi$ can be defined by the equivalence relation $\sim_\pi$ on $[n]$ whose equivalence classes are the blocks $B_j$ of $\pi$, i.e.,

$$i \sim_\pi j \quad \Longleftrightarrow \quad i \text{ and } j \text{ belong to the same block of } \pi.$$

A *crossing* of $\pi$ is a quadruple $i < i' < j < j'$ such that $i \sim_\pi j$, $i' \sim_\pi j'$ and $i \nsim_\pi i'$. A partition $\pi$ is called *noncrossing* if it has no crossings. We represent partitions by diagrams as shown in Fig. 1. Thus a partition is noncrossing if and only if its diagram can be drawn with no intersecting lines. We denote by $\mathrm{NC}(n)$ the set of noncrossing partitions of the $n$-element set $[n]$. Equipped with the refinement order it is a lattice with minimal element $\hat{0}_n = \{\{1\}, \{2\}, \dots, \{n\}\}$ and maximal element $\hat{1}_n = \{\{1, 2, \dots, n\}\}$. We denote the lattice operations as usual $\pi \wedge \rho$ and $\pi \vee \rho$. For a function $h : [n] \to A$ where $A$ is an arbitrary set, we denote by $\ker h$ the partition of $[n]$ induced by the level sets of $h$, i.e., the equivalence relation $i \sim j \Longleftrightarrow h(i) = h(j)$.

Noncrossing partitions are enumerated by the ubiquitous *Catalan numbers*

$$\left|\mathrm{NC}(n)\right| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Möbius function is given by Catalan numbers as well,

$$\mu(\hat{0}_n, \hat{1}_n) = (-1)^{n-1} C_{n-1}.$$

### *2.2. Free cumulants*

Given a noncrossing partition $\pi = \{B_1, B_2, \dots, B_k\} \in \mathrm{NC}(n)$ and random variables $X_1, X_2, \dots, X_n$ in some noncommutative probability space $(\mathcal{A}, \varphi)$, we define the partitioned expectation

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \prod_{B \in \pi} \varphi_B(X_1, X_2, \dots, X_n) \tag{2.1}$$

where for a subset $B \subseteq [n]$ we denote the ordered partial moments by

$$\varphi_B(X_1, X_2, \dots, X_n) = \varphi\Biggl(\overrightarrow{\prod_{i \in B}} X_i\Biggr).$$

Following Speicher [13,10] we define the *free cumulants* $C_n$ by the requirement

$$\varphi_\pi(X_1, X_2, \dots, X_n) = \sum_{\substack{\sigma \in \mathrm{NC}(n)\\ \sigma \leqslant \pi}} C_\pi(X_1, X_2, \dots, X_n)$$

where

$$C_\pi(X_1, X_2, \dots, X_n) = \prod_{B \in \pi} C_B(X_1, X_2, \dots, X_n)$$

similar to (2.1). If we consider a single random variable, we write

$$C_n(X) = C_n(X, X, \dots, X).$$

By Möbius inversion, this is equivalent to defining

$$C_\pi(X_1, X_2, \dots, X_n) = \sum_{\substack{\sigma \in \mathrm{NC}(n) \\ \sigma \leqslant \pi}} \varphi_\pi(X_1, X_2, \dots, X_n)\, \mu(\sigma, \pi).$$

Speicher [13] discovered that freeness is equivalent to the vanishing of mixed cumulants, i.e., in the notation of Definition 2.1

$$C_n(X_1, X_2, \dots, X_n) = 0$$

whenever $X_j \in \mathcal{A}_{i_j}$, $n \geqslant 2$ and at least two $i_j$ are different; see also [8] for an explanation why noncrossing partitions appear. We are going to deal with identically distributed free random variables and apply the above formalism in the following situation.

**Corollary 2.3.** *Let $(X_i)_{i\in\mathbb{N}}$ be identically distributed free copies of a random variable $X$ from a noncommutative probability space $(\mathcal{A}, \varphi)$ and $h : [n] \to \mathbb{N}$ an index map. Then*

$$\varphi(X_{h(1)} X_{h(2)} \cdots X_{h(n)}) = \sum_{\pi \leqslant \ker h} C_\pi(X).$$

For example, the only nonvanishing free cumulant of Wigner's semicircle law is $c_2$ and it follows that in the normalized case where $c_2 = 1$ the $2n$-th moment equals the number of noncrossing pair partitions on $2n$ elements. Similarly the normalized *free Poisson law* is characterized by the property that all free cumulants $c_n = 1$ and thus the $n$-th moment equals the number of noncrossing partitions on $n$ elements, that is, again the Catalan numbers.

## 3. A noncrossing basis for noncommutative invariants

### 3.1. The symbolic method [15]

We look for invariants of $T^m(R_d^*)$, the space of $m$-homogeneous polynomials in the noncommuting variables $a_0, a_1, \dots, a_d$, under the induced action of $SL(2, \mathbb{C})$. Earlier (1.3) we have identified these with $m$-linear forms on $R_d = S^d(V^*)$ and now in order to apply the fundamental theorems we have to relate these to invariants of $V^{*k} \times V^l$ for some $k$ and $l$. This is accomplished by Tambour's *Symbolic Method* [14] which proceeds as follows. Denote by $\varphi : T^d(V^*) \to S^d(V^*)$ the projection ("symmetrizator") which maps a $d$-linear form on $V$ to its symmetrization. Now every $m$-linear form $F$ on $R_d = S^d(V^*)$ is an element of the tensor space $T^m(R_d^*)$ and extends to an $m$-linear form $\varphi^* F$ on $T^d(V^*)$, i.e., an element of $T^m(T^d(V^*))$ by setting, for $z_1, z_2, \dots, z_n \in T^d(V^*)$,

$$\varphi^* F(z_1, z_2, \dots, z_m) = F\bigl(\varphi(z_1), \varphi(z_2), \dots, \varphi(z_m)\bigr)$$

and a fortiori an $md$-linear form $\omega_F$ on $V^*$, called the *symbol*, namely

$$\begin{aligned}
&\omega_F(y_{11}, y_{12}, \dots, y_{1d}, y_{21}, \dots, y_{2d}, \dots, y_{m1}, \dots, y_{md}) \\
&\quad = \varphi^* F(y_{11} \otimes y_{12} \otimes \cdots \otimes y_{1d}, y_{21} \otimes \cdots \otimes y_{2d}, \dots, y_{m1} \otimes \cdots \otimes y_{md}) \\
&\quad = F\bigl(\varphi(y_{11} \otimes y_{12} \otimes \cdots \otimes y_{1d}), \varphi(y_{21} \otimes \cdots \otimes y_{2d}), \dots, \varphi(y_{m1} \otimes \cdots \otimes y_{md})\bigr).
\end{aligned}$$

Now it is immediate that the symbol $\omega_F$ is invariant under permutation of each block of arguments $y_{i1}, y_{i2}, \dots, y_{id}$ and because of this symmetry it will be enough to consider map

$$\tilde{\omega}_F : (y_1, y_2, \dots, y_m) \mapsto \omega_F(y_1, \dots, y_1, y_2, \dots, y_2, \dots, y_m, \dots, y_m)$$

which is $d$-homogeneous in each variable.

**Example 3.1.** Consider the linear functional $a_0 \in T^1(R_d^*)$ which is defined by

$$a_0\left(\sum \binom{d}{k} \xi_k X^k Y^{d-k}\right) = \xi_0.$$

Its symbol evaluated at $y = \eta_1 X + \eta_2 Y \in V^*$ is

$$\begin{aligned}
\omega_{a_0}(y, \ldots, y) &= \varphi^* a_0(y \otimes y \otimes \cdots \otimes y) \\
&= a_0\big((\eta_1 X + \eta_2 Y)^d\big) \\
&= a_0\left(\sum \binom{d}{k} \eta_1^k \eta_2^{d-k} X^k Y^{d-k}\right) \\
&= \eta_2^d.
\end{aligned}$$

So far we have shown one half of the following lemma.

**Lemma 3.2.** *The invariant m-linear forms on $R_d$ are in one-to-one correspondence with the md-linear forms on $V^*$ which are invariant under permutations $\mathfrak{S}_d \times \mathfrak{S}_d \times \cdots \times \mathfrak{S}_d$.*

The opposite process which reconstructs an $m$-linear form from its symbol is called *restitution* and establishes the other half of the lemma. We start with an example.

**Example 3.3.** The $2d$-linear $\mathfrak{S}_d \times \mathfrak{S}_d$-invariant map corresponding to the multi-$d$-homogeneous map

$$\begin{aligned}
&\omega : \left(V^*\right)^2 \to \mathbb{C}, \\
&(y_1, y_2) \mapsto [y_1\ y_2]^d
\end{aligned}$$

is invariant and is the symbol of the following $d$-linear form:

$$\begin{aligned}
[y_1\ y_2]^d &= \det \begin{pmatrix} \eta_{11} & \eta_{21} \\ \eta_{12} & \eta_{22} \end{pmatrix}^d \\
&= (\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^d \\
&= \sum_{k=0}^{d} \binom{d}{k} (-1)^k (\eta_{11}\eta_{22})^{d-k} (\eta_{12}\eta_{21})^k \\
&= \sum_{k=0}^{d} \binom{d}{k} (-1)^k \eta_{11}^{d-k} \eta_{12}^k \eta_{21}^k \eta_{22}^{d-k}.
\end{aligned}$$

This means that

$$F\left(\sum \binom{d}{k} \xi_{1k} X^k Y^{d-k}, \sum \binom{d}{k} \xi_{2k} X^k Y^{d-k}\right) = \sum_{k=0}^{d} \binom{d}{k} (-1)^k \xi_{1,d-k} \xi_{2,k}$$

i.e.,

$$F = \sum_{k=0}^{d} \binom{d}{k} (-1)^k a_{d-k} a_k$$

and for $d = 2$ this is the noncommutative discriminant $a_2 a_0 - a_1 a_1 + a_0 a_2$.

In general, if $\omega : (V^*)^{md}$ is an invariant multilinear functional which is also invariant under permutations from $\mathfrak{S}_d^m$, then by the first fundamental theorem the value

$$\omega(y_1, y_1, \ldots, y_1, y_2, y_2, \ldots, y_2, \ldots, y_m, y_m, \ldots, y_m)$$

must be a linear combination of products of brackets $[y_i\ y_j]$ with $i \neq j$ where each $y_i$ appears exactly $d$ times. Thus by linearity it suffices to construct for each $(m-d)$-homogeneous form

$$\tilde{\omega}(y_1, y_2, \ldots, y_m) = \prod [y_{i_k}\ y_{j_k}]$$

satisfying the condition just stated a noncommutative invariant whose symbol is $\omega$. Now if we decompose $y_i = \eta_{i1} X + \eta_{i2} Y$ we have

$$\tilde{\omega}(y_1, y_2, \ldots, y_m) = \prod (\eta_{i_k 1}\eta_{j_k 2} - \eta_{i_k 2}\eta_{j_k 1})$$

and expanding the product we get a sum of terms of the form

$$\prod_{i=1}^{m} \eta_{i1}^{s_i} \eta_{i2}^{d-s_i}$$

which is the symbol of the noncommutative monomial

$$\prod_{i=1}^{m} a_{s_i}.$$

*3.2. Finding a basis*

We have thus used the first fundamental theorem to determine all invariants; namely, the symbols are spanned by the elementary symbols

$$\prod [y_i\ y_j]$$

where each $y_i$ appears exactly $d$ times. For finding a basis it is convenient to use diagrams.

**Definition 3.4.** An *m-partite partition* of the set $[dm]$ is a partition whose blocks contain at most one element from each interval $\{kd+1, kd+2, \ldots, (k+1)d\}$. To each $m$-partite pair partition $\pi = \{\{i_1, j_1\}, \ldots, \{i_q, j_q\}\}$ we associate the symbol

$$\tilde{\omega}_\pi(y_1, \ldots, y_m) = \prod [y_{i_k}\ y_{j_k}].$$

It is easy to see that different partitions may lead to identical symbols and in particular the corresponding symbols are not linearly independent. Moreover the Plücker relations lead to even more linear dependencies. We shall show that the latter is true if we restrict to noncrossing $m$-partite pair partitions. Moreover in the rest of this section we prove that they form a basis:

**Theorem 3.5.** *The dimension of the space $T^m(R_d^*)^G$ of invariant noncommutative polynomials is equal to the number of m-partite noncrossing pair partitions $\pi \in \mathrm{NC}(md)$.*

The key observation is that the Plücker relation

$$[v_1\ v_3][v_2\ v_4] = [v_1\ v_2][v_3\ v_4] + [v_1\ v_4][v_2\ v_3]$$

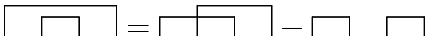
has a pictorial interpretation as follows:

We see that the number of crossings is reduced by one and this means that if we start with an elementary symbol

$$\prod [y_{i_k}\ y_{j_k}]$$

we can associate to it a pairing and successively remove any crossings to obtain a linear combination of noncrossing pairings. Thus the space of symbols is spanned by noncrossing symbols. This strategy is different from the usual *straightening algorithm* where the formula is read as

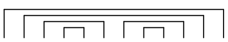
$$[y_1\ y_4][y_2\ y_3] = [y_1\ y_3][y_2\ y_4] - [y_1\ y_2][y_3\ y_4],$$



i.e., nestings are removed. The straightening algorithm has the advantage to be applicable for arbitrary $SL(n,\mathbb{C})$, whereas our approach only works for $SL(2,\mathbb{C})$. The next lemma concludes the proof of Theorem 3.5.

**Lemma 3.6.** *Symbols coming from different noncrossing pairings are linearly independent. The irreducible noncrossing pairings, that is, those in which the left- and rightmost vertices are connected with each other, generate the invariants as a ring.*

**Proof.** This can be shown as in [15]. We order the noncommutative monomials in $T^m(R_d^*)$ lexicographically with respect to the order $a_d > a_{d-1} > \cdots > a_0$ on the letters and we will show that different noncrossing symbols have different leading terms with respect to this order. Let us first consider an example:



$$\begin{aligned} 1\,1\,1\,1\,2\,2\,2\,2\,2\,3\,3\,3\,3 &= [1\ 2]^2\,[1\ 3]^2\,[2\ 3]^2 \\ &= (\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^2(\eta_{11}\eta_{32} - \eta_{12}\eta_{31})^2(\eta_{21}\eta_{32} - \eta_{22}\eta_{31})^2 \\ &= \eta_{11}^4\eta_{21}^2\eta_{22}^2\eta_{32}^4 + \cdots \\ &\simeq a_4a_2a_0 + \cdots . \end{aligned}$$

Consider the $k$-th interval $\{(k-1)d+1, (k-1)d+2, \ldots, kd\}$. An edge adjacent to this interval is called *incoming* if it connects to an element to the left and *outgoing* if it connects to the right. Then the index of the $k$-th factor of the leading term indicates the number of outgoing edges of the $k$-th interval.

Since in a noncrossing partition the incoming edges always come before the outgoing edges, these numbers uniquely determine the partition. Thus different noncrossing partitions have different leading terms.

As in [15] one can show that the invariants coming from noncrossing irreducible symbols (i.e., those with only one outer block) form a free generating set of the ring of noncommutative invariants. $\square$

Note that this also yields an explicit bijection between $m$-partite noncrossing pair partitions and column-strict Young tableaux. This combinatorial coincidence was also found independently in [4] by establishing a bijection with the Young tableaux of Teranishi [15].

## 4. Free stochastic measures and the Hilbert series

In order to find the Hilbert–Poincaré series

$$H_d(z) = H\left(T\left(R_d^*\right)^G; z\right) = \sum_{m=0}^{\infty} \dim T^m\left(R_d^*\right)^G z^m \tag{4.1}$$

for fixed $d$ one usually resorts to integration on the group (Molien's formula) which in our case reads:

**Theorem 4.1.** *(See [2].)*

$$H_d(z) = \frac{2}{\pi}\int_0^\pi \frac{\sin^2 x}{1 - z\sin((d+1)x)/\sin x}\,dx.$$

Our aim here is to provide a different proof of this by establishing a combinatorial link to *free stochastic measures*. The latter have been constructed by Anshelevich [3] following Rota and Wallstrom [11]. Let $X$ be a free infinitely divisible random variable. Then for every $N \in \mathbb{N}$ we can write $X$ as a sum of identically distributed free random variables $X_i^{(N)}$, $i \in \{1,\dots,N\}$ and for every partition $\pi \in \Pi_n$ the *stochastic measure* $\mathrm{St}_\pi$ and the *product measure* $\mathrm{Pr}_\pi$ are defined as the elements

$$\mathrm{St}_\pi = \lim_{N\to\infty} \sum_{\ker h = \pi} X_{h(1)}^{(N)} X_{h(2)}^{(N)} \cdots X_{h(n)}^{(N)},$$

$$\mathrm{Pr}_\pi = \lim_{N\to\infty} \sum_{\ker h \geqslant \pi} X_{h(1)}^{(N)} X_{h(2)}^{(N)} \cdots X_{h(n)}^{(N)}.$$

It can be shown that the limits exist in norm and we will be particularly interested in the special cases $\psi_n = \mathrm{St}_{\hat{0}_n}$ and the so-called diagonal measures $\Delta_n = \mathrm{St}_{\hat{1}_n}$. The following properties hold: $\mathrm{St}_\pi = 0$ unless $\pi$ is noncrossing [3, Thm. 1] and from this it follows immediately that

$$\mathrm{Pr}_\pi = \sum_{\substack{\sigma \in \mathrm{NC} \\ \sigma \geqslant \pi}} \mathrm{St}_\sigma, \qquad \mathrm{St}_\pi = \sum_{\substack{\sigma \in \mathrm{NC} \\ \sigma \geqslant \pi}} \mu(\pi,\sigma)\,\mathrm{Pr}_\sigma.$$

Moreover, by [3, Lemma 1], the expectation of a stochastic measure has a simple expression in terms of cumulants of the original random variable $X$, namely

$$\varphi(\mathrm{St}_\pi) = C_\pi(X).$$

Concerning the joint distribution of $\psi_n$, [3, Prop. 4] tells us that

$$\psi_{k_1}\psi_{k_2}\cdots\psi_{k_m} = \sum_{\substack{\sigma \in \mathrm{NC}(k_1+k_2+\cdots+k_m) \\ \sigma \wedge \hat{1}_{k_1}\hat{1}_{k_2}\cdots\hat{1}_{k_m} = \hat{0}}} \mathrm{St}_\sigma$$

where we recognize the $m$-partite partitions of Definition 3.4. Altogether it follows that

$$\varphi(\psi_{k_1}\psi_{k_2}\cdots\psi_{k_m}) = \sum_{\substack{\sigma \in \mathrm{NC}(k_1+k_2+\cdots+k_m) \\ \sigma \wedge \hat{1}_{k_1}\hat{1}_{k_2}\cdots\hat{1}_{k_m} = \hat{0}}} C_\sigma(X). \tag{4.2}$$

An alternative inductive proof of this formula is given in [9], see also [7] for an application to strong Haagerup inequalities for so-called $R$-diagonal elements.

To conclude our proof of Theorem 4.1 let us from now on assume that $X$ is a standard semicircular element, with $\varphi(X^2) = 1$. Then

$$C_\sigma(X) = \begin{cases} 1, & \sigma \in \mathrm{NC}_2, \\ 0, & \sigma \notin \mathrm{NC}_2 \end{cases}$$

together with Theorem 3.5 implies that

$$\dim T^m\big(R_d^*\big)^G = \varphi\big(\psi_d^m\big). \tag{4.3}$$

It remains to identify the distribution of $\psi_d$. Here we use one more result of Anshelevich [3, Prop. 5] which states that for a centered free infinite divisible random variable we have the orthogonality relation

$$\gamma(\psi_m \psi_n) = \delta_{mn}\varphi(\Delta_2)^n = \delta_{mn}C_2(X)^n$$

and therefore $\psi_k$ can be identified with the orthogonal polynomials of $X$, which in the semicircular case are the Chebyshev polynomials $U_n$ of the second kind and thus [3, Cor. 8]

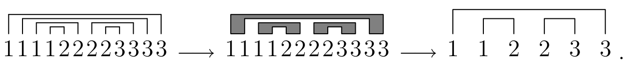
$$\psi_n = X\psi_{n-1} - \psi_{n-2}$$

i.e., $\psi_n = U_n(X)$. Plugging this into (4.3) we obtain

$$\begin{aligned}\dim T^m\big(R_d^*\big)^G &= \varphi\big(U_d(X)^m\big)\\ &= \int_{-2}^{2} U_d(x)^m \sqrt{4-x^2}\,dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(d+1)\theta}{\sin\theta}\right)^m \sin^2\theta\,d\theta\end{aligned}$$

by the standard substitution $U_d(\cos\theta) = \sin((d+1)\theta)/\sin\theta$.

**Remark 4.2.** If $d$ is even then the noncrossing $m$-partite pair partitions are in bijection with all $m$-partite noncrossing partitions without singletons on $md/2$ points via the thickening bijection illustrated in the following example:

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