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PROBABILITY LOGIC WITH CONDITIONAL EXPECTATION*

Sergio FAJARDO

Department of Mathematics, University of Wisconsin, Madison, WI 53705, USA
(Current address: *Dept. of Math., Univ. of Colorado, Boulder, CO 80309, USA*)

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0. Introduction

Probability logics are logics adequate for the study of structures arising in Probability Theory. The newest member of this family is the Probability Logic with Conditional Expectation, denoted $L_{\mathbb{A}E}$, recently introduced by Keisler in [16]. In this paper we develop the model theory of this logic. We answer affirmatively problems 5 and 6 of Keisler's paper, regarding the existence and uniqueness of hyperfinite ('saturated') models and the validity of the Robinson consistency property and Craig interpolation theorem for $L_{\mathbb{A}E}$. Here is a summary of the history of Probability Logics and a description of the contents of this paper.

The development of the model theory of first-order logic has brought up the need for the study of logics with a stronger expressive power. This allows us to incorporate into the realm of logic certain common mathematical structures and concepts that have been left out of first-order logic due to its limited scope. Most of the new logics that have been studied focus on structures arising in algebra and set theory. Topological Logic is an example of a well behaved logic with 'enriched' structures, its models besides a first order part also have a topology on the universe (see Zeigler's survey paper [28]). An up to date account of the research done during the last twenty years in several extensions of first order logic can be found in the forthcoming book [5].

Keisler in [17] began the study of the Probability Logics $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$. These are logics where the quantifiers $\forall x$ and $\exists x$ are not allowed and instead the quantifiers $(Px > r)$ and $\int dx$ are respectively incorporated. The most interesting structures (models) for $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ are of the form (A, X, μ), where μ is a probability measure on A and X is a random variable defined on A. The model theory of the logics has been developed further by Hoover in [11–14] and Keisler in [16]. Later on, inspired by Keisler's work on stochastic differential equations

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[18], Hoover and Keisler (see [16], [19]) introduced Adapted Probability Logic denoted L_{ad}. This is a logic adequate for the study of stochastic processes. Its structures are of the form $\langle A, X = (X(t))_{t\in[0,1]}, (\mathcal{F}_t)_{t\in[0,1]}, \mu\rangle$, where μ is a probability measure on A, X is a measurable stochastic process defined on A with time parameters in $[0, 1]$ and $(\mathcal{F}_t)_{t\in[0,1]}$ is an increasing sequence of σ-algebras of μ-measurable sets. For information about the role of these structures in the theory of stochastic processes see [8] and [9]. Hoover and Keisler in [15] introduced in a probabilistic context two notions of elementary equivalence for L_{ad}; using nonstandard analysis tools and model theoretic techniques they gave direct applications to problems in Probability Theory. Rodenhausen in his thesis [25] presented a set of axioms for L_{ad} and proved a completeness theorem for this logic. His proof was very long and complicated due to the apparent lack of connection between $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ with L_{ad}.

With the introduction of $L_{\mathbb{A}E}$ the gap was filled. This logic extends $L_{\mathbb{A}\int}$ with the addition of a new operator $E[\,|\,]$ that allows one to talk about conditional expectations of random variables with respect to σ-algebras. Its simplest structures are of the form $\langle A, X, \mathcal{F}, \mu\rangle$ with $\langle A, X, \mu\rangle$ as for $L_{\mathbb{A}\int}$ and $\mathcal{F}$ a σ-algebra of μ-measurable sets. Keisler in [16], [20] proved a completeness theorem for $L_{\mathbb{A}E}$, introduced L_{ad} in a two-sorted form of $L_{\mathbb{A}E}$ and using the completeness theorem for this logic gave a simpler and more natural proof of Rodenhausen's completeness theorem for L_{ad}.

In this paper we continue the study of $L_{\mathbb{A}E}$. In Section 1 we give a direct definition of $L_{\mathbb{A}E}$ without assuming a previous knowledge of either of $L_{\mathbb{A}P}$ or $L_{\mathbb{A}\int}$ and list some basic facts about $L_{\mathbb{A}E}$. The reader is assumed to be familiar with some elementary probabilistic concepts such as the definition of conditional expectation and some of its properties; these can be found in any introductory probability textbook (see for example [3] or [6]). We also indicate in this section some possible generalizations of $L_{\mathbb{A}E}$ by allowing more than one conditional expectation operator symbol; this permits us to study in $L_{\mathbb{A}E}$ stochastic processes with discrete time without having to go to L_{ad}.

Sections 2 and 3 contain the main theorems of this paper. We introduce a notion of hyperfinite model for $L_{\mathbb{A}E}$ that naturally extends the corresponding notion of hyperfinite model for $L_{\mathbb{A}\int}$ as presented in [16]. We prove that these models exist (Section 2) and for a special type of hyperfinite models that we call uniform we prove that they are unique (Section 3). Most of our proofs make use of arguments and constructions from Nonstandard Analysis; in particular, we assume the reader is familiar with the definition and main features of a nonstandard universe and also has a knowledge of the Loeb measure construction. The books [27] and [1] have good presentations of nonstandard analysis, [26] concentrates on the study of Probability Theory from a hyperfinite point of view and Cutland's survey paper [7] has a nice introduction to non-standard measure theory aimed at the general mathematician.

In Section 4 we use the existence and uniqueness theorems of hyperfinite models in order to give proofs of the Robinson consistency property and Craig interpolation theorem for $L_{\mathbb{A}E}$.

A knowledge of basic model theory is desirable; with respect to Probability Logic not much is needed but some familiarity with [16] will help to have a better understanding of the whole subject of Probability Logics and in particular of the results presented here. Our results have been previously announced in [10] and were presented at the 1983–84 ASL Annual Meeting held in Boston.

1. Basic definitions and background

$L_{\mathbb{A}E}$ was introduced by Keisler in [16] as an extension of the logic $L_{\mathbb{A}\int}$, which allows us to express in a natural way probabilistic notions involving conditional expectations of random variables with respect to σ-algebras. In this section we present the basic definitions for both logics and all those results about them that are relevant to our exposition. $\mathbb{A}$ is an admissible set contained in HC (Hereditarily countable sets) with $\omega \in \mathbb{A}$. For simplicity we restrict our attention to the case $\mathbb{A} = \mathrm{HC}$ and denote the logic so obtained by $L_{\omega_1 E}$. All our results can be readily generalized to $L_{\mathbb{A}E}$.

Definition 1.1. Throughout this paper we work with a countable set $L = \{X_i^{n_i} : i \in I\}$ of random variable symbols. The n_i represents the number of arguments of X_i and we will not bother to write it unless it is strictly necessary.

The *Logical Symbols* of $L_{\omega_1 E}$ are:

(a) A countable list of individual variables: $u, v, x, y, z, \ldots$

(b) Connectives:

(1) First-order Connectives: $\neg, \bigwedge$.

(2) Function Connectives: The symbol $\bar{F}$, for each $n \in \mathbb{N}$ and $F \in C(\mathbb{R}^n) =$ the set of continuous functions $F : \mathbb{R}^n \to \mathbb{R}$.

(c) Integral Quantifier Symbol: $\int$.

(d) Conditional Expectation Symbol: $E[\,|\,]$.

Definition 1.2. The set of $L_{\omega_1 E}$*-atomic terms* is defined by:

(a) For each random variable symbol X_i, $\boldsymbol{u}$ an n_i-tuple of variables and $r \in \mathbb{Q}^+$, $[X_i(\boldsymbol{u}) \restriction r]$ is an $L_{\omega_1 E}$-atomic term.

(b) If u_1 and u_2 are variables, then $1(u_1 = u_2)$ is an atomic term.

The set of $L_{\omega_1 E}$*-terms* is the least set such that:

(a) Every atomic term is a term.

(b) Each real number r is a term.

(c) If $\tau_1, \ldots, \tau_n$ are terms and $F \in C(\mathbb{R}^n)$, then $\bar{F}(\tau_1, \ldots, \tau_n)$ is a term.

(d) If τ is a term and u is a variable, then $\int \tau \, du$ is a term.

(e) If $\tau(\boldsymbol{u}, v)$ is a term, $v \notin \boldsymbol{u}$ and x is a variable, then $E[\tau(\boldsymbol{u}, v) \mid v](x)$ is a term.

Due to the introduction of the conditional expectation operator we have to be careful in the way we handle the variables that appear in terms. Intuitively, the quantifier $\int$ binds the variable u in $\int du$ and in $E[\tau \mid v]\,(x)$ the occurrence of v is bound and that of x is free. But when confronted with terms of the form $E[\tau(u) \mid u]\,(u)$ we have to decide whether the variable u is bound or free. We are going to use the concept 'free' variable for those variables 'with at least one free occurrence', this idea is made precise with the following definition.

Definition 1.3. Given a term τ and a variable v we say that v is *free in* τ according to the following inductive definition:

(a) If $\tau(u)$ is $[X(\boldsymbol{u}) \restriction r]$, then v is free in τ if and only if $v \in u$.

(b) If $\tau(u_1, u_2)$ is $1(u_1 = u_2)$, then v is free in τ if and only if v is u_1 or v is u_2.

(c) If τ is $\bar{F}(\tau_1, \ldots, \tau_n)$, then v is free in τ if and only if, for some i, v is free in τ_i.

(d) If τ is $\int \theta\, du$, then v is free in τ if and only if v is not u and v is free in θ.

(e) If τ is $E[\theta \mid z]\,(x)$, then v is free in τ if and only if v is x or v is not z and it is free in θ.

If v is not free in τ we say that v is *bound in* τ. With these definitions we can see that u is free in $E[\tau(u) \mid u](u)$ and u is bound in $E[\tau(v, u) \mid u](x)$. A *closed* term is a term with no free variables.

Definition 1.4. The set of $L_{\omega_1 E}$*-formulas* is the least set such that:

(a) For each $L_{\omega_1 E}$-term τ, $[\tau \geqslant 0]$ is an (atomic) formula.

(b) If ϕ is a formula, so is $\neg\phi$.

(c) If Γ is a countable set of formulas with finitely many free variables, then $\bigwedge \Gamma$ is a formula.

A *sentence* is a formula with no free variables.

Remark. In the previous definitions when we delete those clauses that involve conditional expectations (i.e. (d) in Definition 1.1 and (e) in Definition 1.2), we obtain the corresponding notions for $L_{\omega_1 \int}$. Given the way the conditional expectation operator is interpreted (see Definition 1.8) it turns out that it is important to distinguish two types of variables among the free variables that appear on terms. Intuitively we divide them in two groups: In one we have the 'E-bound' variables which are those that when interpreted produce an $\mathcal{F}$-measurable random variable (see Definition 1.8 for undefined concepts) and in the other group we have the 'E-free' variables which are those that when interpreted produce random variables that are not $\mathcal{F}$-measurable. These informal notions are made rigorous with our next definition and the examples following it illustrate different cases.

Definition 1.5. Given an $L_{\omega_1 E}$-term τ and a variable v, we say "v *is* E*-free in* τ" according to the following inductive definition:

(a) If $\tau(\boldsymbol{u})$ is $[X(\boldsymbol{u}) \restriction r]$, then v is E-free in τ if and only if for some i, v is u_i.

(b) If $\tau(u_1, u_2)$ is $1(u_1 = u_2)$, then v is E-free in τ if and only if v is u_1 or v is u_2.

(c) If τ is $\bar{F}(\tau_1, \ldots, \tau_n)$, then v is E-free in τ if and only if for some i, v is E-free in τ_i.

(d) If τ is $\int \theta \, du$, then v is E-free in τ if and only if v is not u and v is E-free in θ.

(e) If τ is $E[\theta \mid z](x)$, then v is E-free in τ if and only if v is not z and v is in θ.

If v is free in τ and it is not E-free in τ, we say that "*v is E-bound in τ*".

Examples. (1) z is E-bound in $E[[X(u, v) \upharpoonright r] \mid v](z)$ and u is E-free.

(2) If we let $\tau(u, v)$ be $E[[X(u, v) \upharpoonright r] \mid v](v)$, then u is E-free in τ and v is E-bound in τ.

(3) If we let $\tau(u, u', v)$ be $[X(u, v) \upharpoonright r] + E[[X(u', z) \upharpoonright r] \mid z](v)$, then u is E-free in τ, u' is E-free in τ and v is E-free in τ.

Notice also that v is E-bound in $E[[X(u', z) \upharpoonright r] \mid z](v)$ and it is E-free in $[X(u, v) \upharpoonright r]$.

(4) v is E-free in $E[[X(v, z) \upharpoonright r] \mid z](v)$.

As mentioned in the introduction we assume the reader is familiar with the elementary probabilistic concepts that appear on the following definitions.

Definition 1.6. (a) A triple $\langle A, \mathscr{G}, \nu \rangle$ is a *probability space* if $\mathscr{G}$ is a σ-algebra on A such that each singleton belongs to $\mathscr{G}$. Usually we just write $\langle A, \nu \rangle$. $\mathscr{G}$ is called the set of ν-measurable sets.

(b) (*Keisler* [16]). Let $\langle A, \mathscr{G}, \nu \rangle$ be a probability space and $n \in \mathbb{N}$. $\langle A^n, \mathscr{G}^{(n)}, \nu^{(n)} \rangle$ is the probability space where $\mathscr{G}^{(n)}$ is the σ-algebra on A^n generated by the rectangles of the form $G_1 \times \cdots \times G_n$ with $G_i \in \mathscr{G}$ for each i and the diagonal sets of the form $D_{ij} = \{\mathbf{x} \in A^n : x_i = x_j\}$. $\nu^{(n)}$ is the unique extension of the product measure $\nu^n = \nu \times \cdots \times \nu$ to $\mathscr{G}^{(n)}$ such that $\nu^{(n)}(D_{ij}) = \sum_{x \in A} \nu(\{x\})^2$.

In general when we say that a random variable is $\nu^{(n)}$-measurable, we mean it is measurable with respect to $\mathscr{G}^{(n)}$.

Definition 1.7. (a) A *random variable model* for L is a structure $\mathscr{U} = \langle A, X_i^{\mathscr{U}}, \nu \rangle_{i \in I}$ where $\langle A, \nu \rangle$ is a probability space and each $X_i^{\mathscr{U}}$ is a real-valued $\nu^{(n_i)}$-measurable random variable (i.e. for every Borel subset B of $\mathbb{R}$, $(X_i^{\mathscr{U}})^{-1}(B)$ belongs to $\mathscr{G}^{(n_i)}$).

(b) A *conditional expectation model* for L is a structure of the form $\mathscr{U} = (\mathscr{U}_0, \mathscr{F}^{\mathscr{U}})$ where $\mathscr{U}_0$ is a random variable model for L and $\mathscr{F}^{\mathscr{U}}$ is a σ-algebra on A contained in $\mathscr{G}$. We write $\mathscr{U} = \langle A, X_i^{\mathscr{U}}, \mathscr{F}^{\mathscr{U}}, \nu \rangle_{i \in I}$.

Definition 1.8. An *interpretation* of $L_{\omega_1 E}$ in a conditional expectation model $\mathscr{U}$ is a function that assigns to each term $\tau(u_1, \ldots, u_n)$ a $\nu^{(n)}$-measurable function

$\tau^{\mathcal{U}} : A^n \to \mathbb{R}$ such that:

(a)

$$(1)\ [X_i(u) \upharpoonright r]^{\mathcal{U}}(\boldsymbol{a}) = \begin{cases} r & \text{if } X_i^{\mathcal{U}}(\boldsymbol{a}) \geqslant r, \\ -r & \text{if } X_i^{\mathcal{U}}(\boldsymbol{a}) \leqslant -r, \\ X_i^{\mathcal{U}}(\boldsymbol{a}) & \text{if } -r \leqslant X_i^{\mathcal{U}}(\boldsymbol{a}) \leqslant r. \end{cases}$$

$$(2)\ [1(u_1 = u_2)]^{\mathcal{U}}(a_1, a_2) = \begin{cases} 1 & \text{if } a_1 = a_2, \\ 0 & \text{if } a_1 \neq a_2. \end{cases}$$

$$(3)\ [\bar{F}(\tau_1, \ldots, \tau_n)]^{\mathcal{U}}(\boldsymbol{a}) = F(\tau_1^{\mathcal{U}}(\boldsymbol{a}), \ldots, \tau_n^{\mathcal{U}}(\boldsymbol{a})).$$

$$(4)\ \left[\int \theta(\boldsymbol{u}, v)\, \mathrm{d}v\right]^{\mathcal{U}}(\boldsymbol{a}) = \int_A \theta^{\mathcal{U}}(\boldsymbol{a}, b)\, \mathrm{d}v(b).$$

(b) $(E[\tau(\boldsymbol{u}, v) \mid v](x))^{\mathcal{U}}(\boldsymbol{a}, b)$ is $\nu^{(n)} \times \mathcal{F}$-measurable and for each $\boldsymbol{a} \in A^n$, $(E[\tau(\boldsymbol{u}, v)\mathbf{g}\, v](x))^{\mathcal{U}}(\boldsymbol{a}, b) = E[\tau^{\mathcal{U}}(\boldsymbol{a}, \cdot) \mid \mathcal{F}^{\mathcal{U}}]\ (b)$ for ν-almost all b.

The following lemma is an immediate consequence of the definition of interpretation.

Lemma 1.9. *For every conditional expectation model $\mathcal{U}$ there exists an interpretation of $L_{\omega_1 E}$ in $\mathcal{U}$. Two interpretations agree almost surely on each term. The values of closed terms and sentences are the same in all interpretations.*

The reader can notice that in Definition 1.8 (a.1) the values of the random variables are truncated. This is a technical point that makes the interpretation of each term a bounded random variable in the model and consequently an *integrable* random variable. Moreover, we can give uniform bounds for the interpretations of terms in arbitrary models as the following definition and lemma show. Hereafter $\tau^{\mathcal{U}}$ is some interpretation in $\mathcal{U}$.

Definition 1.10. The *bound* $\|\tau\|$ of an $L_{\omega_1 E}$-term τ is defined by:

(a) $\|[X_i(\boldsymbol{u}) \mid r]\| = r$,

(b) $\|1(u = v)\| = 1$,

(c) $\|\int \tau\, \mathrm{d}x\| = \|\tau\|$,

(d) $\|E[\tau \mid u](v)\| = \|\tau\|$,

(e) $\|\bar{F}(\tau_1, \ldots, \tau_n)\| = \mathrm{Sup}\{|F(s_1, \ldots, s_n)| : |s_i| \leqslant \|\tau_i\|\}$,

(f) $\|r\| = |r|$.

Lemma 1.11. *Given a structure $\mathcal{U}$ for all $\boldsymbol{a}$ in A we have*: $|\tau^{\mathcal{U}}(\boldsymbol{a})| \leqslant \|\tau\|$.

Proof. Induction on the complexity of terms. □

In [16, Section 3.5], we can find a set of axioms for $L_{\omega_1 \int}$ and a proof of the completeness theorem for $L_{\omega_1 \int}$. These axioms express natural properties of the integral and instead of writing them down we refer the reader to that section for complete details. In the same paper, Section 4.2, we find the following set of axioms and rules of inference for $L_{\omega_1 E}$ (we shorten $E[\tau \mid u](u)$ to $E[\tau \mid u]$).

Definition 1.12. (a) *Axioms* for $L_{\omega_1 E}$.

(A0) All axiom schemes for $L_{\omega_1 \int}$.

(A1) $E[\tau(\boldsymbol{v}, x) \mid x](u) = E[\tau(\boldsymbol{v}, y) \mid y](u)$ where x and y do not occur in $\boldsymbol{v}$.

(A2) $\int E[\sigma(u, \boldsymbol{v}) \mid u]\tau(u)\,du = \int E[\sigma(u, \boldsymbol{v}) \mid u] \cdot E[\tau(u) \mid u]\,du$.

(b) *Rules of inference* for $L_{\omega_1 E}$:

(R1) Modus Ponens: $\phi,\ \phi \to \gamma \vdash \gamma$.

(R2) Conjunction: $\{\phi \to \gamma : \gamma \in \Gamma\} \vdash \phi \to \bigwedge \Gamma$.

(R3) Generalization: $\phi \to [\tau(u, \boldsymbol{x}) \geqslant 0] \vdash \phi \to [\int \tau(u, \boldsymbol{x})\,du \geqslant 0]$ where u is not free in ϕ.

Theorem 1.13 (Keisler [16]). Soundness and completeness for $L_{\omega_1 E}$: *A countable set of sentences of $L_{\omega_1 E}$ has a model if and only if it is consistent.*

Definition 1.14 (Generalizations). In probabilistic practice we usually find situations where it is important to consider two or more σ-algebras in the same model. A particularly well known case is that of discrete stochastic processes where we have structures of the form $\mathcal{U} = \langle A, (X_n)_{n \in \mathbb{N}}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mu \rangle$ such that for all $n \in \mathbb{N}$:

(a) $X_n : A \to \mathbb{R}$ is a random variable.

(b) $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mu$-measurable sets.

$L_{\omega_1 E}$ can be extended in such a way that this type of model is covered. For each $n \in \mathbb{N}$ introduce a conditional expectation operator E_n where the interpretation of $E_n[\tau(\boldsymbol{u}, v) \mid v](x)$ in $\mathcal{U}$ is $E[\tau^{\mathcal{U}}(\boldsymbol{a}, \cdot) \mid \mathcal{F}_n](b)$ a.s.

If for each $n \in \mathbb{N}$ we add to the list of axioms given in Definition 1.12 the new axiom:

$$(\mathrm{A3})_n \quad E_n[\tau(\boldsymbol{v}, u) \mid u] = E_{n+1}[E_n[\tau(\boldsymbol{v}, u) \mid u] \mid u],$$

then we have:

Theorem 1.15 (Generalized soundness and completeness for $L_{\omega_1 E}$). *Let Γ be a countable set of sentences in $L_{\omega_1 E}$ with conditional expectation operators E_n for each $n \in \mathbb{N}$.*

Γ has a model $\mathcal{U} = \langle A, (X_i^{\mathcal{U}})_{i \in I}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mu \rangle$ with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for each $n \in \mathbb{N}$ if and only if Γ is consistent in $L_{\omega_1 E}$ with $(\mathrm{A3})_n$, $n \in \mathbb{N}$.

Proof. Soundness as usual is easy. For the other direction we just add a few things to Keisler's proof of the completeness theorem for $L_{\omega_1 E}$. Instead of defining a single σ-algebra $\mathcal{F}$, for each $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the σ-algebra generated by the sets of the form $\{d \in M : X_\tau(\mathbf{c}, d)^{\mathcal{M}} \geq r\}$ where $\tau(\mathbf{u}, v)$ is of the form $E_n[\sigma(\mathbf{u}, w) \mid w](v)$ with v E-bound in τ and $\mathbf{c}$ is in M.

The proof is finished as before using the axioms $(\mathrm{A3})_n$, $n \in \mathbb{N}$ to guarantee that the $\mathcal{F}_n$'s form an increasing sequence of σ-algebras. □

Definition 1.16. Let $\mathcal{U}$ and $\mathcal{B}$ be $L_{\omega_1 E}$-models. $\mathcal{U}$ is said to be *elementarily equivalent* to $\mathcal{B}$ if for each $L_{\omega_1 E}$-sentence θ, $\mathcal{U} \models \theta$ if and only if $\mathcal{B} \models \theta$. We denote this relation by $\mathcal{U} \equiv_{\omega_1 E} \mathcal{B}$. A trivial but useful observation is

Proposition 1.17. *$\mathcal{U} \equiv_{\omega_1 E} \mathcal{B}$ if and only if for each closed term τ, $\tau^{\mathcal{U}} = \tau^{\mathcal{B}}$.*

It is clear from Definitions 1.2 and 1.4 that the set of $L_{\omega_1 E}$-formulas is uncountable, but a result of Keisler shows that for most arguments it is enough to consider a countable 'dense' set of $L_{\omega_1 E}$-terms. Here are some details.

Definition 1.18. A countable set D of terms is said to be *dense* in $L_{\omega_1 E}$ if for every term $\tau(\mathbf{x})$ of $L_{\omega_1 E}$ and every $\varepsilon > 0$ there is a term $\tau_0(\mathbf{x}) \in D$ such that $\|\tau - \tau_0\| < \varepsilon$.

A particular instance of the famous Stone–Weierstrass theorem which is useful to us is the following:

Theorem 1.19 (Stone–Weierstrass). *The set $\mathcal{D}$ of polynomials with rational coefficients is such that for each $n \in \mathbb{N}$, $[r, s]$, $F \in C(\mathbb{R}^n)$ and $\varepsilon > 0$ there is $F_0 \in \mathcal{D} \cap C(\mathbb{R}^n)$ such that*:

$$\sup\{|F(\mathbf{x}) - F_0(\mathbf{x})| : \mathbf{x} \in [r, s]^n\} < \varepsilon.$$

The next two results are applications of this theorem to $L_{\omega_1 E}$.

Theorem 1.20 (Keisler [17]). *If $\mathcal{D}$ is as above, then the set D of $L_{\omega_1 E}$-terms where the only function connectives allowed come from $\mathcal{D}$ is a countable dense set of terms in $L_{\omega_1 E}$.*

Corollary 1.21 (Keisler [17]). *Let $\mathcal{U}$ and $\mathcal{B}$ be $L_{\omega_1 E}$-models. If D is a dense set of terms such that for each closed term $\tau \in D$, $\tau^{\mathcal{U}} = \tau^{\mathcal{B}}$, then $\mathcal{U} \equiv_{\omega_1 E} \mathcal{B}$.*

2. Existence of hyperfinite models

Keisler has pointed out that hyperfinite models for $L_{\omega_1 P}$ and $L_{\omega_1 \int}$ can be considered as the Probability Logic notion analogous to that of saturated models for first-order logic. In this section we give definitions of hyperfinite models (internal and external) for $L_{\omega_1 E}$ and prove, see Theorems 2.11 and 2.12, that given any model $\mathcal{U}$ for $L_{\omega_1 E}$ we can find a (uniform) hyperfinite model which is elementarily equivalent to $\mathcal{U}$. The main tools that we use in our proofs come from basic nonstandard analysis; [7] and [22] are short introductions to this subject. We also make use of previous existence and uniqueness results of Keisler for $L_{\omega_1 \int}$-hyperfinite models. For detailed proofs of these theorems see [16] and [17].

From now on we use a fixed ω_1-saturated nonstandard universe. We also simplify notation by giving definitions and proofs for $L = \{X\}$ where X is a 2-argument random variable; of course, all results are valid for L as considered in Definition 1.1.

Definition 2.1. (a) A model $\mathcal{N} = \langle N, \hat{X}^{\mathcal{N}}, I, \hat{\mu} \rangle$ is a *$*$-finite model* for $L_{\omega_1 E}$, if:

(1) N is an internal set such that its internal cardinality $|N|$ is hyperfinite (i.e. $|N| \in {}^*\mathbb{N} - \mathbb{N}$).

(2) $\hat{X} : N \times N \to {}^*\mathbb{R}$ is an internal function.

(3) I is an $*$-algebra $\subseteq {}^*\mathcal{P}(N)$ = set of internal subsets N.

(4) $\hat{\mu}$ is the $*$-counting measure on N.

(b) $\mathcal{N}$ as above is said to be a *uniform $*$-finite model* for $L_{\omega_1 E}$ if:

(1) $N = H \times K$ with H, K internal such that $|H|$, $|K| \in {}^*\mathbb{N} - \mathbb{N}$.

(2) I is the $*$-algebra of internal subsets of N of the form $A \times K$ with $A \in {}^*\mathcal{P}(H)$.

(c) A model $\mathcal{M} = \langle N, X, \mathcal{F}, \mu \rangle$ is said to be a (*Uniform*) *Hyperfinite Model* for $L_{\omega_1 E}$ if there exists a (uniform) $*$-finite model $\mathcal{N}$ such that:

(1) $\hat{X}$ is a lifting of X. This means ${}^\circ\hat{X}(\omega, \omega') = X(\omega, \omega')$ a.s. (μ_2), where μ_2 is the $*$-counting measure on $N \times N$.

(2) μ is the Loeb measure obtained from $\hat{\mu}$.

(3) $\mathcal{F}$ is the σ-algebra $\sigma(I)$ generated by I.

Given this situation we say that $\mathcal{N}$ is a *lifting* of $\mathcal{M}$ and write ${}^\circ\mathcal{N} = \mathcal{M}$.

Remark. Observe that if in part (a) of the above definition we leave out the $*$-algebra I we obtain the so-called $*$-finite models for $L_{\omega_1 \int}$. $L_{\omega_1 \int}$-hyperfinite models are defined as in part (c) but leaving out (3).

It is important to notice that for a $*$-finite model $\mathcal{N}$ the random variable $\hat{X}^{\mathcal{N}}$ takes values in ${}^*\mathbb{R}$ and not necessarily in $\mathbb{R}$. We introduce the notion of $*$-interpretation for $*$-finite models which is analogous to the one given in Definition 1.8 for standard $L_{\omega_1 E}$-models.

Definition 2.2. Let $\mathcal{N} = \langle \hat{N}, X, I, \hat{\mu} \rangle$ be a $*$-finite model for $L_{\omega_1 E}$. A *$*$-interpretation* of $L_{\omega_1 E}$ in $\mathcal{N}$ is a function that assigns to each $L_{\omega_1 E}$-term τ an internal function $\hat{\tau}^{\mathcal{N}}$ satisfying the following conditions:

(1) If $\tau(\boldsymbol{u})$ is $[X(\boldsymbol{u}) \upharpoonright r]$, $r \in \mathbb{Q}^+$, then:

$$\hat{\tau}^{\mathcal{N}}(a) = \begin{cases} r & \text{if } \hat{X}^{\mathcal{N}}(\boldsymbol{a}) \geqslant r, \\ -r & \text{if } \hat{X}^{\mathcal{N}}(\boldsymbol{a}) \leqslant -r, \\ \hat{X}^{\mathcal{N}}(\boldsymbol{a}) & \text{if } -r \leqslant \hat{X}^{\mathcal{N}}(\boldsymbol{a}) \leqslant r. \end{cases}$$

(2) If $\tau(u, v)$ is $1(u = v)$, then

$$\hat{\tau}^{\mathcal{N}}(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

(3) If τ is $\bar{F}(\tau_1, \ldots, \tau_n)$, then

$$\hat{\tau}^{\mathcal{N}}(\boldsymbol{a}) = {}^*F(\hat{\tau}_1^{\mathcal{N}}(\boldsymbol{a}), \ldots, \hat{\tau}_n^{\mathcal{N}}(\boldsymbol{a})).$$

(4) If $\tau(\boldsymbol{u})$ is $\int \theta(\boldsymbol{u}, v)\, \mathrm{d}v$, then

$$\hat{\tau}^{\mathcal{N}}(\boldsymbol{a}) = \sum_{b \in N} \theta^{\mathcal{N}}(\boldsymbol{a}, b)\ 1/|N|.$$

(5) If $\tau(\boldsymbol{u}, x)$ is $E[\theta(\boldsymbol{u}, v) \mid v](x)$, then

$$\hat{\tau}^{\mathcal{N}}(\boldsymbol{a}, b) = E[\theta^{\mathcal{N}}(\boldsymbol{a}, \cdot) \mid I](b).$$

Observe that in this definition $\sum$ is internal summation and $\bar{E}[\cdot \mid I]$ is internal conditional expectation with respect to I.

The following lemma is an immediate consequence of the definitions of the nonstandard integral and nonstandard conditional expectation for hyperfinite Loeb spaces (see for example Section 3 of [7]).

Lemma 2.3. *For every $*$-finite model $\mathcal{N}$ there exists a $*$-interpretation of $L_{\omega_1 E}$ in $\mathcal{N}$.*

Lemma 2.4. *If $\mathcal{N}$ is a lifting of $\mathcal{M}$, then for all $L_{\omega_1 E}$-terms τ we have*

$$(\hat{\tau}^{\mathcal{N}}(\omega)) = \tau^{\mathcal{M}}(\omega) \ \text{a.s.}$$

Proof. Induction on the complexity of terms. For the integral quantifier step use Loeb's Lifting Theorem (Proposition 2, p. 118 in [23]) and for the conditional expectation operator use Theorem 12 (ii) from [2]. □

Before we start proving the main theorems of this section, let's take a closer look to the elementary equivalence relation for $L_{\omega_1 E}$-models introduced in Definition 1.16. This gives us an opportunity to present results about $\equiv_{\omega_1 E}$ that are important in their own right and to introduce some concepts that appear in later proofs.

Definition 2.5. Let $\mathcal{D}$ be as in the Stone–Weierstrass Theorem 1.19. Let $(\gamma_n)_{n \in \mathbb{N}}$

be an enumeration of the dense set D of $L_{\omega_1 E}$-terms obtained from $\mathscr{D}$ (Theorem 1.20).

(a) For each $\gamma_m(\boldsymbol{u})$ let $X_{\gamma_m}(\boldsymbol{u})$ be a new random variable symbol. Let $L^n = \{X, X_{\gamma_m} : m \leq n\}$ and $L^\infty = \{X, X_{\gamma_m} : m \in \mathbb{N}\}$.

(b) Let $\mathscr{U} = \langle A, X^{\mathscr{U}}, \mathscr{F}, \nu\rangle$ be an $L_{\omega_1 E}$-model.

(1) The $L^n_{\omega_1 \int}$-model associated to $\mathscr{U}$ is the model $\mathscr{U}_n = \langle A, X^{\mathscr{U}}, (\gamma^{\mathscr{U}}_m)_{m\leq n}, \nu\rangle$ where for each $m \leq n$ $\gamma^{\mathscr{U}}_m$ is the interpretation of $X_{\gamma_n}(\boldsymbol{u})$ in $\mathscr{U}_n$.

(2) The $L^\infty_{\omega_1 \int}$-model associated to $\mathscr{U}$ is the model $\mathscr{U}_\infty = \langle A, X^{\mathscr{U}}, (\gamma^{\mathscr{U}}_m)_{m\in\mathbb{N}}, \nu\rangle$ where for each $m \in \mathbb{N}$ $\gamma^{\mathscr{U}}_m$ is the interpretation of $X_{\gamma_n}(\boldsymbol{u})$ is $\mathscr{U}_\infty$.

(3) Let γ_m be a term from our list.

(i) If γ_m is of the form $E[\theta(\boldsymbol{u}, x) \mid x](v)$ with v E-bound in γ_m let $\mathscr{G}_{\gamma_m}$ be the σ-algebra on A generated by the sets of the form $\{d \in A : \gamma^{\mathscr{U}}_m(\boldsymbol{a}, d) \geq r, \boldsymbol{a} \in A, r \in \mathbb{Q}\}$.

(ii) If γ_m is not of this form let $\mathscr{G}_{\gamma_m} = \{\phi, A\}$.

(iii) For each $n \in \mathbb{N}$. Let $\mathscr{G}_n = \sigma\left(\bigcup_{m\leq n} \mathscr{G}_{\gamma_m}\right)$ and $\mathscr{G} = \sigma\left(\bigcup_{n\in\mathbb{N}} \mathscr{G}_n\right)$.

(4) (i) The $L_{\omega_1 E}$-model associated to $\mathscr{U}_n$ is the model $\mathscr{U}'_n = \langle A, X^{\mathscr{U}}, \mathscr{G}_n, \nu\rangle$.

(ii) The $L_{\omega_1 E}$-model associated to $\mathscr{U}_\infty$ is the model $\mathscr{U}'_\infty = \langle A, X^{\mathscr{U}}, \mathscr{G}, \nu\rangle$.

(c) If instead of $\mathscr{U}$ as in (b) we have an $*$-finite model $\mathscr{N} = \langle N, X^{\mathscr{N}}, I, \nu\rangle$ we can define corresponding internal notions as follows:

(1), (2): $\mathscr{N}_n$ and $\mathscr{N}_\infty$ are defined in the obvious way.

(3) (i) If γ_m is of the form $E[\theta(\boldsymbol{u}, x) \mid x](v)$ with v E-bound in γ_m. Let J_{γ_m} be the $*$-algebra on N generated by the sets of the form $\{d \in N : \hat{\gamma}^{\mathscr{N}}_m(\boldsymbol{a}, d) \geq r, \boldsymbol{a} \in N, r \in {}^*\mathbb{Q}\}$.

(ii) If γ_m is not of this form let $J_{\gamma_m} = \{\phi, N\}$.

(iii) For each $n \in \mathbb{N}$ let J_n be the $*$-algebra generated by $\bigcup_{m\leq n} J_{\gamma_m}$.

(4) The $L_{\omega_1 E}$-*model associated to* $\mathscr{N}_n$ is the model $\mathscr{N}'_n = \langle N, \hat{X}^{\mathscr{N}}, J_n, \hat{\mu}\rangle$.

Observe that we do not define an $L_{\omega_1 E}$-model associated to $\mathscr{N}_\infty$ because the algebra generated by countably many internal algebras is not necessarily an internal algebra.

Proposition 2.6. *Let* $\mathscr{U}$ *and* $\mathscr{B}$ *be* $L_{\omega_1 E}$-*models. The following are equivalent.*

(a) $\mathscr{U} \equiv_{\omega_1 E} \mathscr{B}$.

(b) $\mathscr{U}_\infty \equiv_{L_{\omega_1 \int}} \mathscr{B}_\infty$.

(c) *For each* $n \in \mathbb{N}$, $\mathscr{U}_n \equiv_{L^n_{\omega_1 \int}} \mathscr{B}_n$.

Proof. It follows easily from the definitions. $\square$

Proposition 2.7. *Let* $\mathscr{U}$ *and* $\mathscr{B}$ *be* $L_{\omega_1 E}$-*models. Then*

(a) $\mathscr{U} \equiv_{\omega_1 E} \mathscr{U}'_\infty$.

(b) $\mathscr{U} \equiv_{\omega_1 E} \mathscr{B}$ *if and only if* $\mathscr{U}'_\infty \equiv_{\omega_1 E} \mathscr{B}'_\infty$.

Proof. (a) It is similar to Keisler's proof (Section 4.2, [16]) of the completeness theorem for $L_{\omega_1 E}$, we just have to observe the following:

Given a term $\tau(\boldsymbol{u}, v)$ of the form $E[\theta(\boldsymbol{u}, v) \mid v](v)$, the σ-algebra $\mathcal{G}_\tau$ generated by the sets $\{d \in A : \tau^{\mathcal{U}}(\boldsymbol{a}, d) \geq r, r \in \mathbb{Q}, \boldsymbol{a} \in A\}$ is contained in $\mathcal{G}$. This is because the γ_n's form a dense set of $L_{\omega_1 E}$-terms, by Theorem 1.10 and Lemma 1.11 we have:

For each $m \in \mathbb{N}$ and $L_{\omega_1 E}$-term τ we can find γ_τ from our list such that for all $\boldsymbol{a}$, d in A

$$|\tau^{\mathcal{U}}(\boldsymbol{a}, d) - \gamma_\tau^{\mathcal{U}}(\boldsymbol{a}, d)| \leq \|\tau - \gamma_\tau\| < 1/m.$$

Therefore $\mathcal{G}_\tau \subseteq \sigma(\bigcup_{m \in \mathbb{N}} \mathcal{G}_{\gamma_m}) = \mathcal{G}$.

(b) follows immediately from (a). $\square$

Here is an interesting convergence result for $L_{\omega_1 E}$-models.

Proposition 2.8. *Let $\mathcal{U}$ be an $L_{\omega_1 E}$-model. The sequence $(\mathcal{U}'_n)_{n \in \mathbb{N}}$ converges almost surely to $\mathcal{U}'_\infty$. This means: For each $L_{\omega_1 E}$-term $\tau(\boldsymbol{u})$,*

$$\tau^{\mathcal{U}'_n}(\boldsymbol{\omega}) \xrightarrow[n \to \infty]{} \tau^{\mathcal{U}'_\infty}(\boldsymbol{\omega}) \quad a.s.$$

Proof. By induction on the complexity of τ.

(i) Trivial for atomic terms.

(ii) If τ is $\bar{F}(\tau_1, \ldots, \tau_n)$ use the continuity of F, Fubini's theorem and the hypothesis of induction.

(iii) If $\tau(\boldsymbol{u})$ is $\int \theta(\boldsymbol{u}, v)\, dv$ use the Dominated Convergence Theorem with bound $\|\theta\|$, Fubini's theorem and the hypothesis of induction.

(iv) If $\tau(\boldsymbol{u}, v)$ is $E[\theta(\boldsymbol{u}, v) \mid v](v)$ use Hunt's Convergence Theorem (for the statement and proof of this theorem see [9, p. 39]). We have to recall that the $\mathcal{G}_n$'s form an increasing sequence of σ-algebras on A and that $\|\theta\|$ is a uniform bound for each $\tau^{\mathcal{U}'_n}$. As before use Fubini's Theorem and the hypothesis of induction $\square$

Proposition 2.9. *Let $\mathcal{U}$ and $\mathcal{B}$ be $L_{\omega_1 E}$-models. If for each $n \in \mathbb{N}$, $\mathcal{U}'_n \equiv_{\omega_1 E} \mathcal{B}'_n$, then $\mathcal{U} \equiv_{\omega_1 E} \mathcal{B}$.*

Proof. By Proposition 2.8 for each closed $L_{\omega_1 E}$-term τ we have:

$$\tau^{\mathcal{U}'_n} \xrightarrow[n \to \infty]{} \tau^{\mathcal{U}'_\infty} \quad \text{and} \quad \tau^{\mathcal{B}'_n} \xrightarrow[n \to \infty]{} \tau^{\mathcal{B}'_\infty}. \tag{1}$$

By Proposition 1.17,

$$\text{for each } n \in \mathbb{N}: \quad \tau^{\mathcal{U}'_n} = \tau^{\mathcal{B}'_n}. \tag{2}$$

Therefore, from (1) and (2) we have $\tau^{\mathcal{U}'_\infty} = \tau^{\mathcal{B}'_\infty}$ and so by Proposition 2.7(b) we have $\tau^{\mathcal{U}} = \tau^{\mathcal{B}}$ and by Proposition 1.17, $\mathcal{U} \equiv_{\omega_1 E} \mathcal{B}$. $\square$

A partial converse to this Proposition is given by

Proposition 2.10. *If $\mathcal{M}=\langle M, X^{\mathcal{M}}, \mathcal{F}, \mu\rangle$ and $\mathcal{P}=\langle M, X^{\mathcal{P}}, \mathcal{F}, \mu\rangle$ are hyperfinite models with $\mathcal{M}\equiv_{\omega_1 E}\mathcal{P}$. Then for all $n\in\mathbb{N}$, $\mathcal{M}'_n\equiv_{\omega_1 E}\mathcal{P}'_n$.*

Proof. Let $n\in\mathbb{N}$ be fixed. By Proposition 2.6(c), $\mathcal{M}_n=\langle M, X^{\mathcal{M}}, (\gamma_m^{\mathcal{M}})_{m\leq n}, \mu\rangle$ and $\mathcal{P}_n=\langle M, X^{\mathcal{P}}, (\gamma_m^{\mathcal{P}})_{m\leq n}, \mu\rangle$ are $L^n_{\omega_1\int}$-hyperfinite models such that $\mathcal{M}_n\equiv_{L^n_{\omega_1\int}}\mathcal{P}_n$. By the uniqueness theorem for $L^n_{\omega_1\int}$-hyperfinite models, see [16], we know the following:

There exists an internal bijection $h\colon M\to M$ such that:

(i) $X^{\mathcal{M}}(h(\omega), h(\omega'))=X^{\mathcal{P}}(\omega,\omega')$ a.s., and

(ii) for all $m\leq n$, $\gamma_m^{\mathcal{M}}(h(\boldsymbol{\omega}))=\gamma_m^{\mathcal{P}}(\boldsymbol{\omega})$ a.s.

From (i) and (ii) and the form we defined the σ-algebras that appear in $\mathcal{M}'_n$ and $\mathcal{P}'_n$ we obtain an extra property:

(iii) For all $m\leq n$, $h(\mathcal{G}_m^{\mathcal{F}})=\mathcal{G}_m^{\mathcal{J}}$, where $\mathcal{G}_m^{\mathcal{F}}(\mathcal{G}_m^{\mathcal{J}})$ is the σ-algebra in $\mathcal{M}'_n(\mathcal{P}'_n)$.

Using properties (i)–(iii) it is easy to see that

$$\mathcal{M}'_n=\langle M, X^{\mathcal{M}}, \mathcal{G}_m^{\mathcal{F}}, \nu\rangle\equiv_{\omega_1 E}\mathcal{P}'_n=\langle M, X^{\mathcal{P}}, \mathcal{G}_m^{\mathcal{J}}, \nu\rangle. \qquad \square$$

We are now ready for the existence theorems of this section. We first prove the existence of hyperfinite models for $L_{\omega_1 E}$. The method of proof is suggested by Keisler's proof of the completeness theorem for $L_{\omega_1 E}$. The idea is to 'translate' problems about $L_{\omega_1 E}$-models into problems about integral models and use the tools already developed for logic with integral quantifier.

Theorem 2.11 (Existence of $L_{\omega_1 E}$-hyperfinite models). *Let $\mathcal{U}=\langle A, Y^{\mathcal{U}}, \mathcal{G}, \nu\rangle$ be an $L_{\omega_1 E}$-model and $P\in{}^*\mathbb{N}-\mathbb{N}$. Then there exists a hyperfinite model $\mathcal{M}=\langle N, X^{\mathcal{M}}, \mathcal{F}, \mu\rangle$ with $|N|=P$ and such that $\mathcal{U}\equiv_{\omega_1 E}\mathcal{M}$.*

Proof. From our fixed dense set of $L_{\omega_1 E}$-terms we pick those terms τ of the form $E[\theta(\boldsymbol{u}, x)\restriction x]\ (v)$ and let $(\tau_m)_{m\in\mathbb{N}}$ be an enumeration of them such that if τ is a subterm of θ then τ is listed before θ. For each such $\tau_m(\boldsymbol{u}, v)$ let $X_{\tau_m}(\boldsymbol{u}, v)$ be a new random variable symbol and let $K^n=\{X, X_{\tau_m} : m\leq n\}$ and $K^\infty=\bigcup_{n\in\mathbb{N}}K^n$. We can inductively define a correspondence between $L_{\omega_1 E}$-terms and $K^\infty_{\omega_1\int}$-terms as follows:

(a) If τ is $[X\restriction r]$, then the $K^\infty_{\omega_1\int}$-translation σ_τ of τ is $[X\restriction r]$.

(b) If τ is $1(u=v)$, then σ_τ is $1(u=v)$.

(c) If τ is $\bar{F}(\tau_1,\dots,\tau_n)$, then σ_τ is $\bar{F}(\sigma_{\tau_1},\dots,\sigma_{\tau_n})$.

(d) If τ is $\int\theta(\boldsymbol{u}, v)\,dv$, then σ_τ is $\int\sigma_\theta(\boldsymbol{u}, v)\,dv$.

(e) If τ is $E[\theta(\boldsymbol{u}, x)\mid x](v)$, then σ_τ is $X_\tau(\boldsymbol{u}, v)$.

This correspondence can be extended to the $L_{\omega_1 E}$-formulas in the obvious way. Clearly, if $\phi\in L_{\omega_1\int}$, then $\sigma_\phi=\phi$.

For each $n\in\mathbb{N}$ let $\mathcal{U}_n=\langle A, Y^{\mathcal{U}}, (X_{\tau_m}^{\mathcal{U}_n})_{m\leq n}, \nu\rangle$ be the $K^n_{\omega_1\int}$-model obtained from $\mathcal{U}$ by taking $X_{\tau_m}^{\mathcal{U}_n}=\tau_m^{\mathcal{U}}$. By the existence theorem of $K^n_{\omega_1\int}$-hyperfinite models, see

[16], we can find $\mathcal{M}_n = \langle N, X^{\mathcal{M}_n}, (X^{\mathcal{M}_n}_{\tau_m})_{m \leq n}, \mu\rangle$ hyperfinite such that:

$$\mathcal{U}_n \equiv_{K^n_{\omega_1 \int}} \mathcal{M}_n \quad \text{and} \quad |N| = P. \tag{1}$$

By the lifting theorem for $K^n_{\omega_1 \int}$-hyperfinite models we can find a $*$-finite $\mathcal{N}_n = \langle N, \hat{X}^{\mathcal{N}_n}, (\hat{X}^{\mathcal{N}_n}_{\tau_m})_{m \leq n}, \hat{\mu}\rangle$ such that $\mathcal{N}_n$ lifts $\mathcal{M}_n$, i.e.:

$$\begin{aligned} &\text{(a) } \hat{X}^{\mathcal{N}_n} \text{ lifts } X^{\mathcal{M}_n}, \\ &\text{(b) for all } m \leq n, \quad \hat{X}^{\mathcal{N}_n}_{\tau_m} \text{ lifts } X^{\mathcal{M}_n}_{\tau_m}. \end{aligned} \tag{2}$$

By the Robinson consistency theorem for the logic with integral quantifier (Keisler [16]) we can assume the $\mathcal{M}_n$'s and $\mathcal{N}_n$'s are increasing, i.e.,

$$\text{for all } n \in \mathbb{N}, \quad \mathcal{M}_{n+1} \restriction_{K^n_{\omega_1 \int}} = \mathcal{M}_n \quad \text{and} \quad \mathcal{N}_{n+1} \restriction_{K^n_{\omega_1 \int}} = \mathcal{N}_n. \tag{3}$$

Therefore for each $n \in \mathbb{N}$, $X^{\mathcal{M}_n} = X^{\mathcal{M}_{n+1}} = X$ and $\hat{X}^{\mathcal{N}_n} = \hat{X}^{\mathcal{N}_{n+1}} = \hat{X}$. For each $n \in \mathbb{N}$ let I_n be the $*$-algebra generated by the internal sets

$$\{d \in N : X^{\mathcal{N}_n}_{\tau_m}(c, d) \geq r, r \in {}^*\mathbb{Q}, m \leq n, c \in N\}.$$

Let $n \in \mathbb{N}$, then $\mathcal{U}_n$ satisfies the translations σ_τ of all those $L_{\omega_1 E}$-sentences τ such that σ_τ belongs to $K^n_{\omega_1 \int}$ and $\mathcal{U} \models \tau$. In particular, if γ and $E[\theta \mid x] \in (\tau_m)_{m \leq n}$, then

$$\mathcal{U}_n \models \int X_\gamma X_{E[\theta \mid x]} = \int X_\gamma \sigma_\theta.$$

By (1), $\mathcal{M}_n$ and $\mathcal{U}_n$ satisfy the same translations and by (2), $\mathcal{N}_n$ satisfies these translations with $\approx$ instead of $=$. By this we mean the following: If for two $K^n_{\omega_1 \int}$-terms τ and θ, $\tau^{\mathcal{M}_n} = \theta^{\mathcal{M}_n}$, then $\tau^{\mathcal{N}_n} \approx \theta^{\mathcal{N}_n}$. This is an immediate consequence of (2), since $\hat{\tau}^{\mathcal{N}_n} \approx \tau^{\mathcal{M}_n} = \theta^{\mathcal{M}_n} \approx \hat{\theta}^{\mathcal{N}_n}$. So, for example instead of (4) for $\mathcal{U}_n$, in $\mathcal{N}_n$ we have:

$$\sum \hat{X}^{\mathcal{N}_n} \hat{X}^{\mathcal{N}_n}_{E[\theta \mid x]} \approx \hat{X}^{\mathcal{N}_n}_\gamma \hat{\sigma}^{\mathcal{N}_n}_\theta. \tag{5}$$

Using ω_1-saturation we can find $H \in {}^*\mathbb{N} - \mathbb{N}$ such that $\mathcal{N}_H = \langle N, \hat{X}, (\hat{X}^{\mathcal{N}_H}_{\tau_m})_{m \leq H}, \hat{\mu}\rangle$ satisfies the translations σ_τ of all the ${}^*L_{\omega_1 E}$-axioms τ such that $\sigma_\tau \in K^H_{\omega_1 \int}$ with $\approx$ instead of $=$. For this H we have:

Claim. *The hyperfinite model $\mathcal{M} = \langle N, X, \sigma(I_H), \mu\rangle$ is $L_{\omega_1 E}$-elementarily equivalent to $\mathcal{U}$.*

Proof. Let $\mathcal{N}'_H = \langle N, \hat{X}, I_H, \hat{\mu}\rangle$. We first prove that the function that assigns to each $L_{\omega_1 E}$-term τ the function $\hat{\sigma}^{\mathcal{N}_H}_\tau$ is almost surely infinitely close to $\hat{\tau}^{\mathcal{N}_H}$, the $*$-interpretation of $L_{\omega_1 E}$ in $\mathcal{N}'_H$. We do this by induction on the complexity of τ.

(a) If τ is $[X \restriction r]$ or $1(u = v)$, it is easy.

(b) If τ is $\bar{F}(\tau_1, \ldots, \tau_n)$ we show $\hat{\sigma}^{\mathcal{N}_H}_\tau \approx {}^*F(\hat{\tau}_1^{\mathcal{N}'_H}, \ldots, \hat{\tau}_n^{\mathcal{N}'_H})$ a.s.

$$\hat{\sigma}^{\mathcal{N}_H}_\tau = (\bar{F}(\hat{\sigma}_{\tau_1}, \ldots, \hat{\sigma}_{\tau_n}))^{\mathcal{N}_H} \approx {}^*F(\hat{\sigma}^{\mathcal{N}_H}_{\tau_1}, \ldots, \hat{\sigma}^{\mathcal{N}_H}_{\tau_n}) \approx {}^*F(\hat{\tau}_1^{\mathcal{N}'_H}, \ldots, \hat{\tau}_n^{\mathcal{N}'_H}) \quad \text{a.s.}$$

(c) If τ is $\int \theta(\boldsymbol{u}, v)\,\mathrm{d}v$ we show $\hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}) \approx \sum_{b\in N} \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b)(1/P)$ a.s.

$$\hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}) = \left(\int \hat{\sigma}_\theta\right)^{\mathcal{N}_H} \approx \sum_{b\in N} \hat{\sigma}_\theta^{\mathcal{N}_H}(\boldsymbol{a}, b)\frac{1}{P} \approx \sum_{b\in N} \hat{\theta}_\theta^{\mathcal{N}'_H}(\boldsymbol{a}, b)\frac{1}{P} \text{ a.s.}$$

(d) If $\tau(\boldsymbol{u}, v)$ is $E[\theta(\boldsymbol{u}, x) \mid x](v)$, we want to show that

$$(\hat{\sigma}_\tau(\boldsymbol{a}, b))^{\mathcal{N}_H} = X_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) \approx \bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](b) \text{ a.s.}$$

Suppose this is false, i.e., $\{(\boldsymbol{a}, b) : \hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) \not\approx \bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](b)\}$ has positive measure. Then by Keisler's Fubini Theorem (Section 1 of [18]) we can find $n \in \mathbb{N}$ such that the internal set

$$U' = \{b : \hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) - \bar{E}[\hat{\theta}^{\mathcal{N}_H}(\boldsymbol{a}, \cdot) \mid I_H](b) \geq 1/n \quad \text{or}$$
$$\bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](b) - \hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) \geq 1/n\}$$

has positive measure for a set of $\boldsymbol{a}$'s of positive measure. Without loss of generality we can assume:

$$U = \{b : \hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) - \bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](b) \geq 1/n\} \text{ has positive measure for a set of } \boldsymbol{a}\text{'s of positive measure.} \tag{6}$$

Observe that $U \in I_H$ since both $\hat{\sigma}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, \cdot)$ and $\bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](\cdot)$ are I_H-measurable.

Now we prove that for a set of $\boldsymbol{a}$'s of positive measure

$$^o\left(\int_U (\hat{X}_\tau^{\mathcal{N}_H}(\boldsymbol{a}, b) - \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b))\,\mathrm{d}b\right) > 0. \tag{7}$$

This is clearly a contradiction to the translation of the conditional expectation axiom in $\mathcal{N}'_H$ (i.e. (5) does not hold in $\mathcal{N}'_H$).

$$\int_U (\hat{X}^{\mathcal{N}'_H}(\boldsymbol{a}, b) - \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b))\,\mathrm{d}b = \sum_{b\in U} (\hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b) - \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b))\frac{1}{P}.$$

Since $U \in I_H$ we can find K, $[b_s]_H$ such that:

(i) $U = \bigcup_{s\leq K} [b_s]_H$, $K \in {}^*\mathbb{N} - \mathbb{N}$.

(ii) For each s, $[b_s]_H$ is an I_H-atom.

(iii) If $s \neq s'$, then $[b_s]_H \cap [b_{s'}]_H = \emptyset$.

Then

$$\sum_{b\in U} (\hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b) - \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b))\frac{1}{P} = \sum_{s\leq K}\left(\sum_{b\in[b_s]_H} \hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b) - \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b)\right)\frac{1}{P}$$
$$= \sum_{s\leq K}\left(\sum_{b\in[b_s]_H} \hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b)\frac{1}{P} - \sum_{b\in[b_s]_H} \hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, b)\frac{1}{P}\right)$$
$$= \sum_{s\leq K}\left(\sum_{b\in[b_s]_H} \hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b)\frac{1}{P} - \bar{E}[\hat{\theta}^{\mathcal{N}'_H}(\boldsymbol{a}, \cdot) \mid I_H](b_s)\frac{|[b_s]_H|}{P}\right)$$

(by definition of $\bar{E}[\,\mid I_H]$)

$$= \sum_{s\le K}\left(\frac{|[b_s]_H|}{P}\hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b_s) - \bar{E}[\hat{\theta}^{\mathcal{N}_H}(\boldsymbol{a},\cdot) \mid I_H](b_s)\frac{|[b_s]_H|}{P}\right)$$

(because $\hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a},\cdot)$ is constant in $[b_s]_H$).

$$= \sum_{s\le K}(\hat{X}_\tau^{\mathcal{N}'_H}(\boldsymbol{a}, b_s) - \bar{E}[\theta^{\mathcal{N}'_H}(\boldsymbol{a},\cdot) \mid I_H](b_s))\frac{|[b_s]_H|}{P}$$

$$\geq \sum_{s\le K}\frac{1}{n}\cdot\frac{|[b_s]_H|}{P} = \frac{1}{nP}\sum_{s\le K}|[b_s]_H| = \frac{1}{nP}|U|$$

and by (6), $^\circ(|U|/P)>0$, so $(1/nP)|U|>0$. This proves (7) and we obtain the contradiction.

To complete the proofs of the claim and the theorem observe the following:

Given a closed $L_{\omega_1\mathrm{E}}$-term τ there is some $n\in\mathbb{N}$ such that its translation σ_τ is in $K^n_{\omega_1\mathcal{J}}$. Then $\tau^{\mathcal{U}} = \sigma_\tau^{\mathcal{U}_n}$ and by (1) we have:

$$\sigma_\tau^{\mathcal{U}_n} = \sigma_\tau^{\mathcal{M}_n}. \tag{8}$$

The lifting property (2) implies $\sigma_\tau^{\mathcal{M}_n} \approx \sigma_\tau^{\mathcal{N}_n}$, by the definition of the $*$-interpretation for $\mathcal{N}'_H$, $\hat{\sigma}^{\mathcal{N}_n} \approx \hat{\tau}^{\mathcal{N}_H}$ and by the definition of $^\circ\mathcal{N}'_H$ we have $\tau^{\mathcal{M}} = {}^\circ(\hat{\tau}^{\mathcal{N}'_H}) = \hat{\sigma}_\tau^{\mathcal{N}_n}$. Then from (8) we can conclude that:

$\tau^{\mathcal{U}} = \tau^{\mathcal{M}}$ and since this is true for each $L_{\omega_1\mathrm{E}}$-closed term τ, $\mathcal{U} \equiv_{\omega_1\mathrm{E}} \mathcal{M}$ as we wanted to show. □

Given an $L_{\omega_1\mathrm{E}}$-model $\mathcal{U}$ we want to find a uniform hyperfinite model $\mathcal{N}$ elementarily equivalent to it. We first use the previous theorem to get a hyperfinite model $\mathcal{M}'$ elementarily equivalent to $\mathcal{U}$ and then use Theorem 3.15 from Rodenhausen's thesis [25] conveniently changed to fit our setting in order to get $\mathcal{M}$ uniform hyperfinite elementarily equivalent to $\mathcal{M}'$. The idea is simple but the details are rather technical.

Theorem 2.12 (Existence of uniform hyperfinite models for $L_{\omega_1\mathrm{E}}$). *Let $\mathcal{U}$ be an $L_{\omega_1\mathrm{E}}$-model and $P, Q \in {}^*\mathbb{N}-\mathbb{N}$. Then we can find a uniform hyperfinite model $\mathcal{M}$ with universe $H\times K$ such that $|H|=P$, $|K|=Q$ and $\mathcal{U}\equiv_{\omega_1\mathrm{E}}\mathcal{M}$.*

Proof. Let $|N|\in{}^*\mathbb{N}-\mathbb{N}$ be such that $|N|<P$ and $|N|/Q\approx 0$. As in the proof of Theorem 2.11 we can find a $*$-finite model $\mathcal{N}' = \langle N, \hat{Y}, J, \hat{\mu}_N\rangle$ such that $^\circ\mathcal{N}' \equiv_{\omega_1\mathrm{E}} \mathcal{U}$. ($\hat{\mu}_N$ is the $*$-counting measure on N.)

Now we modify the proof of Theorem 3.15 in [25] to get:

(2) There exists X such that the uniform $*$-finite model $\mathcal{N} = \langle H\times K, \hat{X}, I, \hat{\mu}_{HK}\rangle$ satisfies $^\circ\mathcal{N} \equiv_{\omega_1\mathrm{E}} {}^\circ\mathcal{N}'$.

To prove this let $\{A_s : s \le B\}$ with $B\in{}^*\mathbb{N}$ be an $*$-enumeration of the J-atoms. These atoms are the equivalence classes of the internal equivalence relation $\sim_I$ defined on N as follows: Given $n, n'\in N$, $n\sim_I n'$ iff for all $A\in J$, $n\in A$ iff $n'\in A$.

Recall that I is the $*$-algebra of internal subsets of $H\times K$ of the form $C\times K$ with $C\in {}^*\mathcal{P}(H)$. The I-atoms are of the form $\{h\}\times K$ with $h\in K$, there are P of them and each one has internal cardinality Q.

For each $s\leqslant B$ let $a_s\in {}^*[0,1]$ be defined by

$$a_s=\hat{\mu}_N(A_s)=|A_s|/N. \tag{3}$$

Let $\{E_s:s\leqslant B\}$ be an internal partition of I such that for all $s\leqslant B$

$$\mu_{\mathrm{HK}}(E_s)=\frac{|E_s|}{PQ}\approx a_s,\quad \text{so } |E_s|\approx|A_s|\frac{PQ}{|N|}. \tag{4}$$

Therefore for each $s\leqslant P$, $E_s=\bigcup_{i=1}E_s^i$ where for each i, E_s^i is an I-atom and $|E_s^i|=Q$. For each $s\leqslant B$ we can find an internal surjection $h_s:E_s\to A_s$ such that:

$$\begin{aligned}
&\text{(i) for all } a,\ a'\in A_s,\quad \hat{\mu}_k(h_s^{-1}(a))\approx\hat{\mu}_k(h_s^{-1}(a')),\\
&\text{(ii) for all } i,\ j,\quad |h_s^{-1}(a)\cap E_s^i|=|h_s^{-1}(a)\cap E_s^j|,\\
&\text{(iii) for all } a,\ a'\in A_s,\quad h_s^{-1}(a)\cap h_s^{-1}(a')=\emptyset.
\end{aligned} \tag{5}$$

Let's give an idea of how to find h_s: Given $a\in A_s$ and i we can find $aE_s^i\subseteq E_s^i$ such that

$$\left|\frac{|aE_s^i|}{Q}-\frac{1}{|A_s|}\right|<\frac{1}{Q} \tag{6}$$

and if $a\neq a'$, then $aE_s^i\cap a'E_s^i=\emptyset$.

This is possible because $|A_s|\leqslant|N|$ and $|N|/Q\approx 0$. Then we can define h_s on aE_s^i as follows: If $x\in aE_s^i$ let $h_s(x)=a$.

Now we define h: $H\times K\to N$ by $h=\bigcup_{s\leqslant B}h_s$ and $\hat{X}:(H\times K)\times(H\times K)\to {}^*\mathbb{R}$ by $\hat{X}(\omega,\omega')=\hat{Y}(h(\omega),h(\omega'))$ and claim:

$$\text{for every } L_{\omega_1\mathrm{E}}\text{-term},\quad \tau(\boldsymbol{u}),[\tau(\boldsymbol{u})]^{\mathcal{N}}(\boldsymbol{a})\approx[\tau(\boldsymbol{u})]^{\mathcal{N}'}(h(\boldsymbol{a})). \tag{7}$$

We prove this statement by induction on the complexity of τ. The only non-trivial steps are those that involve integrals and conditional expectation. We give the proof of the integral case. The other case is similar and we leave it to the reader. If $\tau(\boldsymbol{u})$ is $\int\theta(\boldsymbol{u},v)\,\mathrm{d}v$ we want to show

$$\tau^{\mathcal{N}}(\boldsymbol{a})=\sum_{b\in H\times K}\theta^{\mathcal{N}}(\boldsymbol{a},b)\frac{1}{PQ}\approx\sum_{b'\in N}\theta^{\mathcal{N}'}(h(\boldsymbol{a}),b')\frac{1}{|N|}=\tau^{\mathcal{N}'}(h(\boldsymbol{a})). \tag{8}$$

By induction hypothesis $\hat{\theta}^{\mathcal{N}}(\bar{a},b)\approx\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),h(b))$ for all $b\in H\times K$. Taking the maximum difference we can find $R\in{}^*\mathbb{N}-\mathbb{N}$ such that

$$\text{for all } b\in H\times K,\quad |\hat{\theta}^{\mathcal{N}}(\boldsymbol{a},b)-\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),h(b))|<\frac{1}{R}. \tag{9}$$

Then

$$\hat{\tau}^{\mathcal{N}}(\boldsymbol{a})=\sum_{b\in H\times K}\hat{\theta}^{\mathcal{N}}(\boldsymbol{a},b)\frac{1}{PQ}\approx\sum_{b\in H\times K}\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),h(b))\frac{1}{PQ}, \tag{10}$$

so

$$|\hat{\tau}^{\mathcal{N}}(\boldsymbol{a})-\hat{\tau}^{\mathcal{N}'}(h(\boldsymbol{a}))|=\left|\sum_{b\in H\times K}\hat{\theta}^{\mathcal{N}}(\boldsymbol{a},b)\frac{1}{PQ}-\sum_{b'\in N}\hat{\theta}^{\mathcal{N}}(h(\boldsymbol{a}),b')\frac{1}{|N|}\right|$$

$$\approx\left|\sum_{b\in H\times K}\theta^{\mathcal{N}'}(h(\boldsymbol{a}),h(b))\frac{1}{PQ}-\sum_{b'\in N}\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),b')\frac{1}{|N|}\right| \quad \text{by (10)},$$

$$\approx\left|\sum_{b'\in N}\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),b')+\frac{|A_s|}{|N|Q}-\sum_{b'\in N}\hat{\theta}^{\mathcal{N}'}(h(\boldsymbol{a}),b')\frac{1}{|N|}\right|.$$

This is because for each $b'\in A_s$ there are $\approx P(Q+|A_s|)/|N|$ elements in $H\times K$ mapped onto b'. The last term is

$$=\left|\sum_{b'\in N}\theta^{\mathcal{N}'}(h(\boldsymbol{a}),b')\left(\frac{1}{|N|}+\frac{|A_s|}{|N|Q}-\frac{1}{|N|}\right)\right|\leqslant|N|\sup\hat{\theta}^{\mathcal{N}'}\frac{|A_s|}{|N|}$$

$$=\frac{\sup\hat{\theta}^{\mathcal{N}'}|A_s|}{Q}\approx 0 \qquad \text{since } \frac{|A_s|}{Q}\leqslant\frac{|N|}{Q}\approx 0 \quad \text{and} \quad \hat{\theta}^{\mathcal{N}} \text{ is bounded.}$$

This proves (8) and completes the proof of (7).

(2) follows directly from (7) and if we put together (1) and (2) we have: $\mathcal{M}={}^o\mathcal{N}\equiv_{\omega_1 E}{}^o\mathcal{N}'\equiv_{\omega_1 E}\mathcal{U}$ as we wanted to show. □

In Definition 1.14 we pointed out some possible ways of generalizing $L_{\omega_1 E}$ in order to allow more than one conditional operator in the language. There are natural ways of extending the definitions and theorems of this section to cover those generalizations. We now give indications on how to handle the case involving two conditional expectation operators E_1 and E_2 and leave to the interested reader the details of the other cases.

Definition 2.13 (Generalizations). Let $L_{\omega_1 E}$ contain the conditional expectation symbols E_1 and E_2.

(a) $*$-*finite* models are of the form $\mathcal{N}=\langle N,\hat{X},I,J,\hat{\mu}\rangle$ with $I\subseteq J$ $*$-algebras contained in ${}^*\mathcal{P}(N)$ and everything else as in Definition 2.1(a).

(b) *Uniform* $*$-*finite* models are $*$-finite models such that:

(1) $N=H\times K\times L$ with $|H|$, $|K|$, $|L|\in{}^*\mathbb{N}-\mathbb{N}$.

(2) I is the $*$-algebra of internal subsets of N of the form $A\times K\times L$ with $A\in{}^*\mathcal{P}(H)$.

(3) J is the $*$-algebra of internal subsets of N of the form $A\times B\times L$ with $A\in{}^*\mathcal{P}(H)$ and $B\in{}^*\mathcal{P}(K)$.

(c) (*Uniform*) *Hyperfinite* models are defined in the obvious way.

We can sum things up with

Theorem 2.14. *The corresponding versions of Theorems* 11 *and* 12 *of this section hold for the* E_1, E_2 *case.*

Proof. The only place where the generalization may not be immediate is in the definition of the function h in the proof of Theorem 12. For this observe that if $\mathcal{N}$ is a uniform $*$-finite model each $I^{\mathcal{N}}$ splits into $|K|$ many $J^{\mathcal{N}}$-atoms. The idea is to define first $h_0 : I^{\mathcal{N}}$-atoms $\to I^{\mathcal{N}'}$-atoms and then define h_s as before from $J^{\mathcal{N}}$-atoms onto $J^{\mathcal{N}'}$-atoms in such a way that h_s preserves h_0. Again the details are technical. See Rodenhausen [25] for a more complete description of this construction in a slightly different setting.

3. Uniqueness of uniform hyperfinite models

Pursuing the analogy with first order logic, once we have proved the existence of hyperfinite models ('saturated') we go on to prove their uniqueness. This in the sense that if two of them with the same hyperfinite cardinality are elementarily equivalent then they are a.s. isomorphic. In this context a.s. isomorphic means that there exists a probabilistic a.s. isomorphism as in Definition 3.7.

In this section we are going to be dealing with certain definable sets and their probabilities; as in the case of $L_{\omega_1\int}$ and $L_{\omega_1 P}$, it is sometimes more convenient to have at hand both quantifiers $\int dx$ and $(Px \geqslant r)$. We know that there was no problem in doing so by Theorem 5.7 in [17]. We can naturally extend this idea to $L_{\omega_1 E}$ as the following definition and theorem indicate.

Definition 3.1. $L_{\omega_1 EP}$ is the logic obtained from $L_{\omega_1 E}$ by allowing the quantifiers $(Px \geqslant r)$, formally:

(i) The atomic formulas of $L_{\omega_1 EP}$ are the expressions of the form $[\tau(u) \geqslant 0]$ where $\tau(u)$ is an $L_{\omega_1 E}$-term.

(ii) The set of $L_{\omega_1 EP}$-formulas is the least set containing the $L_{\omega_1 EP}$-atomic formulas and closed under $\neg$, $\bigwedge$, $(Px \geqslant r)$.

(iii) $L_{\omega_1 EP}$-models and interpretations are defined in the obvious way.

In this paper we are not interested in studying $L_{\omega_1 EP}$ by itself. We just want to use the following theorem in order to give a natural proof of the main theorem of this section.

Theorem 3.2. *$L_{\omega_1 EP}$ is a conservative definitional extension of $L_{\omega_1 E}$, that is:*

(a) *For any sentence θ in $L_{\omega_1 E}$:*

$$L_{\omega_1 EP} \models \theta \quad \text{if and only if} \quad L_{\omega_1 E} \models \theta.$$

(b) *For any $L_{\omega_1 EP}$-formula $\phi(\boldsymbol{u})$ there is an $L_{\omega_1 E}$-formula $\gamma(\boldsymbol{u})$ such that*

$$L_{\omega_1 EP} \models \phi(\boldsymbol{u}) \leftrightarrow \gamma(\boldsymbol{u}).$$

Proof. It follows from Keisler's quantifier elimination theorem (Theorem 5.7 in

[17]) that shows how to eliminate the probability quantifiers $(P\mathbf{x} \geq r)$ by means of the integral quantifier. $\square$

Lemma 3.3. *Let $\tau(\mathbf{u}, v)$ by an $L_{\omega_1 E}$-term. If v is E-bound in τ, then*

$$\vdash_{L_{\omega_1 E}} E[\tau(\mathbf{u}, v) \mid v] = \tau(\mathbf{u}, v).$$

Proof. Intuitively the lemma says: "Once we have conditioned one variable, we get nothing new by conditioning the same variable later, as long as it remains E-bound." The examples following Definition 1.5 illustrate the situation. It is an easy consequence of the definition of conditional expectation in probability that if v is E-bound in τ, then $E[\tau(\mathbf{u}, v) \mid v] = \tau(\mathbf{u}, v)$ is true in all $L_{\omega_1 E}$-models. Therefore, by the completeness theorem for $L_{\omega_1 E}$, $\vdash_{L_{\omega_1 E}} E[\tau(\mathbf{u}, v) \mid v] = \tau(\mathbf{u}, v)$. $\square$

Lemma 3.4. *Let $\tau(\mathbf{u}, v)$ be an $L_{\omega_1 E}$-term with v E-bound in τ. If $\mathcal{U} = \langle A, X, \mathcal{F}, \nu \rangle$ is an $L_{\omega_1 E}$-model and $S \subseteq \mathbb{R}$ is a Borel set, then $\{(\mathbf{a}, b) \in \mathbf{A} \times A\colon \tau(\mathbf{a}, b) \in S\}$ is $\nu^{(n)} \times \mathcal{F}$-measurable.*

Proof. It is sufficient to prove the lemma for S of the form (c, ∞) with $c \in \mathbb{Q}$. By Lemma 3.3, $\vdash_{L_{\omega_1 E}} E[\tau(\mathbf{u}, v) \mid v] = \tau(\mathbf{u}, v)$, then for $r \in \mathbb{R}$:

$$\tau^{\mathcal{U}}(\mathbf{a}, b) \geq r \quad \text{if and only if} \quad E[\tau^{\mathcal{U}}(\mathbf{a}, \cdot) \mid \mathcal{F}](b) \geq r,$$

so

$$\{(\mathbf{a}, b) \in \mathbf{A} \times A : \tau^{\mathcal{U}}(\mathbf{a}, b) \geq c\} = \{(\mathbf{a}, b) \in \mathbf{A} \times A : E[\tau^{\mathcal{U}}(\mathbf{a}, \cdot) \mid \mathcal{F}](b) \geq c\}$$

and this last set is $\nu^{(n)} \times \mathcal{F}$-measurable by Definition 1.7(b). $\square$

A natural question is: What happens if in Lemma 3.4 we consider more than one E-bound variable in τ at a time? Let's take a look at a simple case, say $\tau(u, v)$ is $E[E[X(u, v) \mid u]v]$ both u and v are E-bound in τ, is τ $\mathcal{F} \times \mathcal{F}$-measurable? We do not know. Applying the lemma on each component of τ, we can see that τ is $\nu \times \mathcal{F}$-measurable and $\mathcal{F} \times \nu$-measurable but in general $\mathcal{F} \times \mathcal{F}$ is strictly contained in $\nu \times \mathcal{F} \cap \mathcal{F} \times \nu$. For example, if singletons do not belong to $\mathcal{F}$ we can have $\{a\} \times A \in \nu \times \mathcal{F}$ and $A \times \{a\} \notin \mathcal{F} \times \nu$ and $\{(a, a)\} \in \mathcal{F} \times \mathcal{F}$. The lesson is: "$E[E[X \mid \mathcal{F}] \mid \mathcal{F}]\ (\cdot, \cdot)$ is not necessarily equal to $E[X \mid \mathcal{F} \times \mathcal{F}](\cdot, \cdot)$."

Theorem 3.5 (Rectangle approximation for $L_{\omega_1 E}$). *Let $\tau(\mathbf{u}, v)$, $\mathcal{U}$ and S be as in Lemma 3.4. Then for all $m \in \mathbb{N}$ there exists a finite union of rectangles $\bigcup_{i=1}^{r_m} A_{1i} \times A_{2i} \times \cdots \times A_{ni} \times B_i$ such that for all $i = 1, \ldots, r_m$, $j = 1, \ldots, n$, A_{ji} is ν-measurable, B_i is $\mathcal{F}$-measurable and*

$$\nu^{(n+1)}\left((\tau)^{-1}(S) \triangle \bigcup_{i=1}^{r_m} A_{1i} \times \cdots \times A_{ni} \times B_i\right) < \frac{1}{m},$$

where

$$(\tau^{\mathcal{U}})^{-1}(S) = \{(\mathbf{a}, b) \in A^n \times A : \tau^{\mathcal{U}}(\mathbf{a}, b) \in S\}.$$

Proof. By Lemma 3.4 we know that $(\tau^{\mathcal{U}})^{-1}(S)$ is $\nu^{(n)}\times\mathcal{F}$-measurable. Then for real $\delta>0$ we can find rectangles $A_i'\times B_i$, $i=1,\ldots,m_\delta$, such that

$$\nu^{(n+1)}\left(\bigcup_{i=1}^{m_\delta} A_i'\times B_i\triangle(\tau^{\mathcal{U}})^{-1}(S)\right)<\delta$$

with A_i' $\nu^{(n)}$-measurable and $B_i\in\mathcal{F}$.

For each i, by Proposition 1.7.2 in [16], there exist A_i ν^n-measurable such that $\nu^{(n)}(A_i'\triangle A_i)=0$. Therefore in (1) we can replace each A_i' by A_i to get:

$$\nu^{(n+1)}\left(\bigcup_{i=1}^{m_\delta} A_i\times B_i\triangle(\tau^{\mathcal{U}})^{-1}(S)\right)<\delta$$

with A_i ν^n-measurable and $B_i\in\mathcal{F}$.

Now, for each ν^n-measurable A_i and each real $\varepsilon>0$ we can find rectangles $A_{ij}^i\times\cdots\times A_{nj}^i$ with A_{kj}^i ν-measurable ($k=1,\ldots,n$; $j=1,\ldots,r_i^\varepsilon$) such that:

$$\nu^n\left(\bigcup_{j=1}^{r_i^\varepsilon} A_{ij}^i\times\cdots\times A_{nj}^i\triangle A_i\right)<\varepsilon.$$

We can then consider approximations to $(\tau^{\mathcal{U}})^{-1}(S)$ of the form

$$\nu^{(n+1)}\left(\bigcup_{i=1}^{m_\delta}\left(\bigcup_{j=1}^{r_i^\varepsilon} A_{ij}^i\times\cdots\times A_{nj}^i\right)\times B_i\triangle(\tau^{\mathcal{U}})^{-1}(S)\right)$$

and if we take δ and ε sufficiently small we can make sure that (4) is less than $1/m$. $\square$

Theorem 3.6. *Let $\mathcal{U}=\langle A,X,\mathcal{F},\nu\rangle$ and $\mathcal{B}=\langle A,Y,\mathcal{F},\nu\rangle$ be L_{ω_1E}-models, $\tau(\boldsymbol{u},v)$ an L_{ω_1E}-term, $S\subseteq\mathbb{R}$ a Borel set and $n\in\mathbb{N}$. If $\mathcal{U}\equiv_{\omega_1E}\mathcal{B}$, then we can find rectangle approximations to $(\tau^{\mathcal{U}})^{-1}(S)$ and $(\tau^{\mathcal{B}})^{-1}(S)$ as given by the previous theorem, say*

$$\nu^{(n+1)}\left((\tau^{\mathcal{U}})^{-1}(S)\triangle\bigcup_{i=1}^{r_m} A_{1i}\times\cdots\times A_{ni}\times B_i\right)<\frac{1}{m}$$

and

$$\nu^{(n+1)}\left((\tau^{\mathcal{B}})^{-1}(S)\triangle\bigcup_{i=1}^{r_m} C_{1i}\times\cdots\times C_{ni}\times D_i\right)<\frac{1}{m}$$

such that:

(1) $r_m=r_m'$.
(2) *For all i, j, $\nu(A_{ji})=\nu(C_{ji})$ and $\nu(B_i)=\nu(D_i)$.*

Proof. As usual it is sufficient to prove the theorem for Borel sets S of the form (c,∞) with $c\in\mathbb{Q}$. It is for this theorem that it is more convenient to work with L_{ω_1EP} than with L_{ω_1E}. For each $c\in\mathbb{Q}$, $[\tau(\boldsymbol{u},v)-c\geqslant 0]$ is an atomic formula and for each $r\in[0,1]$, $(P\boldsymbol{u}>r)[\tau(\boldsymbol{u})\geqslant c]$ is an L_{ω_1EP}-sentence. By Theorem 3.2 we have for each $r \in [0, 1]$ and $L_{\omega_1 E}$-term $\tau(\boldsymbol{u})$:

$$\mathcal{U} \models (P\boldsymbol{u} > r)[\tau(\boldsymbol{u}) \geq c] \quad \text{if and only if} \quad \mathcal{B} \models (P\boldsymbol{u} > r)[\tau(u) \geq c].$$

Therefore $\nu^{(n+1)}((\tau^{\mathcal{U}})^{-1}(S)) = \nu^{(n+1)}((\tau^{\mathcal{B}})^{-1}(S))$. Using $L_{\omega_1 EP}$ sentences we ensure that the projections of $(\tau^{\mathcal{U}})^{-1}(S)$ and $(\tau^{\mathcal{B}})^{-1}(S)$ have the same probabilities. For example:

(i) $\nu(\{a \in A : \tau^{\mathcal{U}}(a, \ldots, a) \geq c\}) = \nu(\{a : \tau^{\mathcal{B}}(a, \ldots, a) \geq c\})$ (i.e. the diagonals have the same probability). To see this we just consider the sentences

$$\theta = (Pu > r)[\tau(u, \ldots, u) \geq c] \quad \text{with } r \in [0, 1].$$

Again we have $\mathcal{U} \models \theta$ iff $\mathcal{B} \models \theta$ and this implies what we want.

(ii) Consider the sentence

$$(Pu_i > r)(Pu_1 \cdots u_{i-1}u_{i+1} \cdots u_n > 0)[\tau(u) \geq c]$$

This type of sentence allows us to show that the i-th projections of $(\tau^{\mathcal{U}})^{-1}(S)$ and $(\tau^{\mathcal{B}})^{-1}(S)$ have the same probabilities.

Using conjunctions and negations of sentences like this one we can make sure that corresponding Boolean combinations of the projections have the same probabilities. The theorem follows from these considerations. □

Definition 3.7. Let $\mathcal{M} = \langle H \times K, X, \sigma(I), \mu\rangle$ and $\mathcal{M}' = \langle H \times K, Y, \sigma(I), \mu\rangle$ be uniform hyperfinite models. $\mathcal{M}$ and $\mathcal{M}'$ are said to be *a.s. isomorphic* if there exists an internal bijection h: $H \times K \to H \times K$ such that:

(i) $X(h(\boldsymbol{\omega})) = Y(\boldsymbol{\omega})$ a.s.

(ii) $h(\sigma(I)) = \sigma(I)$.

We denote this by $\mathcal{M} \cong_{\text{a.s.}} \mathcal{N}$. Finally, the main theorem of this section:

Theorem 3.8. (Uniqueness of Uniform Hyperfinite Models). *Let* $\mathcal{M} = \langle H \times K, X, \sigma(I), \mu\rangle$ *and* $\mathcal{M}' = \langle H \times K, Y, \sigma(I), \mu\rangle$ *be uniform hyperfinite models for* $L_{\omega_1 E}$. *Then*:

$$\mathcal{M} \equiv_{\omega_1 E} \mathcal{M}' \quad \text{if and only if} \quad \mathcal{M} \cong_{\text{a.s.}} \mathcal{M}'.$$

Proof. ($\Rightarrow$) Easy.

($\Leftarrow$) Let $\{S_m : m \in \mathbb{N}\}$ be a countable basis for the Borel σ-algebra on R. Recall the following key properties of hyperfinite models:

(i) If D' is μ-measurable, then there exists an internal D such that $\mu(D \triangle D') = 0$.

(ii) If $D' \in \sigma(I)$, then there exists $D \in I$ such that $\mu(D \triangle D') = 0$.

Using these two properties we can strengthen the conclusions of Theorem 3.5 and 3.6 for the case of two elementary equivalent hyperfinite models $\mathcal{M}$ and $\mathcal{M}'$. Following the notation of those theorems with $\mathcal{M}$ and $\mathcal{M}'$ in place of $\mathcal{U}$ and $\mathcal{B}$ respectively, we have:

(1) We can assume the A_{ji}'s and C_{ji}'s are internal. The B_i's and D_i's are

internal and belong to I. And

(2) Corresponding Boolean combinations of A_{ji}'s and C_i's (B_{ji}'s and D_i's) have the same internal cardinality.

Now using (1) and (2) we can show: For each $(l, m, n) \in \mathbb{N}^3$ there exists an internal bijection $h_l^{mn}: H \times K \to H \times K$ such that:

(3) $h_l^{mn}(I) = I$. And

(4) If $(\gamma_n)_{n \in \mathbb{N}}$ is our fixed dense set of $L_{\omega_1 E}$-terms as described in Definition 2.5 then we have: For each $j \leq n$ and $p \leq m$ and approximations satisfying (1) and (2) and

$$\mu^{(r_j)}\left((\gamma_j^{\mathcal{M}'})^{-1}(S_p) \triangle \bigcup_{i=1}^{\theta_l} A_{1i}^j \times \cdots \times A_{r_j i}^j\right) < \frac{1}{l}$$

and

$$\mu^{(r_j)}\left((\gamma_j^{\mathcal{M}'})^{-1}(S_p) \triangle \bigcup_{i=1}^{\theta_l} C_{1i}^j \times \cdots \times C_{r_j i}^j\right) < \frac{1}{l}$$

where (r_j) is the number of free variables in γ_j, we have

$$h_l^{mn}(A_{si}^j) = C_{si}^j, \quad s = 1, \ldots, r_j.$$

Notice that for each term $\gamma_j(\boldsymbol{u}, v)$ with v E-bound in γ_j in the above approximations we have that the A_{si}^j (C_{si}^j) corresponding to v is an element of I.

The proof of (3) and (4) is technical. By a saturation argument we can find $h: H \times K \to H \times K$ internal bijection satisfying (3) and (4) for all $l, m, n \in \mathbb{N}$. For each $n \in \mathbb{N}$ h satisfies: $\gamma_n^{\mathcal{M}}(h(\omega)) = \gamma_n^{\mathcal{M}'}(\boldsymbol{\omega})$ a.s. and in particular $X(h(\boldsymbol{\omega})) = Y(\boldsymbol{\omega})$ a.s.; therefore h is an a.s. isomorphism between $\mathcal{M}$ and $\mathcal{M}'$ and this is precisely what we wanted to show. □

At the end of the previous section we indicated how to extend the existence theorems for hyperfinite models to the case where we allowed more than one conditional expectation operator. Again, all the theorems in this section can be naturally extended to cover those cases. Details are left to the reader.

4. Applications: Robinson consistency and Craig interpolation for $L_{\omega_1 E}$

In first-order logic there are several proofs of the Robinson Consistency theorem, among them a particularly simple one uses recursively saturated models (see for example [21]). For the probability logics $L_{\omega_1 P}$, $L_{\omega_1 \int}$ and L_{ad} Hoover in [11, 12], Keisler in [16], Hoover and Keisler in [15, 16] have given proofs that resemble that construction. Using the existence and uniqueness results of the previous sections we can give a proof of Robinson consistency for $L_{\omega_1 E}$ along the same lines. As before, the generalizations of the theorems to $L_{\omega_1 E}$ with more than one conditional expectation symbol go through in a natural way.

Theorem 4.1 (Robinson Consistency for $L_{\omega_1 E}$). *Let L^1 and L^2 be two languages with $L = L^1 \cup L^2$ and $L^0 = L^1 \cap L^2$. Let $\mathcal{U}$ be an $L^1_{\omega_1 E}$-structure and $\mathcal{B}$ an $L^2_{\omega_1 E}$-structure such that $\mathcal{U} \upharpoonright_{L^0} \equiv_{\omega_1 E} \mathcal{B} \upharpoonright_{L^0}$. Then there exists an $L_{\omega_1 E}$-model C such that:*

$$C \upharpoonright_{L^1} \equiv_{\omega_1 E} \mathcal{U} \quad \text{and} \quad C \upharpoonright_{L^2} \equiv_{\omega_1 E} \mathcal{B}.$$

Proof. By Theorem 2.12 we can find $L'_{\omega_1 E}$ and $L^2_{\omega_1 E}$ uniform hyperfinite models $\mathcal{M}_1$ and $\mathcal{M}_2$ such that:

(i) $\mathcal{M}_1 = \langle H \times K, \ldots, \sigma(I), \mu \rangle, \qquad \mathcal{M}_2 = \langle H \times K, \ldots, \sigma(I), \mu \rangle.$

(ii) $\mathcal{M}_1 \equiv_{\omega_1 E} \mathcal{U}$ and $\mathcal{M}_2 \equiv_{\omega_1 E} \mathcal{B}$.

We then have $\mathcal{M}_1 \upharpoonright_{L^0} \equiv_{\omega_1 E} \mathcal{M}_2 \upharpoonright_{L^0}$ and using the uniqueness Theorem 3.8 we can assume $\mathcal{M}_1 \upharpoonright_{L^0} \equiv_{\omega_1 E} \mathcal{M}_2 \upharpoonright_{L^0}$, so we can take as C the common expansion to L of $\mathcal{M}_1$ and $\mathcal{M}_2$. $\square$

We cannot give the usual first-order logic argument in order to prove the Craig interpolation theorem for $L_{\omega_1 E}$ from the Robinson Consistency theorem for $L_{\omega_1 E}$. Keisler in [16] explains the difference between Probability Logics and extensions of first order logic as considered by Mundici in [24]. Nevertheless, using Robinson Consistency and a Henkin construction Hoover in [11, 12] proved an interpolation theorem for $L_{\omega_1 P}$ and Hoover and Keisler, see [16], gave a proof of Interpolation for L_{ad}. Hoover in [14] proves an almost sure interpolation theorem $L_{\omega_1 P}$ for which we have no analog in $L_{\omega_1 E}$.

Theorem 4.2 (Craig Interpolation for $L_{\omega_1 E}$). *If ϕ is an $L^1_{\omega_1 E}$-sentence and γ is an $L^2_{\omega_1 E}$-sentence such that $\models \phi \to \gamma$, then there is an $L^0_{\omega_1 E} = L^1_{\omega_1 E} \cap L^2_{\omega_1 E}$ sentence θ such that $\models \phi \to \theta$ and $\models \theta \to \gamma$.*

Proof. Use a Henkin argument in conjunction with Theorem 4.1 as in [16]. $\square$

To conclude, we remind the reader that all the above results hold for the logics $L_{\mathbb{A}E}$ with $\mathbb{A}$ an admissible set and $\omega \in \mathbb{A}$.

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