JOURNAL OF COMBINATORIAL THEORY (B) 13, 83–90 (1972)

Enumeration of Almost Cubic Maps

A. M. MATHAI AND P. N. RATHIE

Department of Mathematics, McGill University, Montreal, Quebec, Canada

Communicated by W. T. Tutte

Received July 20, 1971

This paper deals with the enumeration of rooted planar maps in which the root vertex is of arbitrary valence and all other vertices are trivalent. A formula, in explicit form, is given and closed form expressions are given for several cases of interest. Some interesting summation formulae are also obtained towards the end of the paper.

1. INTRODUCTION

The aim of this article is the enumeration of rooted trivalent planar maps in which the root vertex has arbitrary valence and all other vertices are trivalent. This is done by considering planar maps with the root face of arbitrary valence and all other faces of valence 3. Tutte [4] has solved the problem of rooted planar maps with root face a k-gon and all other faces p-gons (p-even). When p is odd the only cases solved are $p = 3$, $k = 1, 2$, and 3 by Mullin [2]. Mullin, Nemeth, and Schellenberg [3] considered the case $p = 3$ for all k and obtained numerical tables for the number of admissible maps of the type (n, m) for $n, m = 1, 2,..., 14$. An admissible map is a rooted planar map with root face of arbitrary valence and all other faces trivalent and it is of type (n, m) if it has n interior faces and exterior faces of valence m. We follow Tutte [5] for the definitions of planar map, face, root, valence, etc. The explicit expression given in [3] for the number of admissible maps of the type (n, m) does not seem to contain sufficient number of conditions for the validity of the results. In this article we obtain the explicit expression for the number of admissible maps of the type (n, m) and a number of other interesting results.

582b/13/1-6

2. Determination of $t_{n,m}$

Let $t_{n,m}$ be the number of admissible maps of type (n, m). Then the formal series,

$$T(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t_{n,m} x^n y^m, \tag{1}$$

is the generating function for these numbers. A functional equation for T is given below,

$$T = 1 + y^2 T^2 + xy^{-1}(T - yL - 1), \tag{2}$$

where

$$L \equiv L(x) = \sum_{n=0}^{\infty} t_{n,1} x^n \tag{3}$$

is the generating function for maps whose root face is a loop. The functional equation (2) can be easily obtained from [5, p. 69].

Equation (2) may be rewritten as

$$T = [1 - xy(y - x)^{-1} L] + y^3(y - x)^{-1} T^2, \tag{4}$$

which on applying Lagrange's expansion [6, p. 132] gives

$$T = \sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(n+1)!} [y^3(y - x)^{-1}]^n [1 - xy(y - x)^{-1} L]^{n+1}. \tag{5}$$

Now expanding the terms inside the sum by using binomial expansion and keeping in mind that the powers of x and y are non-negative, we get

$$T = \sum_{n=0}^{\infty} \sum_{k=0}^{n+1} \sum_{j=0}^{2n} \frac{(2n)!}{n!\,(n+1)!} \binom{n+1}{k} \binom{n+k+j-1}{j} (-1)^k x^j y^{2n-j} (xL)^k$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \sum_{k=1}^{n+1} \frac{(2n)!}{n!\,(n+1)!} \binom{n+1}{k} \binom{n+k+j-1}{j} (-1)^k x^j y^{2n-j} (xL)^k$$

$$+ \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \frac{(2n)!}{n!\,(n+1)!} \binom{n+j-1}{j} x^j y^{2n-j}. \tag{6}$$

An expression for $(xL)^k$ is given in [3] that can lead to wrong values for $t_{n,m}$. The correct expression in a simplified form is given below:

$$(xL)^k = \sum_{m=k}^{\infty} \sum_{i=0}^{\min(k-1,m-k)} \sum_{q=0}^{m-k-i} \left(\frac{k}{m}\right)\binom{k-1}{i}\binom{m+q-1}{q}(-3)^i\, 2^{2m-2k-i-q}$$
$$\times \left[\left(\frac{3}{2}\right)\binom{2m-i-q-2}{m+k-2} - \left(\frac{1}{2}\right)\binom{2m-i-q}{m+k}\right] x^{2m},$$
$$\text{for} \quad m \geqslant k \geqslant 1. \quad (7)$$

The binomial coefficient $\binom{n}{r}$ is written with the convention $\binom{n}{r} = 0$, when $r > n$.

Thus (6) with the help of (7) takes the following form:

$$T = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \sum_{k=1}^{n+1} \sum_{m=k}^{\infty} f(n, k, j, m)\, x^{2m+j} y^{2n-j} + \sum_{n=0}^{\infty} \sum_{j=0}^{2n} f(n, 0, j, 0)\, x^j y^{2n-j}, \quad (8)$$

where

$$f(n, k, j, m) = \frac{(2n)!\, k}{n!\,(n+1)!\, m} \binom{n+1}{k}\binom{n+k+j-1}{j}(-1)^k$$
$$\times \sum_{i=0}^{\min(k-1,m-k)} \sum_{q=0}^{m-k-i} \binom{k-1}{i}\binom{m+q-1}{q}(-3)^i\, 2^{2m-2k-i-q}$$
$$\times \left[\left(\frac{3}{2}\right)\binom{2m-i-q-2}{m+k-2} - \left(\frac{1}{2}\right)\binom{2m-i-q}{m+k}\right], \quad (9)$$

and

$$f(n, 0, j, 0) = \frac{(2n)!}{n!\,(n+1)!} \binom{n+j-1}{j}. \quad (10)$$

It is easy to see by actual expansion of the sums that

$$\sum_{k=1}^{n+1} \sum_{m=k}^{\infty} f(n, k, j, m)\, x^{2m} = \sum_{k=1}^{n+1} \left(\sum_{m=k}^{n+1} + \sum_{m=n+2}^{\infty}\right) f(n, k, j, m)\, x^{2m}$$
$$= \left(\sum_{m=1}^{n+1} \sum_{k=1}^{m} + \sum_{k=1}^{n+1} \sum_{m=n+2}^{\infty}\right) f(n, k, j, m)\, x^{2m}. \quad (11)$$

Hence (8) and (11) yield

$$T = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \sum_{m=n+2}^{\infty} \sum_{k=1}^{n+1} f(n, k, j, m)\, x^{2m+j} y^{2n-j}$$

$$+ \sum_{n=0}^{\infty} \sum_{j=0}^{2n} \sum_{m=1}^{n+1} \sum_{k=1}^{m} f(n, k, j, m)\, x^{2m+j} y^{2n-j}$$

$$+ \sum_{n=0}^{\infty} \sum_{j=0}^{2n} f(n, 0, j, 0)\, x^{j} y^{2n-j}, \tag{12}$$

where $f(n, k, j, m)$ and $f(n, 0, j, 0)$ are given by (9) and (10), respectively. The third term in (12) can be absorbed in the second, if necessary, with the convention that, when $k = 0$, $m = 0$ and $k/m = 1$.

Now comparing the coefficients of $x^{2s}y^{2r}$ and $x^{2s+1}y^{2r-1}$ for $r, s = 0, 1,\ldots$ separately in (12), we obtain

$$t_{2s,2r} = \sum_{p=0}^{[\frac{1}{2}(s-r-2)]} \sum_{k=1}^{r+p+1} f(r+p, k, 2p, s-p)$$

$$+ \sum_{p=[\frac{1}{2}(s-r)]}^{s-1} \sum_{k=1}^{s-p} f(r+p, k, 2p, s-p) + f(r+s, 0, 2s, 0), \tag{13}$$

and

$$t_{2s+1,2r-1}$$

$$= \sum_{p=0}^{[\frac{1}{2}(s-r-2)]} \sum_{k=1}^{r+p+1} f(r+p, k, 2p+1, s-p)$$

$$+ \sum_{p=[\frac{1}{2}(s-r)]}^{s-1} \sum_{k=1}^{s-p} f(r+p, k, 2p+1, s-p) + f(r+s, 0, 2s+1, 0), \tag{14}$$

where the convention $[x]$ = greatest integer in $x \leqslant x$ is followed and f's are given by (9) and (10). Also, when $[\frac{1}{2}(s-r)]$ is negative it is interpreted as zero and the usual convention that $\sum_a^b (\cdot) = 0$, if $b < a$, is followed.

When $s = 0$ in (13) and (14), only the last term, namely, $f(r+s, 0, 2s, 0)$ and $f(r+s, 0, 2s+1, 0)$, respectively, will exist which will readily give $t_{0,2r}$ and $t_{1,2r-1}$. When $r > s-2$, then in (13) and (14) there will be no contribution from the first term on the right-hand side of each.

3. Some Interesting Particular Cases

In this section, we will derive closed form expressions for $t_{n,m}$ in several cases

(i) Since the right-hand side of (12) gives the coefficients of $x^p y^q$ only when $p + q$ is even,

$$t_{p,q} = 0, \tag{15}$$

whenever $p + q$ is odd.

(ii) Writing (2) in the form,

$$y^3T^2 + (x - y)T - xyL - x + y = 0 \tag{16}$$

and substituting the expression for T from (1) in (16), we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\left(\sum_{s=0}^{m} \sum_{r=0}^{n} t_{r,s}t_{n-r,m-s}\right) x^n y^{m+3} + t_{n,m}x^{n+1}y^m - t_{n,m}x^n y^{m+1}\right]$$

$$- \sum_{n=0}^{\infty} t_{n,1}x^{n+1}y - x + y = 0. \tag{17}$$

Now, comparing the coefficients of $x^{n+1}y^0$ in (17), we get

$$t_{n,0} = 0, \tag{18}$$

for $n \geqslant 1$.

(iii) Taking $s = 0$, 1 and 2, respectively, in (13) and (14) and utilizing (9) and (10), we have

$$t_{0,2r} = (2r)!/\{r!(r+1)!\}, \qquad r = 0, 1, 2, \ldots . \tag{19}$$

$$t_{1,2r-1} = (2r)!/\{(r-1)!\,(r+1)!\} = rt_{0,2r}, \qquad r = 1, 2, \ldots . \tag{20}$$

$$t_{2,2r} = 2(2r)!/\{(r-1)!\,r!\} = 2r(r+1)\,t_{0,2r}, \qquad r = 1, 2, \ldots . \tag{21}$$

$$t_{3,2r-1} = 2(r+2)(2r)!/\{3(r-1)!\,r!\} = (2/3)\,r(r+1)(r+2)\,t_{0,2r}, \qquad r = 1, 2, \ldots . \tag{22}$$

$$\begin{aligned} t_{4,2r} &= 2(r^2 + 7r + 16)(2r)!/\{3(r-1)!\,r!\} \\ &= (2/3)\,r(r+1)(r^2 + 7r + 16)\,t_{0,2r}, \qquad r = 1, 2, \ldots . \end{aligned} \tag{23}$$

$$\begin{aligned} t_{5,2r-1} &= 2(r^3 + 11r^2 + 50r + 58)(2r)!/\{15(r-1)!\,r!\} \\ &= (2/15)\,r(r+1)(r^3 + 11r^2 + 50r + 58)\,t_{0,2r}, \qquad r = 1, 2, \ldots . \end{aligned} \tag{24}$$

From (19) to (24), we can conjecture that

$$t_{2n,2m} = P_{2n}t_{0,2m} \tag{25}$$

and

$$t_{2n-1,2m-1} = Q_{2n-1}t_{0,2m}\,, \tag{26}$$

where P_{2n} and Q_{2n-1} are polynomials in m of degree $2n$ and $2n - 1$, respectively.

(iv) Mullin [2] has shown that

$$t_{2s+1,1} = 2^{3s}\Gamma(3s/2 + 1)/\{(s + 1)!\,\Gamma(s/2 + 2)\}, \qquad s = 0, 1, 2, \ldots . \tag{27}$$

Comparing the coefficients of $x^{2s+1}y^2$ and $x^{2s}y^3$ in (17), we get

$$t_{2s-1,3} = t_{2s,2} = t_{2s+1,1}\,, \tag{28}$$

for $s \geqslant 1$.

The relation $t_{2s-1,3} = t_{2s+1,1}$ can also be proved from (14). For that, taking $r = 1$ in (14) gives an expression for $t_{2s+1,1}$. Now replacing s by $s - 1$, r by $r + 1$, j by $j + 2$ and then putting $r = 1$ in (14) we get an expression for $t_{2s-1,3}$. We can account for the combination of n, k, j, m with $j = 0$ by putting $p = -1$ in the sum. It is easily seen from these two expressions that $t_{2s-1,3} = t_{2s+1,1}$.

Again, comparing the coefficients of $x^{2s}y^4$ in (17) and utilizing (28), we have

$$t_{2s,4} = t_{2s+3,1} - 2t_{2s+1,1}\,. \tag{29}$$

Now $t_{2s,4}$, $t_{2s-1,3}$ and $t_{2s,2}$ can be put into closed forms by using (27).

In a similar manner, more coefficients can be put in closed forms. In general, comparing the coefficients of x^ny^{m+3} in (17), we have the following recurrence relation

$$\sum_{s=0}^{m}\sum_{r=0}^{n} t_{r,s}t_{n-r,m-s} + t_{n-1,m+3} - t_{n,m+2} = 0 \tag{30}$$

for $n \geqslant 0$, $m \geqslant 0$. When $n = 0$, the term $t_{n-1,m+3}$ in (30) is taken as zero.

The recurrence relation (30) can also be used for computing $t_{n,m}$ for various values of n and m.

4. Summation Formulae

This section deals with some interesting summation formulae obtained either from the recurrence relation (30) or by equating the two types of expressions obtained for the same coefficient in T.

(i) The equation (30) for $n = 0$ with the help of (15) and (19) gives

$$\sum_{s=0}^{m} \frac{(2s)!\,(2m-2s)!}{s!\,(s+1)!\,(m-s)!\,(m-s+1)!} = \frac{(2m+2)!}{(m+1)!\,(m+2)!}. \tag{31}$$

(ii) Taking $n = 1$, $m = 2r - 1$ in (30) and utilizing (15), (19) and (20), we have

$$\sum_{s=0}^{k-1} \frac{(2s)!\,(2k-2s)!}{s!\,(s+1)!\,(k-s-1)!\,(k-s+1)!} = \frac{k(2k+2)!}{2(k+1)!\,(k+2)!}. \tag{32}$$

More results of the type (31) and (32) can be obtained from (30) by taking $n = 2, 3,\ldots$ and using equations (19) onwards.

(iii) The equations (13) and (14) for $r = 1$, (28), (9), (10) and the result $\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}$, give

$$\begin{aligned}
&\left[\sum_{p=0}^{[\frac{1}{2}(s-3)]} \sum_{k=1}^{p+2} + \sum_{p=[\frac{1}{2}(s-1)]}^{s-1} \sum_{k=1}^{s-p}\right] \frac{(2p+2)!\,k}{(p+1)!\,(p+2)!\,(s-p)} \\
&\quad \times \binom{p+2}{k}\binom{3p+k}{2p+1}(-1)^{k+1} \sum_{i=0}^{\min(k-1,s-p-k)} \sum_{q=0}^{s-p-k-i} \binom{k-1}{i} \\
&\quad \times \binom{s-p+q-1}{q}(-3)^i\, 2^{2s-2p-2k-i-q} \\
&\quad \times \left[\binom{\frac{3}{2}}{}\binom{2s-2p-i-q-2}{s-p+k-2} - \binom{\frac{1}{2}}{}\binom{2s-2p-i-q}{s-p+k}\right] \\
&= \frac{(2s+2)!}{(s+1)!\,(s+2)!}\binom{3s}{2s+1}.
\end{aligned} \tag{33}$$

(iv) Lastly, we will show that,

$$\sum_{r=1}^{m-1} 2^{2r-1} t_{2,2m-2r} = t_{3,2m-1} - t_{2,2m}, \tag{34}$$

or, alternatively,

$$\begin{aligned}
&\sum_{r=1}^{m-1} 2^{2r}(2m-2r)!\,/\,\{(m-r-1)!\,(m-r)!\} \\
&\qquad = 2(2m)!\,/\,\{3(m-2)!\,m!\}.
\end{aligned} \tag{35}$$

The left-hand side of (35) on using $n! = \Gamma(n+1)$, the duplication formula for gamma functions [1, p. 5 (15)] and [1, p. 104 (46), p. 52 (6)] proves the desired result on interpreting with the help of (21) and (22).

ACKNOWLEDGMENT

The authors would like to thank Professor R. C. Mullin for introducing them to enumeration problems in graph theory as well as for some useful discussions.

REFERENCES

1. A. ERDÉLYI *et al.*, "Higher Transcendental Functions," Vol. I, McGraw-Hill, New York, 1953.
2. R. C. MULLIN, On the average number of trees in certain maps, *Canad. J. Math.* **18** (1966), 33–41.
3. R. C. MULLIN, E. NEMETH, AND P. J. SCHELLENBERG, The enumeration of almost cubic maps.
4. W. T. TUTTE, A census of planar maps, *Canad. J. Math.* **15** (1963), 249–271.
5. W. T. TUTTE, On the enumeration of planar maps, *Bull. Amer. Math. Soc.* **74** (1968), 64–74.
6. E. T. WHITTAKER AND G. N. WATSON, "A Course of modern Analysis," Cambridge University Press, Cambridge, 1962.