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Hermite–Fejér-Related Interpolation and Product Integration

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Convergence of product integration rules based on Hermite–Fejér interpolation with end conditions is shown for all Riemann-integrable functions when the interpolation points are zeros of generalized Jacobi polynomials even in cases where the corresponding Hermite–Fejér operator is indefinite.

1. INTRODUCTION

This paper is concerned with the convergence of product integration (PI) rules based on Hermite–Fejér (HF) interpolatory polynomials with end conditions. We consider PI rules of the form

$$I(kf) := \int_{-1}^{1} k(x) f(x)\, dx \simeq I(kH_{npq}(w; f)), \tag{1}$$

where $k \in L_1(J)$, $J := [-1, 1]$, $f \in R(J)$, the set of all (bounded) Riemann-integrable functions on J and H_{npq} is an HF interpolating polynomial with end conditions previously studied by the authors [8, 10] and Knoop [2]. Here, p and q are nonnegative integers and w is a generalized Jacobi (GJ) weight function of the form

$$w(x) := \psi(x)\, w_{\alpha\beta}(x), \qquad \alpha, \beta > -1, \tag{2}$$

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where $\psi > 0$ and $\psi' \in \text{Lip } 1$ on J and $w_{\alpha\beta}$ is the Jacobi weight function

$$w_{\alpha\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}. \tag{3}$$

PI based on HF polynomials was first studied explicitly in [1] although certain results already appear implicitly in [4]. Further works on the subject appear in [3, 6, 7].

In this paper, we are interested in giving conditions which ensure that

$$I(kH_{npq}(f)) \to I(kf) \quad \text{as} \quad n \to \infty \text{ for all } k \in L_1(J) \tag{4}$$

whenever $f \in R(J)$. Now, in general, convergence results for PI follow readily from convergence results for HF interpolation of the form

$$\|f - H_{npq}(f)\|_{\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

However, such convergence results hold only for $f \in C(J)$. If we wish to deal with $R(J)$, we must introduce additional devices. The simplest situation occurs when H_{npq} is a nonnegative operator. In this case, it has been shown in [6] that if (4) holds for all $f \in C(J)$, then it holds for all $f \in R(J)$. However, there are situations where (5) holds for all $f \in C(J)$ but H_{npq} is an indefinite operator. This occurs, for example, when $w = w_{\alpha\beta}$ and

$$a := \alpha - p, \qquad b := \beta - q$$

satisfy the conditions

$$-1.5 \leqslant a,\ b < 0, \qquad |a - b| \leqslant 1 \tag{6}$$

as shown in [2, 10]. (If $p = 0$, the inequality for a is $a > -1$ and similarly for q and b. This holds also in (7) and (8).) If

$$-1 \leqslant a,\ b < 0 \tag{7}$$

then H_{npq} is a positive operator so that there are no problems but (6) is a weaker condition.

To deal with indefinite operators, we introduce the HF-related operator S_{npq} which is positive and "close" in some sense to H_{npq}. We then show that if $w = w_{\alpha\beta}$ and

$$-1.5 < a,\ b < 0, \qquad |a - b| \leqslant 1 \tag{8}$$

then

$$I(kS_{npq}(f)) \to I(kf) \quad \text{as} \quad n \to \infty \text{ for all } k \in L_1(J) \tag{9}$$

whenever $f \in C(J)$ and consequently, whenever $f \in R(J)$. Using this result, we can prove that if $w = w_{\alpha\beta}$ and (8) holds, then (4) holds for all $f \in R(J)$.

We cannot extend this to the case where (6) holds since it has been shown in [1] that in the particular case $p=q=2$, $k\equiv 1$, and a or $b=-1.5$, (4) does not hold for all $f\in R(J)$.

Returning to our work, we notice that once we have this methodology, namely, of using close operators for which convergence is known, we apply it to prove (4) for all GJ weights for all $f\in R(J)$ and all a, b satisfying (8). On the way, we show that (5) holds for all $f\in C(J)$ and all a, b satisfying (6). This is in itself a noteworthy result since the best that has been shown explicitly to date in [5] is that (5) holds when $p=q=0$ and (7) holds, although implicit in [5] are results for any p and q together with (7).

In Section 2, we give the necessary background information. In Sections 3 and 4, we discuss the convergence to $I(kf)$ of $I(kS_{npq})$ and $I(kH_{npq})$, respectively, when $w=w_{\alpha\beta}$. In Section 5, we discuss the case where w is an arbitrary GJ weight function and in Section 6, we make some concluding remarks.

2. Background Material

Let p, q be integers $\geqslant 0$, let $w\in GJ$, and let $X_n := \{x_{in}: i=1, ..., n\}$ be the set of zeros of $p_n(w)$, the polynomial of degree n belonging to the sequence of polynomials orthonormal with respect to w. We order the zeros in decreasing order so that

$$x_{0n} := 1 > x_{1n} > \cdots > x_{nn} > x_{n+1,n} := -1.$$

The HF [2] polynomial with end conditions, $H_{npq}(w;f)=H_{npq}(f)$, interpolating a real-valued function f on J at the points X_n is a polynomial of degree $2n+p+q-1$ satisfying the conditions

$$\begin{aligned}
H_{npq}(f; x_{in}) &= f_{in} := f(x_{in}), && i=1, ..., n\\
H'_{npq}(f; x_{in}) &= 0, && i=1, ..., n\\
H_{npq}(f; x_{0n}) &= f_0 := f(x_{0n}) && \text{if } p>0\\
H_{npq}(f; x_{n+1,n}) &= f_{n+1} := f(x_{n+1,n}) && \text{if } q>0\\
H^{(r)}_{npq}(f; x_{0n}) &= 0, && r=1, ..., p-1 \text{ if } p>1\\
H^{(s)}_{npq}(f; x_{n+1,n}) &= 0, && s=1, ..., q-1 \text{ if } q>1.
\end{aligned}$$

An explicit form for H_{npq} is given by

$$H_{npq}(f; x) := \sum_{i=1}^{n} v_{inpq}(x)\, a_{inpq}(x)\, f_{in} + \psi_{npq}(x)\, f_0 + \bar{\psi}_{npq}(x)\, f_{n+1}, \tag{10}$$

where

$$a_{inpq}(x) := \left(\frac{1-x}{1-x_{in}}\right)^p \left(\frac{1+x}{1+x_{in}}\right)^q l_{in}^2(x), \qquad i=1, ..., n \tag{11}$$

$$l_{in}(x) := \frac{p_n(w; x)}{(x-x_{in})\, p_n'(w; x_{in})}, \qquad i=1, ..., n \tag{12}$$

$$v_{inpq}(x) := 1 + \left(\frac{p}{1-x_{in}} - \frac{q}{1+x_{in}} - \frac{p_n''(w; x_{in})}{p_n'(w; x_{in})}\right)(x-x_{in}) \tag{13}$$

and ψ_{npq}, $\bar{\psi}_{npq}$ are polynomials of degree $2n+p+q-1$ which satisfy the conditions

$$\begin{aligned}
&\psi_{npq}(x_{in}) = \bar{\psi}_{npq}(x_{in}) = 0, && i=1, ..., n\\
&\psi'_{npq}(x_{in}) = \bar{\psi}'_{npq}(x_{in}) = 0, && i=1, ..., n\\
&\psi_{npq}(x_{0n}) = 1 \quad \text{if} \quad p>0, && \bar{\psi}_{npq}(x_{n+1,n}) = 1 \quad \text{if} \quad q>0\\
&\psi^{(r)}_{npq}(x_{0n}) = 0, && r=1, ..., p-1 \text{ if } p>1\\
&\psi^{(r)}_{npq}(x_{n+1,n}) = 0, && r=0, ..., p-1\\
&\bar{\psi}^{(s)}_{npq}(x_{n+1,n}) = 0, && s=1, ..., q-1 \text{ if } q>1\\
&\bar{\psi}^{(s)}_{npq}(x_{0n}) = 0, && s=0, ..., q-1.
\end{aligned}$$

The explicit forms of the functions ψ_{npq} and $\bar{\psi}_{npq}$ are not needed in the sequel. What is important are the facts shown in [2] that

$$\psi_{npq}, \bar{\psi}_{npq} \geqslant 0 \qquad \text{on } J \tag{14}$$

and that

$$\sum_{i=1}^{n} v_{inpq}(x)\, a_{inpq}(x) + \psi_{npq}(x) + \bar{\psi}_{npq}(x) = 1. \tag{15}$$

Note that for $p, q \in \{0, 2\}$, the polynomials $H_{npq}(f)$ coincide with the classical HF polynomials. Thus $H_{n00}(f)$ is the classical HF polynomial based on X_n while $H_{n22}(f)$ is the classical HF polynomial based on $X_n \cup \{x_{0n}, x_{n+1,n}\}$. Similar identifications can be made for $H_{n20}(f)$ and $H_{n02}(f)$.

Closely related to the operator $H_{npq}(f)$ is the positive operator $S_{npq}(f)$ defined by

$$S_{npq}(f; x) := \sum_{i=1}^{n} f_{in}\, a_{inpq}(x). \tag{16}$$

In the rest of this paper we assume that the values of p and q are fixed throughout the discussion. Hence we write H_n, S_n, v_{in}, a_{in}, ψ_n, and $\bar{\psi}_n$. We also use the notation $A_n(x) = o(1)$ to mean that $A_n(x) \to 0$ as $n \to \infty$ uniformly in J. Similarly, the notation $A_n(x) = o(1)_\delta$, means that $A_n(x) \to 0$

as $n \to \infty$ uniformly in $J_\delta := [-1+\delta, 1-\delta]$ where δ is any number in $(0, 1)$. We also write $I_\delta(g) := \int_{-1+\delta}^{1-\delta} g(x)\,dx$ and $\bar{I}_\delta(g) := I(g) - I_\delta(g)$.

The importance of positivity of S_n for product integration lies in the following result:

PROPOSITION 1. *Consider the product integral $I(kf)$ where $k \in L_1(J)$ and let B_n be an operator on J of the form*

$$B_n(f; x) := \sum_{i=1}^{n} b_{in}(x) f_{in}, \qquad \text{where} \quad b_{in} \geqslant 0 \text{ on } J, \qquad i = 1, ..., n.$$

If

$$I(kB_n(f)) \to I(kf) \qquad \text{as} \quad n \to \infty \tag{17}$$

for all $f \in C(J)$ and all $k \in L_1(J)$, then (17) *holds for all $f \in R(J)$.*

Proof. This is similar to the proof of Theorem 1 in [6].

We now introduce the concept of (p, q)-ρ-normality where the set $X_n \subset (-1, 1)$ is said to be (p, q)-ρ-normal for some $\rho > 0$ if $v_{inpq} \geqslant \rho > 0$ on J for $i = 1, ..., n$. By (14), this implies that the HF operator H_{npq} based on a (p, q)-ρ-normal set is a positive operator. An example of a (p, q)-ρ-normal set is given by the set of zeros of the Jacobi polynomial $p_n(w_{\alpha\beta})$ where α, β satisfy (7) and $\rho > 0$ is some number depending on α, β, p, q.

In a previous work [8], the authors showed that if $\{X_n\}$ is a sequence of (p, q)-ρ-normal sets, with the same value of ρ, then, for all $f \in C(J)$

$$f(x) - H_n(f; x) = o(1) \tag{18}$$

$$f(x) - S_n(f; x) = o(1)_\delta. \tag{19}$$

Clearly, for all $f \in C(J)$, (4) holds and by Proposition 1, (4) holds for all $f \in R(J)$. We show a similar result for $I(kS_n(f))$ as a corollary of the following proposition:

PROPOSITION 2. *Assume that for all $f \in C(J)$,* (19) *holds and that*

$$\|S_n(f)\|_\infty \leqslant A(f) \qquad \text{for all } n. \tag{20}$$

Then (9) *holds for all $f \in R(J)$.*

Proof. Let $\varepsilon > 0$ and assume first that $f \in C(J)$. We have that

$$\begin{aligned} |I(k(S_n(f) - f))| &\leqslant I_\delta(|k|\ |S_n(f) - f|) + \bar{I}_\delta(|k|\ (|S_n(f)| + |f|)) \\ &\leqslant (\max_{x \in J_\delta} |S_n(f; x) - f(x)|)\, I(|k|) + (A(f) + F)\, \bar{I}_\delta(|k|), \end{aligned}$$

where $F := \|f\|_\infty$. Since $k \in L_1(J)$, we can choose δ such that $\bar{I}_\delta(|k|) < \varepsilon/2(A(f)+F)$ and then from (19), (9) holds. We then apply Proposition 1 to prove (9) for $f \in R(J)$.

COROLLARY 1. *If $\{X_n\}$ is a sequence of (p, q)-ρ-normal sets, then* (9) *holds for all $f \in R(J)$.*

Proof. We need only show (20) for all $f \in C(J)$. But by (14), (15), and the nonnegativity on J of the functions a_{in}, we have that

$$\sum_{i=1}^{n} a_{in}(x) \leqslant 1/\rho$$

which implies that $\|S_n(f)\|_\infty \leqslant \|f\|_\infty/\rho$ which proves Corollary 1.

We thus have that both (4) and (9) hold for all $f \in R(J)$ if $\{X_n\}$ are (p, q)-ρ-normal sets. Since the convergence properties of H_n are better than those of S_n, it makes more sense to use H_n for product integration even though the algorithm for evaluating $I(kS_n(f))$ is simpler than that for $I(kH_n(f))$ [6]. In the next section, we consider interpolation based on sets which are not (p, q)-ρ-normal in which case we can only show at the moment that (4) will hold only for $f \in C(J)$. However, we show that (9) still holds for all $f \in R(J)$.

In Section 4, we return to $I(kH_n(f))$ and using the results of Section 3, show that (4) also holds for all $f \in R(J)$ when $w = w_{\alpha\beta}$.

3. CONVERGENCE OF $I(kS_n(f))$ FOR $w = w_{\alpha\beta}$

In [2, 10] it was shown that if X_n is the set of zeros of $p_n(w_{\alpha\beta})$ that satisfy (6) then for all $f \in C(J)$, (18) and consequently (4) holds. However, if a, b satisfy (6) but not (7) so that the X_n are not (p, q)-ρ-normal, then one cannot apply Proposition 1 to show that (4) holds for all $f \in R(J)$. In this section we show that, with slightly stronger conditions on a and b, both (19) and (20) hold for all $f \in C(J)$ which will imply, by Proposition 2, that (9) holds for all $f \in R(J)$.

THEOREM 1. *Let p and q be nonnegative integers, let $w = w_{\alpha\beta}$, and let a, b satisfy* (8). *Then* (19) *and* (20) *hold for every $f \in C(J)$. Consequently,* (9) *holds for all $f \in R(J)$.*

Proof. Since (18) holds, it follows that

$$f(x) - H_n(f; x) = o(1)_\delta.$$

Thus, to show (19), it suffices to show that

$$H_n(f;x)-S_n(f;x)=\sum_{i=1}^{n}(v_{in}(x)-1)\,a_{in}(x)\,f_{in}+\psi_n(x)\,f_0+\bar{\psi}_n(x)\,f_{n+1}$$
$$=o(1)_\delta \tag{21}$$

or that

$$\psi_n(x),\ \bar{\psi}_n(x)=o(1)_\delta \tag{22}$$

and

$$A_n(x):=\sum_{i=1}^{n}|(1-v_{in}(x))|\,a_{in}(x)=o(1)_\delta. \tag{23}$$

We first show (23) using the methodology of [9], and the results of [10, (4.1); 4, Lemma 2].

Consider first $A_n(x)$ for some $x\in J_\delta$ for a fixed $\delta>0$. Then, for all $n\geqslant n_0$, if $x\approx x_{jn}$ then $j\sim n$. Let $x\geqslant 0$, say, and set $K:=n-k+1$. We then have that

$$A_n(x)\leqslant C\left[\frac{n^{2a}}{j^{2a+1}}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{k^{2a+3}}{n^{2a+4}}\frac{n^4}{|k+j|^2\,|k-j|^2}\frac{(k+j)\,|k-j|}{n^2}\frac{n^2}{k^2}\right.$$
$$\left.+\frac{n^{2a}}{j^{2a+1}}\sum_{K=1}^{n}\frac{K^{2b+3}}{n^{2b+4}}\frac{n^2}{j^2}\right]\sim\frac{1}{j^{2a+1}}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{k^{2a+1}}{|k+j|\,|k-j|}$$
$$+\frac{n^{2a-2b-2}}{j^{2a+1}}\sum_{K=1}^{n}K^{2b+1}:=A_{1n}+A_{2n}$$

$$A_{1n}\sim\frac{1}{j^{2a+1}}\left\{\begin{bmatrix}1/j^2 & \text{if}\quad 2a+1<-1\\ j^{2a+2}/j^2 & \text{if}\quad 2a+1>-1\\ \log j/j^2 & \text{if}\quad 2a+1=-1\end{bmatrix}+j^{2a}\log j+j^{2a}\right\}$$

$$=o(1)_\delta\qquad\text{if}\quad -1.5<a<0$$

$$A_{2n}\sim\left\{\begin{matrix}n^{2a}/j^{2a+1} & \text{if}\quad b>-1\\ n^{2a}\log n/j^{2a+1} & \text{if}\quad b=-1\\ n^{2a-2b-2}/j^{2a+1} & \text{if}\quad b<-1\end{matrix}\right\}$$

$$=o(1)_\delta\qquad\text{if}\quad -1.5<a<0,\quad -1.5<b,\quad a-b\leqslant 1.$$

Thus, if (8) holds then (23) holds. Hence

$$H_n(f; x) - S_n(f; x) - \psi_n(x) f_0 - \bar{\psi}_n(x) f_{n+1} = o(1)_\delta \tag{24}$$

whence, setting $f = 1$,

$$1 - \sum_{i=1}^{n} a_{in}(x) - \psi_n(x) - \bar{\psi}_n(x) = o(1)_\delta. \tag{25}$$

If we now show (22), we will have proved (19). To this end, we first prove the following lemma.

LEMMA 1. *For any* $\eta > 0$, $\sum_{|x - x_{in}| \geq \eta} a_{in}(x) = o(1)_\delta$.

Proof. We first prove that

$$B_n(x) := \sum_{i=1}^{n} a_{in}(x) \, |x - x_{in}| = o(1)_\delta. \tag{26}$$

In fact, if $x \approx x_{jn}$, then using the same methodology and notation as above,

$$B_n(x) \leq \frac{C}{n} \left[\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{k^{2a+3}}{n^{2a+4}} \frac{n}{|k-j|} + \sum_{K=1}^{n} \frac{K^{2b+3}}{n^{2b+4}} \right] = o(1)_\delta,$$

where we have used the fact that $j \sim n$. Hence

$$B_n(x) = \sum_{|x - x_{in}| \geq \eta} |x - x_{in}| \, a_{in}(x) + \sum_{|x - x_{in}| < \eta} |x - x_{in}| \, a_{in}(x) = o(1)_\delta.$$

Since $a_{in}(x) \geq 0$ in J, it follows that

$$o(1)_\delta = \sum_{|x - x_{in}| \geq \eta} |x - x_{in}| \, a_{in}(x) \geq \eta \sum_{|x - x_{in}| \geq \eta} a_{in}(x)$$

which proves Lemma 1.

We now choose η such that $\gamma := 1 - \delta + \eta < 1$ and define $F \in C(J)$ by

$$F(x) = \begin{cases} 0, & x \leq \gamma \\ (x - \gamma)/(\delta - \eta), & \gamma \leq x \leq 1. \end{cases}$$

Then by (24), $H_n(F; x) - S_n(F; x) - \psi_n(x) = o(1)_\delta$. Since $F = 0$ in J_δ, $H_n(F; x) = o(1)_\delta$ and $0 \leq S_n(F; x) = \sum_{x_{in} \geq \gamma} F_{in} a_{in}(x) \leq (1/\eta) \sum_{x_{in} \geq \gamma} a_{in}(x)$ $|x - x_{in}| \leq (1/\eta) \, B_n(x) = o(1)_\delta$. Hence $\psi_n(x) = o(1)_\delta$. Similarly $\bar{\psi}_n(x) = o(1)_\delta$ proving (19).

To prove (20), let $x \geqslant -1/2$, say. Then

$$\sum_{i=1}^{n} a_{in}(x) \leqslant C_1 \frac{n^{2a}}{j^{2a+1}} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{k^{2a+3}}{n^{2a+4}} \frac{n^4}{|k+j|^2\,|k-j|^2}$$

$$+ C_2 \frac{n^{2a}}{j^{2a+1}} \sum_{K=1}^{n} \frac{K^{2b+3}}{n^{2b+4}} := A_{3n}(x) + A_{4n}(x).$$

$$A_{3n}(x) \sim \frac{1}{j^{2a+1}} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{k^{2a+3}}{|k+j|^2\,|k-j|^2} \sim \frac{1}{j^{2a+1}} \left(\frac{j^{2a+4}}{j^4} + \frac{j^{2a+3}}{j^2} + j^{2a} \right)$$

$$= O(1).$$

(Here, as well as in the estimate above for A_{1n}, we have decomposed the sum into three summations, from 1 to $j/2$, from $j/2$ to $2j$, and from $2j+1$ to n.)

$$A_{4n}(x) \sim \frac{n^{2a-2b-4}}{j^{2a+1}} n^{2b+4} \sim \frac{n^{2a}}{j^{2a+1}} = o(1) \qquad \text{since} \quad a<0 \text{ and } b > -1.5.$$

A similar result holds for $x \leqslant -1/2$. This proves (20). The last statement in Theorem 1 then follows from Proposition 2.

We remark that we have shown that (20) holds for all bounded f and not only for continuous f.

4. Convergence of $I(kH_n(f))$ for $w = w_{\alpha\beta}$

We are now in position to prove a theorem corresponding to Theorem 1 for $I(kH_n(f))$.

THEOREM 2. *Let p and q be nonnegative integers, let $w = w_{\alpha\beta}$ satisfy* (8). *Then* (4) *holds for all $f \in R(J)$.*

Proof. We need only show that

$$\|H_n(f)\|_\infty = O(1) \tag{27}$$

for then we have that

$$|I(k(f - H_n(f)))| \leqslant |I(k(f - S_n(f)))| + |I(k(S_n(f) - H_n(f)))|$$

$$:= I_1 + I_2,$$

$I_1 = o(1)$ by Theorem 1, and $I_2 = o(1)$ as in the proof of Proposition 2 using (21), (20), and (27).

The proof of (27) is given in the following lemma which is a little stronger than required in the above proof.

LEMMA 2. *Assume that* $\|f\|_\infty = O(1)$ *and that*

$$-1.5 \leqslant a,\ b \leqslant 0, \qquad |a-b| \leqslant 1, \tag{28}$$

then (27) *holds.*

Proof. Assume first that (8) holds. Then, if $x \geqslant 0$, say

$$\sum_{k=1}^{n} |v_{kn}(x)|\, a_{kn}(x) \leqslant \sum_{k=1}^{n} |v_{kn}(x) - 1|\, a_{kn}(x) + \sum_{k=1}^{n} a_{kn}(x)$$
$$:= A_n(x) + \bar{B}_n(x).$$

By the formulae in the proof of Theorem 1, it follows that $A_n(x) = O(1)$. Further, with $K := n - k - 1$

$$\bar{B}_n(x) \leqslant C \left[\frac{n^{2a}}{j^{2a+1}} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{k^{2a+3}}{n^{2a+4}} \frac{n^4}{|k+j|^2\,|k-j|^2} + \frac{n^{2a}}{j^{2a}} \sum_{K=1}^{n} \frac{K^{2b+3}}{n^{2b+4}} \right\}$$
$$:= C_n + D_n.$$

Here, $D_n \leqslant A_{2n} = O(1)$, where A_{2n} has been defined above, and

$$C_n \sim \frac{1}{j^{2a+1}} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{k^{2a+3}}{|k+j|^2\,|k-j|^2} \sim \frac{1}{j^{2a+1}} \left\{ \frac{j^{2a+4}}{j^4} + j^{2a+1} + j^{2a} \right\} = O(1).$$

Now, let $a = 0$, say. Then $b \geqslant -1$ so that X_n is (p, q)-normal, i.e., $v_{inpq} \geqslant 0$ on J for $i = 1, \ldots, n$. Then (27) follows from (14) and (15). This completes the proof of Lemma 2 and hence of Theorem 2.

5. CONVERGENCE RESULTS FOR $w \in GJ$

In this section we prove convergence theorems for (4) and (9) when $w = \psi w_{\alpha\beta} \in GJ$ with $\psi \not\equiv 1$. In this section $H_n(f)$ will mean $H_n(f; w)$ and $\bar{H}_n(f)$ will denote $H_n(f; w_{\alpha\beta})$, and similarly for other quantities; for example, the sets $\{x_{kn}\}$ and $\{\bar{x}_{kn}\}$ are the zeros of $p_n(x; w)$ and $p_n(x; w_{\alpha\beta})$, respectively. We also need the Christoffel function $\lambda_n(w)$ defined by

$$\lambda_n(w; x) := \left(\sum_{k=0}^{n-1} p_k^2(x; w) \right)^{-1}.$$

Our first theorem corresponds to the (p, q)-ρ-normal case.

THEOREM 3. *If* (7) *holds, then* (4) *holds for all* $f \in R(J)$.

Proof. By [10, Sect. 4.2], for any $\alpha, \beta > -1$

$$\bar{v}_{kn}(x) = 1 - \frac{a - b + (a + b + 2)\, \bar{x}_{kn}}{1 - \bar{x}_{kn}^2} (x - \bar{x}_{kn}), \qquad 1 \leq k \leq n.$$

By [10, (3.5); 5, (3.9)], for any $\alpha, \beta > -1$

$$\begin{aligned} v_{kn}(x) &= 1 + \left\{ \frac{p}{1 - x_{kn}} - \frac{q}{1 + x_{kn}} + \frac{\lambda_n'(w; x_{kn})}{\lambda_n(w'; x_{kn})} \right] (x - x_{kn}) \\ &= (\text{by [5, Lemma 4.3]}) \quad 1 + \left[\frac{p}{1 - x_{kn}} - \frac{q}{1 + x_{kn}} \right. \\ &\quad \left. - \frac{\alpha - \beta + (\alpha + \beta + 2)\, x_{kn}}{1 - x_{kn}^2} + C_{kn} \right] (x - x_{kn}) \\ &= 1 - \frac{a - b + (a + b + 2)\, x_{kn}}{1 - x_{kn}^2} (x - x_{kn}) + C_{kn}(x - x_{kn}) \\ &:= \tilde{v}_{kn}(x; w) + C_{kn}(x - x_{kn}), \qquad \text{where} \quad C_{kn} = O(1), \quad 1 \leq k \leq n. \end{aligned}$$

If we now define

$$\tilde{H}_n(f; w; x) := \sum_{k=1}^{n} \tilde{v}_{kn}(x)\, a_{kn}(w; x)\, f_{kn} + \psi_n(x)\, f_0 + \bar{\psi}_n(x)\, f_{n+1}$$

and

$$\mathscr{E}_n(f; w; x) := \sum_{k=1}^{n} C_{kn} f_{kn} a_{kn}(w; x)(x - x_{kn}),$$

then

$$H_n(f; w; x) = \tilde{H}_n(f; w; x) + \mathscr{E}_n(f; w; x), \qquad \alpha, \beta > -1. \qquad (28)$$

If (7) holds, it is easy to see that $\tilde{v}_{kn} > 0$ on J, $1 \leq k \leq n$ so that $\tilde{H}_n$ is a positive operator. If we can show that $\mathscr{E}_n(f) = o(1)$ for bounded f, then it will follow that H_n is close to a positive operator on the set of functions bounded on J. This results follow from our next lemma.

LEMMA 3. *If* (7) *holds, then*

$$B_n(x) := \sum_{k=1}^{n} a_{kn}(w; x)\,|x - x_{kn}| = o(1).$$

Proof. As in the proof of Lemma 1

$$B_n(x) \leqslant C\,\frac{1}{n^2 j^{2a+1}} \sum_{\substack{k=1\\ k\neq j}}^{n} \frac{k^{2a+3}}{|k+j|\,|k-j|} + \frac{n^{2a-2b-4}}{j^{2a+1}} \sum_{K=1}^{n} K^{2b+3}$$

$$:= I_1 + I_2, \qquad \text{where} \quad x \approx x_{jn} \text{ and } K := n - k + 1,$$

$$I_1 \sim \frac{1}{n^2 j^{2a+1}} \left[\frac{j^{2a+4}}{j^2} + j^{2a+2} \log j + j^{2a} \right] < C\,\frac{j \log j}{n^2} = o(1)$$

$$I_2 \sim \frac{n^{2a}}{j^{2a+1}} = o(1).$$

We now show convergence of $H_{npq}(f)$ for continuous functions.

LEMMA 4. *If* (7) *holds, then for all* $f \in C(J)$, (5) *holds.*

Proof. We use an improvement of [10, Theorem 3.1] (cf. [8, (21) and (22)]) that (5) holds if

$$\sum_{k=1}^{n} |v_{kn}(x)\, a_{kn}(x)| = O(1) \tag{29}$$

$$\sum_{k=1}^{n} |a_{kn}(x)(x - x_{kn})| = o(1). \tag{30}$$

Equation (30) is the result in Lemma 3. To show (29) we write

$$\sum_{k=1}^{n} |a_{kn}(x)\, v_{kn}(x)| \leqslant \sum_{k=1}^{n} a_{kn}(x)\, \tilde{v}_{kn}(x) + |\bar{\mathscr{E}}_n(x)|. \tag{31}$$

The sum in the right hand side of (31) can be shown to be $O(1)$ as in the proof of Lemma 1 and the fact that $|\bar{\mathscr{E}}_n(x)| = o(1)$ follows from (30), which proves (5).

From (25) and Lemmas 3 and 4, it follows that if (7) holds, then

$$\|\tilde{H}_n(f) - f\|_\infty = o(1)$$

for $f \in C(J)$. Hence, since $\tilde{H}_n$ is a positive operator,

$$I(k(\tilde{H}_n(f)-f))=o(1)$$

for all $f \in R(J)$. Since

$$|I(k(H_n(f)-f))| \leqslant |I(k(\tilde{H}_n(f)-f))| + |I(k\mathscr{E}_n(f))|$$

and $\mathscr{E}_n(f)=o(1)$ for all $f \in R(J)$, our theorem is proved.

We conclude this section with two theorems for $w \in GJ$ corresponding to Theorems 1 and 2.

THEOREM 4. *Let p and q be nonnegative integers, let* $w=\psi w_{\alpha\beta} \in GJ$, *and let a, b satisfy* (8). *Then* (9) *holds for all* $f \in R(J)$.

Proof. This is as in the proof of Theorem 1, using $\bar{H}_n$ instead of H_n.

THEOREM 5. *With the same hypotheses as in Theorem* 4, (4) *holds for all* $f \in R(J)$.

Proof. This is as in Theorem 2, using H_n or $\tilde{H}_n$ instead of $\bar{H}_n := H_n(w_{\alpha\beta})$.

6. CONCLUDING REMARKS

In this paper, we have proved convergence results for product integration which are valid for all $k \in L_1(J)$. If we place some restrictions on k, we can get stronger results. Thus, for the case $p=q=0$, it has been shown in [4, Theorem 5] that if $w \in GJ$ and $f \in C(J)$, then

$$\|w(f-H_n(f))\|_1=o(1). \tag{32}$$

Hence, if $\|k/w\|_\infty \leqslant C$,

$$I(kH_n(f)) \rightarrow I(kf) \tag{33}$$

as $n \rightarrow \infty$ for all $f \in C(J)$. A similar situation occurs for $p=q=0$ when w is a Freud or Erdös weight on the real line and we use the L_1 convergence results in [3].

We also have the results in [1] for the case where $\|k\|_\infty \leqslant C$, namely that if $w \in GJ$, $p=q=0$, and $-1<\alpha$, $\beta<1$ or if $p=q=2$ and $1/2<\alpha$, $\beta<1$ then (33) holds for all $f \in R(J)$.

Finally, using [4, Theorem 2], we have a convergence result for unbounded functions, namely, that if $w \in GJ$, $p = q = 0$, and $f \in R(J_\delta)$ for every $\delta \in (0, 1)$ and satisfies

$$|f(x)| \leqslant C w_{\gamma\delta}(x) \qquad \text{in } J,$$

where γ, δ are arbitrary, then $\|w(f - H_n(f))\|_1 = o(1)$ provided that

$$w w_{\gamma\delta}(1 - x^2)^{1/4} \in L_1(J).$$

Hence, if $\|k/w\|_\infty \leqslant A$ then (33) can hold for functions with singularities at the endpoints.

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