Nonexistence results for a class of fractional elliptic boundary value problems

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Abstract

In this paper we study a class of fractional elliptic problems of the form

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$. We prove nonexistence of positive solutions when Ω is star-shaped and f is supercritical. We also derive a nonexistence result for subcritical f in some unbounded domains. The argument relies on the method of moving spheres applied to a reformulated problem using the Caffarelli–Silvestre extension (Caffarelli and Silvestre (2007) [11]) of a solution of the above problem.
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Keywords: Fractional Laplacian; Nonexistence; Star-shaped domain; Caffarelli–Silvestre extension

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1. Introduction

Let $s \in (0, 1)$ and $N > 2s$. In the present paper, we are concerned with the nonexistence of positive functions solving the fractional elliptic semilinear problem

$$
\begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\tag{1.1}
$$

in a domain $\Omega \subset \mathbb{R}^N$. Problems of this type received immensely growing attention recently, while different versions of the nonlocal operator $(-\Delta)^s$ related to Dirichlet boundary conditions are studied (see e.g. [5,8,12,15,29,32]). The version we consider in (1.1) is the one most commonly considered in analysis and probability theory. In probabilistic terms, it can be defined as the generator of the $2s$-stable process in Ω killed upon leaving Ω. For our purposes, it is more convenient to give an analytic definition. We define $(-\Delta)^s$ for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ by

$$
(-\Delta)^s \varphi(x) = P.V. \int_{\mathbb{R}^N} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy
\tag{1.2}
$$

for $x \in \mathbb{R}^N$, where P.V. stands for the principle value integral. We point out that this definition differs from the standard definition by a multiplicative constant. Via Fourier transform, (1.2) is equivalent to

$$
C_{N,s} \widehat{(-\Delta)^s \varphi}(\xi) = |\xi|^{2s} \widehat{\varphi}(\xi) \quad \text{for } \xi \in \mathbb{R}^N
$$

with the normalization constant $C_{N,s} = s(1 - s)\pi^{-N/2} 2^{2s} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)}$, see e.g. [7, Remark 3.11]. Thanks to Lemma 2.1 below, for any $\varphi \in C_c^\infty(\mathbb{R}^N)$ we have the estimate

$$
\left| (-\Delta)^s \varphi(x) \right| \leqslant C \frac{\|\varphi\|_{C^2(\mathbb{R}^N)}}{1 + |x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N,
\tag{1.3}
$$

where C only depends on the support of φ. Let $\mathcal{L}_s^1$ denote the space of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that $\int_{\mathbb{R}^N} \frac{|u|}{1+|x|^{N+2s}} \, dx < \infty$, and let Ω be an open set of $\mathbb{R}^N$. We define the Hilbert space $\mathcal{D}^{s,2}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to the norm $\| \cdot \|_{\mathcal{D}^{s,2}}$ induced by the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{D}^{s,2}}$ given by

$$
\langle u, v \rangle_{\mathcal{D}^{s,2}} = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy.
\tag{1.4}
$$

We note that if Ω is a bounded Lipschitz domain, then $\mathcal{D}^{s,2}(\Omega)$ coincides with the Sobolev space $\{u \in H^s(\mathbb{R}^N): u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$. We also observe that – for any $u \in \mathcal{D}^{s,2}(\Omega)$ – the Hölder and the Hardy–Littlewood–Sobolev inequalities imply that

$$
\int_{\mathbb{R}^N} \frac{|u|}{1 + |x|^{N+2s}} \, dx \leqslant C \|u\|_{\mathcal{D}^{s,2}} \quad \text{for all } u \in \mathcal{D}^{s,2}(\Omega)
$$

with a constant $C > 0$. In other words, $\mathcal{D}^{s,2}(\Omega)$ is continuously embedded in $\mathcal{L}_s^1$. As a consequence, by recalling (1.3) we may define $(-\Delta)^s u$ for every $u \in \mathcal{D}^{s,2}(\Omega)$ as a distribution by

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx = \langle u, \varphi \rangle_{\mathcal{D}^{s,2}} \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

In particular, given $f \in L^1_{loc}(\Omega)$, we note that $u \in \mathcal{D}^{s,2}(\Omega)$ solves the problem $(-\Delta)^s u = f$ if and only if

$$\langle u, \varphi \rangle_{\mathcal{D}^{s,2}} = \int_\Omega f\varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.5}$$

Throughout the paper, when we refer to solution of (1.1), we mean distributional solutions $u \in \mathcal{D}^{s,2}(\Omega)$ in the sense of (1.5) with $f = f(\cdot, u(\cdot)) \in L^1_{loc}(\Omega)$. In order to state the main result of the present paper, we need to introduce a definition of a star domain which is slightly more general than usually considered in the literature. We say that an open set $\Omega \subset \mathbb{R}^N$ is star-shaped (or a star domain) with respect to the origin $0 \in \overline{\Omega}$ if for every $x \in \Omega$ we have $tx \in \Omega$ for $0 < t \leqslant 1$. So in contrast to the standard definition, we also allow the star center to lie on the boundary of Ω. This will be crucial in deriving results in unbounded domains. In particular, the punctured open unit ball $B_1(0) \setminus \{0\}$ is star-shaped with respect to the origin according to our definition. Our main result is the following.

Theorem 1.1. *Assume that Ω is bounded and star-shaped with respect to the origin $0 \in \overline{\Omega}$. Suppose that $f : \overline{\Omega} \setminus \{0\} \times [0, \infty) \to \mathbb{R}$ is locally Lipschitz in its second variable uniformly in compact subsets of $\overline{\Omega} \setminus \{0\}$ and is supercritical in the sense that*

$$\begin{cases} \text{the function } \lambda \mapsto \lambda^{-(N+2s)/(N-2s)} f(\lambda^{-2/(N-2s)} x, \lambda u) \\ \text{is non-decreasing on } [1, \infty) \quad \text{for every } x \in \Omega \setminus \{0\}, \; u \geqslant 0. \end{cases} \tag{1.6}$$

Then (1.1) has no positive solution $u \in C(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{s,2}(\Omega)$.

We remark that for C^1-nonlinearities $f : \overline{\Omega} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}$ the supercriticality assumption (1.6) is equivalent to

$$H_f(x, u) \geqslant 0 \quad \text{for all } (x, u) \in \Omega \setminus \{0\} \times [0, \infty), \tag{1.7}$$

where

$$H_f(x, u) := u \frac{\partial}{\partial u} f(x, u) - \frac{N+2s}{N-2s} f(x, u) - \frac{2}{N-2s} x \cdot \nabla_x f(x, u). \tag{1.8}$$

As a first consequence of Theorem 1.1 we have the following Pohozaev type result.

Corollary 1.2. *Assume that Ω is bounded and star-shaped with respect to the origin, and let $V \in C^1(\Omega \setminus \{0\})$ satisfy*

$$sV(x) + \frac{1}{2}\nabla V(x) \cdot x \geqslant 0 \quad \text{for all } x \in \Omega \setminus \{0\}.$$

Let $u \in \mathcal{D}^{s,2}(\Omega) \cap C(\mathbb{R}^N \setminus \{0\})$, $u \geqslant 0$ in $\mathbb{R}^N$ be such that

$$\begin{cases} (-\Delta)^s u + V(x)u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.9}$$

for some $p \geqslant \frac{N+2s}{N-2s}$. Then $u = 0$ in $\mathbb{R}^N$.

In the case where Ω is the unit ball in $\mathbb{R}^N$ and $V \equiv 0$, this gives an affirmative answer to a conjecture of Birkner, López-Mimbela and Wakolbinger, see [3, p. 91]. We note that existence results for problem (1.9) in the subcritical range $1 < p < \frac{N+2s}{N-2s}$ and for more general subcritical nonlinearities have been obtained recently by the first author in [17] and by Servadei and Valdinoci in [29].

In our next result the linear term is related to the relativistic Hardy inequality, see [18] and [17].

Corollary 1.3. *Assume that Ω is bounded and star-shaped with respect to the origin and let $u \in \mathcal{D}^{s,2}(\Omega) \cap C(\mathbb{R}^N \setminus \{0\})$, $u \geqslant 0$ in $\mathbb{R}^N$ be such that*

$$\begin{cases} (-\Delta)^s u - \gamma |x|^{-2s} u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.10}$$

for some $\gamma \in \mathbb{R}$ and $p \geqslant \frac{N+2s}{N-2s}$. Then $u = 0$ in $\mathbb{R}^N$.

Our next result is concerned with a singular nonlinearity.

Corollary 1.4. *Assume that Ω is bounded and star-shaped with respect to the origin and let $u \in \mathcal{D}^{s,2}(\Omega) \cap C(\mathbb{R}^N \setminus \{0\})$, $u \geqslant 0$ in $\mathbb{R}^N$ be such that*

$$\begin{cases} (-\Delta)^s u = |x|^{-\sigma} u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.11}$$

for some $\sigma \in \mathbb{R}$ and $p \geqslant \max\{1, \frac{N+2s-2\sigma}{N-2s}\}$. Then $u = 0$ in $\mathbb{R}^N$.

This result should be seen in the context of the criticality of $q = \frac{2(N-\sigma)}{N-2s} = \frac{N+2s-2\sigma}{N-2s} + 1$ for the embedding of the Sobolev space $\mathcal{D}^{s,2}(\Omega)$ in the weighted space $L^q(\Omega; |x|^{-\sigma})$. More precisely, if $N > \max(\sigma, 2s)$ and the underlying domain is bounded, $\mathcal{D}^{s,2}(\Omega)$ is continuously embedded in $L^q(\Omega; |x|^{-\sigma})$ if and only if $q \leqslant \frac{2(N-\sigma)}{N-2s}$, and the embedding is compact iff $q < \frac{2(N-\sigma)}{N-2s}$. Note also that the existence of the embeddings in the subcritical range follows from the fact that

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2(N-\sigma)/(N-2s)}(\mathbb{R}^N; |x|^{-\sigma}),$$

and this latter embedding can be seen as a version of the Stein–Weiss inequality [31].

Our next result is concerned with a class of unbounded domains. Slightly extending a notion from [27], we say that an open set Ω is *star-shaped with respect to infinity* if there exists a point $e \in \mathbb{R}^N \setminus \overline{\Omega}$ such that for every point $x \in \Omega$ the half-line $\{e + t(x - e): t \geqslant 1\}$ is contained in Ω. Up to suitable translation, it is equivalent to require $0 \notin \overline{\Omega}$ and that $\mathbb{R}^N \setminus \overline{\Omega}$ is star-shaped with respect to 0 in the sense defined earlier.

Theorem 1.5. *Assume that Ω is star-shaped with respect to infinity. Let $u \in \mathcal{D}^{s,2}(\Omega) \cap C(\mathbb{R}^N)$ be nonnegative and such that*

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{1.12}$$

for some $1 \leqslant p \leqslant \frac{N+2s}{N-2s}$. Then $u = 0$ in $\mathbb{R}^N$.

In fact, we will deduce Theorem 1.5 from Theorem 1.1 via a variant of the classical Kelvin transform, see Sections 2 and 3 below for details.

Theorem 1.5 in particular applies to the cone-like domains $\Omega_\tau := \{x \in \mathbb{R}^N \setminus \{0\}: \frac{x_N}{|x|} > \tau\}$ for $\tau \in (-1, 1)$. Here one may take $e = -e_N$, where e_N is the N-th coordinate vector, in the definition of star-shapedness at infinity. Since the half-space $\mathbb{R}^N_+$ is a particular case with $\tau = 0$, we deduce the following corollary.

Corollary 1.6. *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N_+) \cap C(\mathbb{R}^N)$ be nonnegative and such that*

$$\begin{cases} (-\Delta)^s u = u^p & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+ \end{cases} \tag{1.13}$$

for some $1 \leqslant p \leqslant \frac{N+2s}{N-2s}$. Then $u = 0$ in $\mathbb{R}^N$.

We remark that Theorem 1.5 does not apply to the case $\Omega = \mathbb{R}^N$. Indeed, in this case the critical problem with $p = \frac{N+2s}{N-2s}$ admits positive solutions which have been classified completely in [14]. Moreover, in the case $\Omega = \mathbb{R}^N$, $s \in [1/2, 1)$ and $1 < p < \frac{N+2s}{N-2s}$, a nonexistence result has been obtained very recently and independently in [15] by de Pablo and Sánchez, see also [23] for $s = 1/2$ and $1 < p < \frac{N+2s}{N-2s}$.

In order to explain our approach to obtain the nonexistence results, we need to compare (1.1) with the classical problem

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

For (1.14), the analogue of Theorem 1.1 is true, and for strictly star-shaped C^1-domains Ω and C^1-nonlinearities f on $\Omega \times [0, \infty)$ satisfying additionally $f(\cdot, 0) = 0$ it can be derived from the Pohozaev type integral identity

$$\int\limits_{\Omega}\int\limits_{0}^{u(x)} H_f(x,t)\,dt\,dx + \frac{1}{N-2}\int\limits_{\partial\Omega} u_\nu^2 x \cdot \nu\,d\sigma(x) = 0, \tag{1.15}$$

see e.g. [26, Theorem 5.2]. Here H_f is defined as in (1.8). Indeed, by (1.7) and the star-shapedness of Ω, the LHS of (1.15) is nonnegative, and by unique continuation it is strictly positive if $u \not\equiv 0$. The above integral identity can be derived by multiplying (1.14) with the functions u and $x \mapsto x \cdot \nabla u$ respectively and integrating by parts. The same strategy does not work for (1.1) since the problem is nonlocal and does not allow a simple integration by parts formula as in the case $s = 1$. More severely, in the case $0 < s < 1$ solutions of (1.1) are not of class C^1 up to the boundary even if the underlying domain is smooth. In particular, if $x \mapsto f(x, u(x)) \geqslant 0$ is a nonnegative nontrivial function on Ω, then any solution u of (1.1) fails to possess a finite normal derivative u_ν on $\partial\Omega$, see e.g. [3, Lemma 4.3].

The approach we follow here is inspired by Reichel and Zou [27] who used the technique of moving spheres to prove nonexistence results for cooperative elliptic systems. The moving sphere method can be seen as a variant of the method of moving hyperplanes (see e.g. [1,2,20, 21,28]) and has been widely used to classify positive solutions of nonlinear elliptic problems, see e.g. [24] and the references therein. For the special case where the underlying domain is the entire space $\mathbb{R}^N$, it has also been applied to problems involving the fractional Laplacian, see the aforementioned recent paper [15] of de Pablo and Sánchez and also [14]. Unlike as in [27], we are not able to implement a moving sphere argument directly in the present setting, so instead – as in [15] – we first transform (1.1) to a local problem by considering the Caffarelli–Silvestre extension of a solution u on $\mathbb{R}^{N+1}_+$, see [11] and also [7,17]. This extension satisfies $w = u$ on Ω and solves in some weak sense (see Section 2 for details) the boundary value problem

$$\begin{cases} \operatorname{div}\!\left(t^{1-2s}\nabla w\right) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\ -c_{N,s}\,\lim\limits_{t\to 0^+} t^{1-2s}\dfrac{\partial w}{\partial t} = f(x, w) & \text{on } \Omega, \end{cases} \tag{1.16}$$

with the positive normalization constant $c_{N,s} = \frac{\pi^{N/2}\Gamma(s)}{2s\,\Gamma(\frac{N+2s}{2})}$ (note that this constant is different from the one noted e.g. in [7, Remark 3.11] due to our normalization of $(-\Delta)^s$). Here and in the following we write $z = (x, t) \in \mathbb{R}^{N+1}_+$ with $x \in \mathbb{R}^N$ and $t > 0$, and we identify $\mathbb{R}^N$ with $\partial\mathbb{R}^{N+1}_+$, so that Ω is contained in $\partial\mathbb{R}^{N+1}_+$. We will then apply the moving sphere argument to the local problem (1.16) in place of (1.1). We note that the Caffarelli–Silvestre extension of a solution of (1.1) has received considerable attention in recent years due to its usefulness in the context of many different problems, see e.g. [9,10,13,16,30].

We should mention that – in contrast to the nonexistence results for (1.14) based on the Pohozaev type identity – our approach does not extend to sign changing solutions. The existence resp. nonexistence of sign changing solutions of (1.1) under supercriticality and star-shapedness assumptions therefore remains an open problem.

Finally, we would like to compare (1.1) with the related problem

$$\begin{cases} A^s u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Here A stands for the negative Laplacian as a self-adjoint operator in $L^2(\Omega)$ with domain

$$\left\{u \in H_0^1(\Omega) \colon \Delta u \in L^2(\Omega) \text{ as a distribution}\right\},$$

and A^s is the corresponding power in spectral theoretic sense. Although problems (1.1) and (1.17) look similar, there are crucial differences as discussed e.g. in [17]. In particular, solutions of (1.17) have in general much better boundary regularity than solutions of (1.1), and this can also be seen when comparing the corresponding extended problems. We point out that in [5,8, 12,15,32] a variant of the Caffarelli–Silvestre extension for solutions of (1.17) was considered which preserves the regularity properties up to the boundary. Moreover, nonexistence results for (1.17) have recently been proved in [5,32] via a Pohozaev type integral identity for the extended problem. As we pointed out before, such an approach is not available for (1.1) resp. (1.16) due to the lack of boundary regularity of solutions.

The paper is organized as follows. In Section 2 we discuss a suitable weak notion of solution of (1.16), and we study how problems (1.1) and (1.16) transform under a Kelvin type transform. We also formulate two versions of boundary maximum principles related to a linearized version of problem (1.16). Since this section deals with all technical aspects of the problem, the remaining parts of the proofs of our main results are relatively short, and they are contained in Section 3.

2. Some preliminaries

Throughout the paper, we consider $s \in (0, 1)$ and assume that $N > 2s$. In this section we collect preliminary tools related to (1.1) and the reformulated version (1.16). We also need to introduce some definitions concerning notions of weak solutions. If $\Omega \subset \mathbb{R}^N$ is an open set and $f \in L_{loc}^1(\Omega)$, we say that $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is a distributional solution of $(-\Delta)^s u = f$ in Ω if

$$\langle u, \varphi \rangle_{\mathcal{D}^{s,2}} = \int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{2.1}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}^{s,2}}$ is defined in (1.4). Note that by considering $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ we do not prescribe u on $\mathbb{R}^N \setminus \Omega$ here. We start with the following result.

Lemma 2.1. *Let Ω be a bounded open set. Then there exists a constant $C = C(N, s, \Omega) > 0$ such that for all $\varphi \in C_c^2(\Omega)$, $x \in \mathbb{R}^N$ and $\varepsilon \in (0, 1)$ we have*

$$\left| \int\limits_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy \right| \leqslant \frac{C\|\varphi\|_{C^2(\mathbb{R}^N)}}{1 + |x|^{N+2s}}. \tag{2.2}$$

Proof. For $x \in \mathbb{R}^N$ and $\varepsilon > 0$, integration by parts yields

$$\int\limits_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \, dy$$

$$= \int\limits_0^1 \int\limits_{|x-y|>\varepsilon} \nabla \varphi\big(x + t(y - x)\big) \cdot \frac{x - y}{|x-y|^{N+2s}} \, dy \, dt$$

$$= \frac{1}{N + 2s - 2} \left(\int_0^1 \int_{|x-y|=\varepsilon} \nabla\varphi\big(x + t(y - x)\big) \cdot (y - x)|x - y|^{-N-2s+1} \, d\sigma(y) \, dt \right.$$

$$\left. + \int_0^1 t \int_{|x-y|>\varepsilon} \Delta\varphi\big(x + t(y - x)\big)|x - y|^{-N-2s+2} \, dy \, dt \right),$$

whereas

$$\int_0^1 \int_{|x-y|=\varepsilon} \nabla\varphi\big(x + t(y - x)\big) \cdot (y - x)|x - y|^{-N-2s+1} \, d\sigma(y) \, dt$$

$$= \varepsilon^{1-2s} \int_0^1 \int_{S^{N-1}} \nabla\varphi(x + t\varepsilon\sigma) \cdot \sigma \, d\sigma \, dt$$

$$= \varepsilon^{1-2s} \int_0^1 \int_{S^{N-1}} \nabla\varphi(x) \cdot \sigma \, d\sigma \, dt + \varepsilon^{2-2s} \int_0^1 t \int_0^1 \int_{S^{N-1}} D^2\varphi(x + \varepsilon t\tau\sigma)[\sigma] \cdot \sigma \, d\sigma \, d\tau \, dt$$

and, by oddness,

$$\int_{S^{N-1}} \nabla\varphi(x) \cdot \sigma \, d\sigma = \sum_{i=1}^N \frac{\partial\varphi}{\partial x^i}(x) \int_{S^{N-1}} \sigma^i \, d\sigma = 0.$$

Consequently,

$$\int_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy = \frac{1}{N + 2s - 2} \int_0^1 t \int_{|x-y|>\varepsilon} \Delta\varphi\big(x + t(y - x)\big)|x - y|^{-N-2s+2} \, dy \, dt$$

$$+ \frac{\varepsilon^{2(1-s)}}{N + 2s - 2} \int_0^1 t \int_0^1 \int_{S^{N-1}} D^2\varphi(x + \varepsilon t\tau\sigma)[\sigma] \cdot \sigma \, d\sigma \, d\tau \, dt, \quad (2.3)$$

while

$$\left| \int_0^1 t \int_0^1 \int_{S^{N-1}} D^2\varphi(x + \varepsilon t\tau\sigma)[\sigma] \cdot \sigma \, d\sigma \, d\tau \, dt \right| \leqslant C_1 \|\varphi\|_{C^2(\mathbb{R}^N)} \quad (2.4)$$

with a constant $C_1 > 0$ depending only on N and s. We now fix $R > 0$ such that $\Omega \subset B(0, R)$, and we first consider $x \in \mathbb{R}^N \setminus B(0, 4R)$. Then $|x - y| \geqslant R + \frac{|x|}{2}$ for $y \in \Omega$ and therefore

$$\left| \int\limits_{|x-y|>\varepsilon} \frac{\varphi(x) - \varphi(y)}{|x - y|^{N+2s}} \, dy \right| \leqslant \int\limits_{|y|\leqslant R} \frac{|\varphi(y)|}{(R + \frac{|x|}{2})^{N+2s}} \, dy \leqslant C_2 \frac{\|\varphi\|_{C^2(\mathbb{R}^N)}}{1 + |x|^{N+2s}} \tag{2.5}$$

with a constant $C_2 > 0$ depending only on R, N and s. Next we consider $x \in B(0, 4R)$ and note that, for every $t \in (0, 1)$,

$$|x - y| \leqslant \frac{R + |x|}{t} \leqslant \frac{5R}{t} \quad \text{if } |x + t(y - x)| \leqslant R,$$

and

$$\Delta\varphi\big(x + t(y - x)\big) = 0 \quad \text{if } |x + t(y - x)| \geqslant R.$$

Hence for $x \in B(0, 4R)$ we have

$$\left| \int\limits_0^1 t \int\limits_{|x-y|>\varepsilon} \Delta\varphi\big(x + t(y - x)\big)|x - y|^{-N-2s+2} \, dy \, dt \right|$$

$$\leqslant \|\varphi\|_{C^2(\mathbb{R}^N)} \int\limits_0^1 t \int\limits_{|x+t(y-x)|<R} |x - y|^{-N-2s+2} \, dy \, dt$$

$$\leqslant \|\varphi\|_{C^2(\mathbb{R}^N)} \int\limits_0^1 t \int\limits_{|x-y|\leqslant \frac{5R}{t}} |x - y|^{-N-2s+2} \, dy \, dt$$

$$\leqslant \|\varphi\|_{C^2(\mathbb{R}^N)} |S^{N-1}| \left| \int\limits_0^1 t \int\limits_0^{\frac{5R}{t}} r^{1-2s} \, dr \, dt \right|$$

$$= \|\varphi\|_{C^2(\mathbb{R}^N)} |S^{N-1}| \frac{(5R)^{2s-2}}{2 - 2s} \int\limits_0^1 t^{-1+2s} \, dt$$

$$= C_3 \|\varphi\|_{C^2(\mathbb{R}^N)}, \tag{2.6}$$

with a constant $C_3 > 0$ depending only on R, N and s. Combining (2.3), (2.4), (2.5) and (2.6), we find that there exists a constant $C > 0$ depending only on R', N and s such that (2.2) holds, as claimed. $\quad\square$

Next, we consider the conformal diffeomorphism

$$\kappa : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}, \qquad \kappa(x) = \frac{x}{|x|^2}. \tag{2.7}$$

It is easy to see that

$$\left|\kappa(x) - \kappa(y)\right| = \frac{|x - y|}{|x||y|} \quad \text{for every } x, y \in \mathbb{R}^N \setminus \{0\}, \tag{2.8}$$

and that the Jacobian determinant of κ satisfies

$$\left|\det J_\kappa(x)\right| = |x|^{-2N}.$$

In the following, for a measurable function u on $\mathbb{R}^N$, we a.e. define Ku on $\mathbb{R}^N$ by

$$Ku(x) = |x|^{2s-N} u\big(\kappa(x)\big).$$

The map K is usually called *Kelvin transform*, and it is a well-known tool in potential theory and partial differential equations. It has also been studied in detail in a probabilistic framework for stable processes, see [4] and the references therein. Here we need the following property of K.

Lemma 2.2. *The map K defines an isometry on $\mathcal{D}^{s,2}(\mathbb{R}^N)$, i.e. for every $u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ we have $Ku, Kv \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and*

$$\langle u, v \rangle_{\mathcal{D}^{s,2}} = \langle Ku, Kv \rangle_{\mathcal{D}^{s,2}}. \tag{2.9}$$

Proof. Since $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ is dense in $C_c^\infty(\mathbb{R}^N)$ with respect to the $\mathcal{D}^{s,2}(\mathbb{R}^N)$-norm as a consequence of our general assumption $N > 2s$ (see [25, p. 397]), it suffices to show (2.9) for $u, v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. By changing variables and using (2.8), we have

$$\begin{aligned}
\langle u, v \rangle_{\mathcal{D}^{s,2}} &= \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \\
&= \int_{\mathbb{R}^{2N}} \frac{(u(\kappa(x)) - u(\kappa(y)))(v(\kappa(x)) - v(\kappa(y)))}{|x - y|^{N+2s}|x|^{-N-2s}|y|^{-N-2s}} |x|^{-2N}|y|^{-2N} \, dx \, dy \\
&= \int_{\mathbb{R}^{2N}} \frac{(u(\kappa(x)) - u(\kappa(y)))(v(\kappa(x)) - v(\kappa(y)))}{|x - y|^{N+2s}} |x|^{-N+2s}|y|^{-N+2s} \, dx \, dy.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\big(u\big(\kappa(x)\big) - u\big(\kappa(y)\big)\big)\big(v\big(\kappa(x)\big) - v\big(\kappa(y)\big)\big)|x|^{-N+2s}|y|^{-N+2s} \\
&= \big(Ku(x) - Ku(y)\big)\big(Kv(x) - Kv(y)\big) + Ku(x)v\big(\kappa(x)\big)\big[|y|^{2s-N} - |x|^{2s-N}\big] \\
&\quad + Ku(y)v\big(\kappa(y)\big)\big[|x|^{2s-N} - |y|^{2s-N}\big].
\end{aligned}$$

We therefore have

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy$$

$$= \int_{\mathbb{R}^{2N}} \frac{(Ku(x) - Ku(y))(Kv(x) - Kv(y))}{|x - y|^{N+2s}} \, dx \, dy$$

$$+ 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{Ku(x)v(\kappa(x))[|y|^{2s-N} - |x|^{2s-N}]}{|x - y|^{N+2s}} \, dy \, dx.$$

It thus remains to prove that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{Ku(x)v(\kappa(x))[|y|^{2s-N} - |x|^{2s-N}]}{|x - y|^{N+2s}} \, dy \, dx = 0. \qquad (2.10)$$

To show this, we consider $f \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ defined by $f(x) = Ku(x)v(\kappa(x))$. Since

$$\int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{f(x)|y|^{2s-N}}{|x - y|^{N+2s}} \, dy \, dx < \infty, \qquad \int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{f(x)|x|^{2s-N}}{|x - y|^{N+2s}} \, dy \, dx < \infty,$$

we have by Fubini's theorem

$$\int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{f(x)[|y|^{2s-N} - |x|^{2s-N}]}{|x - y|^{N+2s}} \, dy \, dx = \int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{|x|^{2s-N}(f(y) - f(x))}{|x - y|^{N+2s}} \, dy \, dx.$$

Note that $x \mapsto |x|^{2s-N} \in \mathcal{L}_s^1$. By Lemma 2.1, we have

$$\left| \int_{|x-y|>\varepsilon} \frac{f(x) - f(y)}{|x - y|^{N+2s}} \, dy \right| \leqslant \frac{C}{1 + |x|^{N+2s}} \quad \text{for all } \varepsilon \in (0, 1),$$

and therefore the dominated convergence theorem implies that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{|x-y|>\varepsilon} \frac{f(x)[|y|^{2s-N} - |x|^{2s-N}]}{|x - y|^{N+2s}} \, dy \, dx = \int_{\mathbb{R}^N} |x|^{2s-N}(-\Delta)^s f(x) \, dx.$$

Since $x \mapsto |x|^{2s-N}$ is the Riesz potential of order $2s$, we have (up to a constant)

$$\int_{\mathbb{R}^N} |x|^{2s-N}(-\Delta)^s f(x) \, dx = \big\langle (-\Delta)^s |x|^{2s-N}, f \big\rangle = \langle \delta, f \rangle = 0$$

in distributional sense, because f is supported away from the origin and δ is the Dirac mass at the origin. Hence we have proved (2.10) and the lemma then follows. $\square$

As a consequence, we get the following result, which is closely related to [4, Theorem 2]. We note that, unlike in the present paper, probabilistic techniques are used in [4].

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^N$ be an open set and*

$$\tilde{\Omega} := \kappa\big(\Omega \setminus \{0\}\big) \subset \mathbb{R}^N \setminus \{0\}.$$

Let $f \in L^1_{loc}(\Omega)$, and let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ solve $(-\Delta)^s u = f$ in Ω in distributional sense. Then $\tilde{u} = Ku$ is contained in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and solves $(-\Delta)^s \tilde{u} = \tilde{f}$ in distributional sense in $\tilde{\Omega}$, where $\tilde{f} \in L^1_{loc}(\tilde{\Omega})$ is given by $\tilde{f}(x) = |x|^{-(N+2s)} f(\frac{x}{|x|^2})$.

Moreover, if $u \in \mathcal{D}^{s,2}(\Omega)$, then $\tilde{u} \in \mathcal{D}^{s,2}(\tilde{\Omega})$.

Proof. Suppose first that $u \in \mathcal{D}^{s,2}(\Omega)$. Since, as noted before, $C_c^\infty(\Omega \setminus \{0\})$ is dense in $\mathcal{D}^{s,2}(\Omega)$, there exists a sequence $(\psi_n)_n$ in $C_c^\infty(\Omega \setminus \{0\})$ with $\|u - \psi_n\|_{\mathcal{D}^{s,2}} \to 0$ as $n \to \infty$. By (2.2), we then also have $\|Ku - K\psi_n\|_{\mathcal{D}^{s,2}} \to 0$ as $n \to \infty$. Since $K\psi_n \in \mathcal{D}^{s,2}(\tilde{\Omega})$ for all n, this implies $Ku \in \mathcal{D}^{s,2}(\tilde{\Omega})$.

Next we assume that $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ solves $(-\Delta)^s u = f$ in Ω in distributional sense. Applying the argument above to $\Omega = \mathbb{R}^N$ yields $\tilde{u} \in \mathcal{D}^{s,2}(\mathbb{R}^N \setminus \{0\}) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$. Moreover, for given $\tilde{\varphi} \in C_c^\infty(\tilde{\Omega})$, we may now write $\tilde{\varphi} = K\varphi$ with $\varphi \in C_c^\infty(\tilde{\Omega})$. By Lemma 2.2, we then have

$$\langle \tilde{u}, \tilde{\varphi} \rangle_{\mathcal{D}^{s,2}} = \langle u, \varphi \rangle_{\mathcal{D}^{s,2}} = \int_\Omega f\varphi \, dx = \int_{\tilde{\Omega}} (f \circ \kappa)(\varphi \circ \kappa) |\det J_\kappa| \, dx$$

$$= \int_{\tilde{\Omega}} f\big(\kappa(x)\big)\varphi\big(\kappa(x)\big)|x|^{-2N} \, dx = \int_{\tilde{\Omega}} \tilde{f}\tilde{\varphi} \, dx.$$

This shows the claim. $\quad\square$

Next, we introduce some notations related to the reformulated problem (1.16). As before, we write $z = (x, t) \in \mathbb{R}^{N+1}_+$ with $x \in \mathbb{R}^N$ and $t \in (0, \infty)$. Let $D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ denote the space of all functions $w \in H^1_{loc}(\mathbb{R}^{N+1}_+)$ such that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla w|^2 \, dz < \infty.$$

Formally introducing the operator $L_s := \mathrm{div}(t^{1-2s}\nabla)$ on $\mathbb{R}^{N+1}_+$, we say that a function $w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is *weakly L_s-harmonic* if

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla w \nabla \varphi \, dz = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^{N+1}_+).$$

By standard elliptic regularity, every weakly L_s-harmonic function $w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ belongs to $C^\infty(\mathbb{R}^{N+1}_+)$ and satisfies $\mathrm{div}(t^{1-2s}\nabla w) \equiv 0$ pointwise in $\mathbb{R}^{N+1}_+$. Moreover, w does not attain an interior maximum or minimum point in $\mathbb{R}^{N+1}_+$ unless w is constant. Note also that we have a well-defined continuous trace map

$$D^{1,2}\big(\mathbb{R}_+^{N+1}; t^{1-2s}\big) \to \mathcal{D}^{s,2}\big(\mathbb{R}^N\big)$$

(see e.g. [5]), and for the sake of simplicity we denote the trace of a function in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ with the same letter as the function itself. If $\varphi, \psi \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ and φ is weakly L_s-harmonic, we have the identity

$$c_{N,s} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla\varphi \nabla\psi \, dz = \int_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{N+2s}} \, dx \, dy \qquad (2.11)$$

with $c_{N,s}$ as in (1.16). Now, for an open set $\Omega \subset \mathbb{R}^N$, we denote by $D(\Omega, s)$ the closed subspace of functions in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ such that their trace on $\mathbb{R}^N$ is contained in $\mathcal{D}^{s,2}(\Omega)$. It is easy to see that every function $u \in \mathcal{D}^{s,2}(\Omega)$ has a unique weakly harmonic extension $H(u) \in D(\Omega, s)$ which can be found by minimizing the functional

$$w \mapsto \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla w|^2 \, dz$$

among all functions $w \in D(\Omega, s)$ satisfying $w = u$ on $\mathbb{R}^N$. Using this fact in the special case $\Omega = \mathbb{R}^N$ (in which $D(\mathbb{R}^N, s) = D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$) together with (2.11), we find that

$$\int_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{N+2s}} \, dx \, dy \leqslant c_{N,s} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla\varphi|^2 \, dz \qquad (2.12)$$

for all $\varphi \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$. Moreover, since $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$, there exists a constant $C > 0$ such that

$$\|\varphi\|^2_{L^{\frac{2N}{N-2s}}(\mathbb{R}^N)} \leqslant C \int_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{N+2s}} \, dx \, dy \quad \text{for all } \varphi \in \mathcal{D}^{s,2}\big(\mathbb{R}^N\big). \qquad (2.13)$$

Another fact we need is the following:

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let $u \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ be such that its trace – also denoted by u – is continuous in $\overline{\Omega}$ and satisfies $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Then $u \in D(\Omega, s)$.*

Proof. Consider $G \in C^\infty(\mathbb{R})$ such that

$$G(r) = 0 \quad \text{if } |r| \leqslant 1, \qquad G(r) = r \quad \text{if } |r| \geqslant 2 \quad \text{and} \quad |G'(r)| \leqslant 3 \quad \text{if } 1 \leqslant |r| \leqslant 2.$$

Then the functions u_n defined by $u_n(t, x) = \frac{1}{n} G(nu(t, x))$ are clearly contained in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ for $n \in \mathbb{N}$. Passing to traces, we therefore have $u_n \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Note that by the dominated convergence theorem we have $u_n \to u$ in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$. In addition, since the support of the trace of u_n in $\mathbb{R}^N$, is contained in the compact subset of Ω

$$\left\{ x \in \Omega \colon \ \big|u_n(x)\big| \geqslant \frac{1}{n} \right\},$$

it follows that $u_n \in \mathcal{D}^{s,2}(\Omega)$ by the density result in [22, Theorem 1.4.2.2]. To conclude we observe that $u_n \to u$ in $\mathcal{D}^{s,2}(\Omega)$ and this holds true thanks to the continuity of the trace operator $D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$. $\quad\square$

We remark that the continuity assumption in Lemma 2.4 is not needed if Ω has a continuous boundary, see [22, Theorem 1.4.2.2].

Next, let $q_s := \frac{2N}{N+2s}$ be the conjugate of $\frac{2N}{N-2s}$. If $f \in L^{q_s}(\Omega)$ is given and $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ satisfies $(-\Delta)^s u = f$ in Ω in distributional sense, then, as a consequence of the embedding $\mathcal{D}^{s,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2s}}(\Omega)$, it also satisfies this equation in weak sense, i.e.

$$\langle u, \psi \rangle_{\mathcal{D}^{s,2}} = \int_{\Omega} f\psi \, dx \quad \text{for all } \psi \in \mathcal{D}^{s,2}(\Omega).$$

Moreover, by (2.11), the weakly L_s-harmonic extension $w = H(u) \in D(\Omega, s)$ of u then satisfies

$$c_{N,s} \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla w \nabla \psi \, dz = \int_{\Omega} f\psi \, dx \quad \text{for all } \psi \in D(\Omega, s). \tag{2.14}$$

We may summarize the discussion in the following statement.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set and $f \in L^{q_s}(\Omega)$. A function $w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ satisfies (2.14) if and only if w is weakly L_s-harmonic and its trace – also denoted by $w \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ – solves $(-\Delta)^s w = f$ in Ω in distributional sense.*

If this holds, we say that w weakly solves the problem

$$\begin{cases} \operatorname{div}\!\big(t^{1-2s} \nabla w\big) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -c_{N,s} \lim_{t \to 0} t^{1-2s} w_t = f & \text{on } \Omega. \end{cases} \tag{2.15}$$

Next, we examine how problems of type (2.15) transform under generalized Kelvin inversions.

Proposition 2.6. *Let $w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$, let $\Omega \subset \mathbb{R}^N$ be an open set and let $f \in L^{q_s}(\Omega)$. Moreover, for fixed $\rho > 0$, consider*

$$\Omega_\rho := \left\{ \frac{\rho^2 x}{|x|^2} \colon x \in \Omega \setminus \{0\} \right\} \subset \mathbb{R}^N,$$

and let $w_\rho : \mathbb{R}^{N+1}_+ \to \mathbb{R}$, $f_\rho : \Omega_\rho \to \mathbb{R}$ be defined by

$$w_\rho(z) := \left(\frac{\rho}{|z|} \right)^{N-2s} w\!\left(\frac{\rho^2 z}{|z|^2} \right) \quad \text{and} \quad f_\rho(x) = \left(\frac{\rho}{|x|} \right)^{N+2s} f\!\left(\frac{\rho^2 x}{|x|^2} \right).$$

Then we have:

(i) $w_\rho \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$, and $f_\rho \in L^{q_s}(\Omega_\rho)$.

(ii) *If w weakly solves the problem*

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -c_{N,s}\lim_{t\to 0} t^{1-2s}\dfrac{\partial w}{\partial t} = f & \text{on } \Omega, \end{cases} \tag{2.16}$$

then w_ρ weakly solves the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla w_\rho) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -c_{N,s}\lim_{t\to 0} t^{1-2s}\dfrac{\partial w_\rho}{\partial t} = f_\rho & \text{on } \Omega_\rho. \end{cases} \tag{2.17}$$

Proof. Let $w \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ and $f \in L^{q_s}(\Omega)$. Note that $w_\rho(z) = \rho^{2s-N}w_1(\frac{z}{\rho^2})$ and $f_\rho(x) = \rho^{-(N+2s)}w_1(\frac{x}{\rho^2})$ for every $\rho > 0$, $z \in \mathbb{R}_+^{N+1}$ and $x \in \mathbb{R}^N \setminus \{0\}$. Hence it suffices to prove the claims in the case $\rho = 1$, and we put $\tilde{w} = w_1$, $\tilde{f} = f_1$ and $\tilde{\Omega} = \Omega_1$. Recalling the properties of the map κ defined in (2.7), we then find

$$\int_{\tilde{\Omega}} |\tilde{f}|^{q_s}\,dx = \int_{\tilde{\Omega}} |x|^{-2N}\left|f\left(\frac{x}{|x|^2}\right)\right|^{q_s}\,dx = \int_{\tilde{\Omega}} |J_\kappa|\,|f\circ\kappa|^{q_s}\,dx = \int_{\Omega} |f|^{q_s}\,dx.$$

To simplify the notations, we set

$$\tau : \overline{\mathbb{R}_+^{N+1}} \setminus \{0\} \to \overline{\mathbb{R}_+^{N+1}} \setminus \{0\}, \qquad \tau(z) = \frac{z}{|z|^2},$$

so that the restriction of τ to $\mathbb{R}^N \setminus \{0\}$ coincides with κ. We note that the Jacobian J_τ of τ satisfies

$$J_\tau^T(z)J_\tau(z) = |z|^{-4}I,$$

where I denotes the $(n+1) \times (n+1)$-identity matrix, and $\det J_\tau(z) = |z|^{-2N-2}$ for every $z \in \mathbb{R}_+^{N+1}$.

Next, we write $\tilde{w} = g \circ \tau$ with $g(z) = |z|^{N-2s}w(z)$. Moreover, we let $\varphi \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}} \setminus \{0\})$ be arbitrary, and define $\tilde{\varphi} \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}} \setminus \{0\})$ by $\tilde{\varphi} = h \circ \tau$ with $h(z) = |z|^{N-2s}\varphi(z)$. Considering first the special case where $w \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}} \setminus \{0\})$, we then calculate

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s}\nabla\tilde{w}\nabla\tilde{\varphi}\,dz = \int_{\mathbb{R}_+^{N+1}} t^{1-2s}\big[J_\tau(z)\nabla g(\tau(z))\big] \cdot \big[J_\tau(z)\nabla h(\tau(z))\big]\,dz$$

$$= \int_{\mathbb{R}_+^{N+1}} t^{1-2s}|z|^{-4}\nabla g(\tau(z))\nabla h(\tau(z))\,dz$$

$$= \int_{\mathbb{R}^{N+1}_+} |z|^{-2N-2} \left(\frac{t}{|z|^2}\right)^{1-2s} |z|^{2(N-2s)} \nabla g\big(\tau(z)\big) \nabla h\big(\tau(z)\big) \, dz$$

$$= \int_{\mathbb{R}^{N+1}_+} \big|\det J_\tau(z)\big| \left(\frac{t}{|z|^2}\right)^{1-2s} |\tau(z)|^{2(2s-N)} \nabla g\big(\tau(z)\big) \nabla h\big(\tau(z)\big) \, dz$$

$$= \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |z|^{2(2s-N)} \nabla g(z) \nabla h(z) \, dz.$$

Noting that

$$\nabla g(z) = (N - 2s)|z|^{N-2s-2} z w(z) + |z|^{N-2s} \nabla w(z)$$

and

$$\nabla h(z) = (N - 2s)|z|^{N-2s-2} z \varphi(z) + |z|^{N-2s} \nabla \varphi(z),$$

we then conclude that

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \tilde{w} \nabla \tilde{\varphi} \, dz = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla w \nabla \varphi \, dz + I_1 + I_2 + I_3$$

with

$$I_1 = (N - 2s)^2 \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |z|^{-2} w(z) \varphi(z) \, dz,$$

$$I_2 = (N - 2s) \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |z|^{-2} w(z) z \nabla \varphi(z) \, dz,$$

$$I_3 = (N - 2s) \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |z|^{-2} \varphi(z) z \nabla w(z) \, dz.$$

Since $\operatorname{div}_z[t^{1-2s} |z|^{-2} z] = (N - 2s) t^{1-2s} |z|^{-2}$, it follows that

$$I_1 + I_2 + I_3 = (N - 2s) \int_{\mathbb{R}^{N+1}_+} \operatorname{div}_z \big(t^{1-2s} |z|^{-2} z w(z) \varphi(z)\big) \, dz = 0$$

and therefore

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla \tilde{w} \nabla \tilde{\varphi} \, dz = \int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla w \nabla \varphi \, dz. \qquad (2.18)$$

By [19], we have that $C_c^\infty(\overline{\mathbb{R}_+^{N+1}} \setminus \{0\})$ is dense in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ thus we deduce that (2.18) also holds for arbitrary $w, \varphi \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$, while $\tilde{w}, \tilde{\varphi}$ are also contained in $D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$. In particular, (i) is proved.

Moreover, (2.18) implies that $\tilde{w}$ is weakly L_s-harmonic if w is weakly L_s-harmonic. In addition, considering the traces of w and $\tilde{w}$ respectively, Corollary 2.3 implies that $(-\Delta)^s \tilde{w} = \tilde{f}$ in distributional sense in $\tilde{\Omega}$ if $(-\Delta)^s w = f$ in distributional sense in Ω. Hence (ii) follows from Lemma 2.5. $\quad\square$

We will need the following version of a strong maximum principle which is essentially a reformulation of [7, Proposition 4.11].

Lemma 2.7. *Let E be an open subset of $\mathbb{R}^N$, and let $w \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ be a weak solution of*

$$
\begin{cases}
\operatorname{div}(t^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\
-c_{N,s} \lim_{t \to 0^+} t^{1-2s} \dfrac{\partial w}{\partial t} = g & \text{on } E
\end{cases}
$$

for some $g \in L^{q_s}(E) \cap C(E)$. Suppose furthermore that w is continuous and nonnegative on $E \times [0, r]$ for some $r > 0$, and that

$$
g(x) \geqslant 0 \quad \text{for every } x \in E \quad \text{with} \quad w(x) = 0. \tag{2.19}
$$

If $w \not\equiv 0$ in E, then w is strictly positive in E and therefore $\inf_K w > 0$ for any compact set $K \subset E$.

Proof. If $w \not\equiv 0$ on E, then $w > 0$ in $E \times (0, r)$, since w is L_s-harmonic and nonnegative in this set. Suppose by contradiction that $w(x_0) = 0$ for some $x_0 \in E$. Then $g(x_0) < 0$ by [7, Proposition 4.11], which contradicts (2.19). $\quad\square$

We will also need the following "small volume" maximum principle:

Lemma 2.8. *Let $\gamma > 0$. Then there exists $\delta = \delta(N, s, \gamma) > 0$ with the following property. If*

(i) *$F \subset \mathbb{R}_+^{N+1}$ is an open subset with $\partial F \cap \mathbb{R}^N \neq \varnothing$,*
(ii) *E is a bounded open subset of $\mathbb{R}^N$ with $E \subset \partial F$,*
(iii) *$c \in L^\infty(E)$ is given with $\|c\|_{L^\infty(E)} \leqslant \gamma$,*
(iv) *$w \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$ is a weak solution of*

$$
\begin{cases}
\operatorname{div}(t^{1-2s} \nabla w) \leqslant 0 & \text{in } \mathbb{R}_+^{N+1}, \\
-c_{N,s} \lim_{t \to 0^+} t^{1-2s} \dfrac{\partial w}{\partial t} \geqslant c(x) w & \text{on } E,
\end{cases} \tag{2.20}
$$

i.e.,

$$c_{N,s} \int_F t^{1-2s} \nabla w \nabla \varphi \, dz \geqslant \int_E c(x) w \varphi \, dx \qquad (2.21)$$

for all nonnegative $\varphi \in D(E, s)$,

(v) *w is continuous on $\overline{F}$ and satisfies $w \geqslant 0$ on $\partial F \setminus E$,*

(vi) *$|\{x \in E : w < 0\}| \leqslant \delta$,*

then $w \geqslant 0$ in F.

Proof. We consider the function

$$v : \overline{\mathbb{R}_+^{N+1}} \to \mathbb{R}, \qquad v(x) = \begin{cases} \max(-w(x), 0), & x \in \overline{F}, \\ 0, & x \in \overline{\mathbb{R}_+^{N+1}} \setminus \overline{F}. \end{cases}$$

It can be deduced from assumptions (i) and (ii) that the relative boundary of $\overline{F}$ in $\overline{\mathbb{R}_+^{N+1}}$ is contained in $\overline{\partial F \setminus E}$, so that v is continuous on $\overline{\mathbb{R}_+^{N+1}}$ by assumption (v). Moreover, $v \equiv 0$ on $\mathbb{R}^N \setminus E$. As a consequence, $v \in H^1_{loc}(\mathbb{R}_+^{N+1})$ by [6, Theorem 9.17 and Remark 19], and

$$\int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 \, dz \leqslant \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla w|^2 \, dz < \infty.$$

Hence $v \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s})$, and Lemma 2.4 implies that $v \in D(E, s)$. We also note that combining (2.12) and (2.13) yields a constant $C = C(N, s) > 0$ such that

$$\|v\|^2_{L^{2N/(N-2s)}(\mathbb{R}^N)} \leqslant C \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 \, dz.$$

Applying (2.21) to v, we then obtain

$$c_{N,s} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 \, dz = -c_{N,s} \int_{\mathbb{R}_+^{N+1}} t^{1-2s} \nabla w \cdot \nabla v \, dz \leqslant -\int_E c(x) w v \, dx$$

$$= \int_E c(x) v^2 \, dx \leqslant \|c\|_{L^\infty(E)} |\{x \in E : w < 0\}|^{N/2s} \|v\|^2_{L^{2N/(N-2s)}(\mathbb{R}^N)}$$

$$\leqslant \gamma \delta^{N/2s} C \int_{\mathbb{R}_+^{N+1}} t^{1-2s} |\nabla v|^2 \, dz.$$

Hence, if $\delta < (\frac{c_{N,s}}{\gamma C})^{2s/N}$, then $v \equiv 0$ in $\mathbb{R}_+^{N+1}$ and therefore $w \geqslant 0$ in F, as claimed. $\quad\square$

3. Proof of the main results

In this section we complete the proof of our main results. We begin with the

Proof of Theorem 1.1. We suppose by contradiction that there exists a nontrivial solution $u \in C(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{s,2}(\Omega)$ of (1.1), and we let $w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ denote the corresponding L_s-harmonic extension of u which weakly solves the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -c_{N,s} \lim_{t \to 0} t^{1-2s} w_t = f(x, w) & \text{on } \Omega', \end{cases}$$

for every open subset $\Omega' \subset \Omega$ which is relatively compact in $\mathbb{R}^N \setminus \{0\}$. Here, as before, we also write w in place of u for the trace on $\mathbb{R}^N$. We clearly have $w \in C(\overline{\mathbb{R}^{N+1}_+} \setminus \{0\})$. Let $R := \sup\{|x|: x \in \Omega\} > 0$. For $\rho \in (0, R)$, we consider the Kelvin transform w_ρ of w as defined in Proposition 2.6. We also put

$$F_\rho := \{z \in \mathbb{R}^{N+1}_+: |z| > \rho\}, \qquad E_\rho := \{x \in \Omega: |x| > \rho\} \quad \text{and} \quad \tilde{E}_\rho := \left\{\frac{\rho^2 x}{|x|^2}: x \in E_\rho\right\}.$$

By definition of R and since Ω is star-shaped with respect to the origin, E_ρ and $\tilde{E}_\rho$ are nonempty open subsets of Ω which are relatively compact in $\mathbb{R}^N \setminus \{0\}$ for $\rho \in (0, R)$, so that the restrictions of the map $x \mapsto f(x, w(x))$ to E_ρ and $\tilde{E}_\rho$ are bounded and continuous. By Proposition 2.6, the difference function $v_\rho = w_\rho - w \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ weakly solves the problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v_\rho) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -c_{N,s} \lim_{t \to 0} t^{1-2s} [v_\rho]_t = g_\rho & \text{on } E_\rho, \end{cases}$$

where g_ρ is the bounded and continuous function on E_ρ given by

$$g_\rho(x) = \left(\frac{\rho}{|x|}\right)^{N+2s} f\left(\frac{\rho^2 x}{|x|^2}, \left(\frac{\rho}{|x|}\right)^{2s-N} w_\rho(x)\right) - f(x, w(x)).$$

Moreover, by the supercriticality assumption (1.6) we have

$$g_\rho(x) \geqslant f(x, w_\rho(x)) - f(x, w(x)) = c_\rho(x) v_\rho(x) \quad \text{for } x \text{ in } E_\rho$$

with

$$c_\rho : E_\rho \to \mathbb{R}, \qquad c_\rho(x) = \begin{cases} \frac{f(x, w_\rho(x)) - f(x, w(x))}{w_\rho(x) - w(x)} & \text{if } w(x) \neq w_\rho(x), \\ 0 & \text{if } w(x) = w_\rho(x). \end{cases}$$

We also note that, since f is assumed to be locally Lipschitz in its second variable, we have $c_\rho \in L^\infty(E_\rho)$ for $0 < \rho < R$. Moreover, for $\tau \in (0, R)$ we have

$$\gamma_\tau := \sup_{\rho \in [\tau, R)} \|c\|_{L^\infty(E_\rho)} < \infty. \tag{3.1}$$

We now define

$$\rho_* := \inf\{\bar{\rho} \in (0, R): v_\rho \geqslant 0 \text{ in } F_\rho \text{ for } \rho \in [\bar{\rho}, R)\}.$$

Since $|E_\rho \cap \{w_\rho < 0\}|$ is small provided ρ is sufficiently close to R, Lemma 2.8 implies that $\rho_* < R$. We claim that $\rho_* = 0$. Indeed, suppose by contradiction that $\rho_* > 0$. By continuity, we then have $v_{\rho_*} \geqslant 0$ in F_{ρ_*}. Moreover, $v_{\rho_*} \not\equiv 0$ in E_{ρ_*} since

$$v_{\rho_*}(x) > 0 \quad \text{for every } x \in \partial\Omega \quad \text{with} \quad |x| > \rho_*.$$

By Lemma 2.7, we obtain $v_{\rho_*} > 0$ in E_{ρ_*}. We now fix $\tau \in (0, \rho_*)$ and choose $\delta > 0$ as in Lemma 2.8 according to $\gamma = \gamma_\tau$ as defined in (3.1). Moreover, we choose a compact set $K \subset E_{\rho_*}$ such that $|E_{\rho_*} \setminus K| < \delta$. Then $\inf_K w_{\rho_*} > 0$, and by continuity we also have

$$K \subset E_\rho, \qquad |E_\rho \setminus K| < \delta \quad \text{and} \quad \inf_K w_\rho > 0$$

for $\rho \in (\tau, \rho_*)$ sufficiently close to ρ_*. Therefore Lemma 2.8 implies that $v_\rho \geqslant 0$ in F_ρ for $\rho \in (\tau, \rho_*)$ sufficiently close to ρ_*. This contradicts the definition of ρ_*. We conclude that $\rho_* = 0$, as claimed. As a consequence, for every $x \in \Omega$ and $x \neq 0$ we have

$$\left(\frac{\rho}{|x|}\right)^{N-2s} w\left(\frac{\rho^2 x}{|x|^2}\right) \geqslant w(x) \quad \text{for all } \rho \in (0, |x|). \tag{3.2}$$

Furthermore, since $w \in \mathcal{D}^{s,2}(\Omega) \subset L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$ we have

$$\int_{S^{N-1}} \int_0^\infty w^{\frac{2N}{N-2s}}(r\sigma)\, dr\, d\sigma = \int_{\mathbb{R}^N} w^{\frac{2N}{N-2s}}\, dx < \infty$$

and therefore, by Fubini's theorem,

$$\int_0^\infty w^{\frac{2N}{N-2s}}(r\sigma_0)\, dr < \infty \quad \text{for a.e. } \sigma_0 \in S^{N-1}. \tag{3.3}$$

We now pick $\sigma_0 \in S^{N-1}$ and $r_0 > 0$ such that $r_0\sigma_0 \in \Omega$ and (3.3) holds for σ_0. By (3.2) we then have

$$\left(\frac{\rho}{r_0}\right)^{N-2s} w\left(\frac{\rho^2 \sigma_0}{r_0}\right) \geqslant w(r_0\sigma_0) > 0 \quad \text{for } \rho \in (0, r_0)$$

and consequently

$$w(r\sigma_0) \geqslant Cr^{(2s-N)/2} \quad \text{for } r \in (0, r_0) \text{ with a constant } C > 0.$$

This implies

$$\int_0^{r_0} w^{\frac{2N}{N-2s}}(r\sigma_0)\, dr = \infty$$

contrary to (3.3). The contradiction shows that there does not exist a nontrivial solution $u \in C(\mathbb{R}^N \setminus \{0\}) \cap \mathcal{D}^{s,2}(\Omega)$ of (1.1) under the assumptions of Theorem 1.1, as claimed. $\quad\square$

Proof of Corollary 1.2. Problem (1.9) is a special case of (1.1) with $f(x, u) = u^p - V(x)u$, and for this nonlinearity we calculate

$$H_f(x, u) = \left(p - \frac{N+2s}{N-2s}\right)u^p + \frac{4u}{N-2s}\left(sV(x) + \frac{1}{2}x \cdot \nabla V(x)\right)$$

so that (1.7) is satisfied by the assumptions on p and V. Moreover, any nontrivial, nonnegative solution of (1.9) is strictly positive in $\Omega \setminus \{0\}$, which follows by applying Lemma 2.7 to the L_s-harmonic extension of u and the sets $E_\varepsilon := \{x \in \Omega : |x| > \varepsilon\}$ for $\varepsilon > 0$ small. Hence nontrivial, nonnegative solutions of (1.9) do not exist by Theorem 1.1. $\quad\square$

Proof of Corollary 1.3. Problem (1.10) is a special case of (1.9) with $V(x) = \gamma|x|^{-2s}$, so the result follows from Corollary 1.2. $\quad\square$

Proof of Corollary 1.4. Problem (1.11) is a special case of (1.1) with $f(x, u) = |x|^{-\sigma}u^p$, and for this nonlinearity we calculate

$$H_f(x, u) = \left(p - \frac{N+2s-2\sigma}{N-2s}\right)|x|^{-\sigma}u^p.$$

Hence (1.7) is satisfied by the assumptions on p and σ. Moreover, by the same argument as in the proof of Corollary 1.2 above, nontrivial and nonnegative solutions of (1.11) must be strictly positive in $\Omega \setminus \{0\}$ and therefore cannot exist by Theorem 1.1. $\quad\square$

We finally give the proof of our nonexistence result in (unbounded) domains being star-shaped at infinity.

Proof of Theorem 1.5. The definition of star-shapedness at infinity implies that, after a suitable translation, the image $\tilde{\Omega} := \kappa(\Omega)$ of the domain Ω under the map κ defined in (2.7) is bounded and star-shaped with respect to $0 \in \partial\tilde{\Omega}$. Moreover, if $u \in \mathcal{D}^{s,2}(\tilde{\Omega}) \cap C(\mathbb{R}^N)$ is a nonnegative solution of (1.12), then Corollary 2.3 implies that $\tilde{u} = Ku \in \mathcal{D}^{s,2}(\tilde{\Omega}) \cap C(\mathbb{R}^N \setminus \{0\})$ solves $(-\Delta)^s\tilde{u} = |x|^{-\sigma}|\tilde{u}|^p$ in $\tilde{\Omega}$ with $\sigma = N + 2s - p(N-2s)$. Since the assumption $p \leqslant \frac{N+2s}{N-2s}$ yields $p \geqslant \frac{N+2s-2\sigma}{N-2s}$, Corollary 1.4 implies that $\tilde{u} \equiv 0$ and hence also $u \equiv 0$, as claimed. $\quad\square$

4. Note added in proof

After the paper was accepted, Xavier Ros and Joaquim Serra sent us a preprint where – under additional regularity assumptions on the domain's boundary – they prove a very interesting Pohozaev type identity where the normal derivatives on the boundary are replaced by other suitable quantities.

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