Abstract

We characterize functions in Morrey space by $p$-Carleson measures. We then reveal a simple relation between $Q$ space and Morrey space, that is $Q$ space can be viewed as a fractional integration of the Morrey space. Therefore, many results for Morrey space can be translated onto $Q$ space. For example, we show that $Q$ space is a dual space by identifying its predual.

1. Introduction

Let $U$ be the upper half complex plane and $\partial U = R$ be the boundary of $U$. For $z = x + iy \in U$ and a measurable function $f$ on $R$, denote by

$$f(z) = f \ast P_y(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-u)^2 + y^2} f(u) \, du,$$

the Poisson extension of $f$ onto $U$. For $-1 < p < \infty$, the space $Q_p$ is defined to be all measurable functions $f$ on $R$ satisfying

$$\int_{\mathbb{R}} \frac{|f(x)|^p}{1+x^2} \, dx < \infty$$

and

$$\|f\|_{Q_p} = \sup_{w = u + iv \in U} \left( \int_U \frac{(yw)^p}{|z-w|^{2p}} |\nabla f(z)|^2 \, dx \, dy \right)^{1/2} < \infty.$$
We say that \( f \) is in \( Q_{p,0} \); if in addition
\[
\lim_{t \to 0,} \int_U \frac{(yv)^p}{|z-w|^{2p}} |\nabla f(z)|^2 \, dx \, dy = 0.
\]

The analytic version of the \( Q_p \) space is introduced in [AL,AXZ]. It is a M"obius invariant function space lying between the classical Dirichlet space (\( Q_0 \)) and the analytic BMO space (\( Q_1 \)). It is proved, in [AL], that the analytic \( Q_p \) is the Bloch space if \( 1 < p < \infty \) and, in [NX], that \( Q_p \) contains only constants if \(-1 < p < 0\). Therefore, the interesting range of \( p \) for \( Q_p \) space is \( 0 < p < 1 \).

Many studies related to \( Q_p \) space and characterizations of \( Q_p \) space have been done in recent years. For example, functions in \( Q_p \) space can be characterized by \( p \)-Carleson measures, wavelets and the atomic decomposition using Bergman metrics. We refer the readers to [A,AL,AXZ,ANZ,ASX,EJPX,L,NX,WX] and Xiao’s recent book [X] for more information.

Morrey space was introduced in 1938 by Morrey [M]. For \( 0 \leq \lambda \leq 1 \), we say \( f \) is in Morrey space \( \mathcal{L}^{2,\lambda} \) if
\[
\inf_c \frac{1}{|I|^{\lambda}} \int_I |f - c|^2 \leq M
\]
holds for any bounded interval \( I \) on \( \mathbb{R} \). Here \( |I| \) is the length of \( I \). We say that \( f \) is in \( \mathcal{L}^{2,\lambda}_{0} \), if in addition
\[
\lim_{|I| \to 0} \inf_c \frac{1}{|I|^{\lambda}} \int_I |f - c|^2 = 0.
\]

Clearly, when \( \lambda = 0 \), \( \mathcal{L}^{2,\lambda} = L^2 / constant \) and when \( \lambda = 1 \), \( \mathcal{L}^{2,\lambda} = BMO \) and \( \mathcal{L}^{2,\lambda}_{0} = VMO \). Morrey space has been studied heavily in the past. The results are used mainly in harmonic analysis and PDE. We refer the readers to [P.Z,M].

Let \( I \) be a bounded interval on \( \mathbb{R} \) with center \( a \) and length \( |I| \). The Carleson box in \( U \), based on \( I \), is defined by
\[
S(I) = \{z = x + iy \in U : |x-a| < \frac{1}{2} |I|, \ 0 < y < |I|\}.
\]

A nonnegative measure \( \mu \) on \( U \) is called a bounded \( p \)-Carleson measure if
\[
\sup_{I \in \mathbb{R}} \frac{\mu(S(I))}{|I|^p} < \infty.
\]

We say that \( \mu \) is a compact \( p \)-Carleson measure, if in addition
\[
\lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0.
\]
Denote by $\hat{f}(\xi) = \int_R f(x) e^{-ix\xi} \, dx$ the Fourier transformation of $f$ at $\xi \in \mathbb{R}$. For $t > 0$, the $t$-integration of $f$, usually called $t$-Riesz potential of $f$ if $0 < t < 1$, is defined by

$$\mathcal{I}_t f(\xi) = (i|\xi|)^{-t} \hat{f}(\xi) \quad \forall \xi \in \mathbb{R}.$$ 

The inverse of the $t$-integration is the $t$-derivative which can be denoted by $\mathcal{I}_{-t}$, i.e.

$$\mathcal{I}_{-t} f(\xi) = (i|\xi|)^{t} \hat{f}(\xi) \quad \forall \xi \in \mathbb{R}.$$ 

In this note we characterize functions in Morrey space in terms of $p$-Carleson measures. We then reveal a simple relation between $Q$ space and Morrey space, that is $Q$ space can be viewed as a fractional integration of the Morrey space. Therefore, many results for Morrey space can be translated onto $Q$ space. For example, we show that $Q$ space is a dual space by identifying its predual. We show also that the Hilbert transformation maps $Q_p$ space into itself. The tools used in our approach are standard tools in complex and harmonic analysis.

The main results are:

**Theorem 1.** Suppose $0 < p \leq 1$ and $g$ is a measurable function on $\mathbb{R}$. Then

(i) $g \in \mathcal{L}^{2,p}$ if and only if the measure $\int_y |\nabla g(z)|^2 y \, dx \, dy$ is a bounded $p$-Carleson measure.

(ii) $g \in \mathcal{L}^{2,p}_0$ if and only if the measure $\int_y |\nabla g(z)|^2 y \, dx \, dy$ is a compact $p$-Carleson measure.

**Theorem 2.** Suppose $0 < p \leq 1$ and $q = \frac{1-p}{2}$. Then

(i) $Q_p = \mathcal{I}_q \mathcal{L}^{2,p}$, or equivalently $f \in Q_p$ if and only if $\mathcal{I}_q f \in \mathcal{L}^{2,p}$.

(ii) $Q_{p,0} = \mathcal{I}_q \mathcal{L}^{2,p}_0$, or equivalently $f \in Q_{p,0}$ if and only if $\mathcal{I}_q f \in \mathcal{L}^{2,p}_0$.

Similar results for Morrey spaces and $Q$ spaces on the unit circle are also true. We choose to work on $\mathbb{R}$ or $U$ because the related formulas are neater than the same type of formulas on the unit circle or the unit disc.

2. **Preliminaries**

Throughout this paper, the letter “$C$” denote absolute positive constant, which is not necessarily the same at each occurrence but is independent of the essential variables and quantities. The notation $A \asymp B$ means that $A$ and $B$ are comparable, i.e. $\frac{1}{C} \leq A \leq C$. We will always use $I$ for a bounded interval on $\mathbb{R}$ and $|I|$ for the length
of $I$. For nonnegative integer $n$, we use $I_n$ for the interval which has the same center as $I$ but length $2^n|I|$.

We gather some standard estimates for the Poisson kernel in the following theorem.

**Theorem A.** For fixed $w = u + iv \in U$, let $I$ be the interval on $\mathbb{R}$ with center $u$ and length $2v$. Let $n$ be a nonnegative integer.

(i) The following estimates hold:

$$|z - \bar{w}| \approx \begin{cases} 
  v & \text{if } z \in S(I), \\
  2^nv & \text{if } z \in S(I_{n+1}) \setminus S(I_n).
\end{cases}$$

(ii) The following estimates hold:

$$P_v(u - x) \approx \begin{cases} 
  v^{-1} & \text{if } z \in S(I), \\
  2^{-2n}v^{-1} & \text{if } z \in S(I_{n+1}) \setminus S(I_n).
\end{cases}$$

**Theorem B.** Suppose $0 < p \leq 1$ and $f$ is a measurable function on $\mathbb{R}$. Then

(i) $f \in Q_p$ if and only if the measure $|\nabla f(z)|^2 y^p \, dx \, dy$ is a bounded $p$-Carleson measure.

(ii) $f \in Q_{p,0}$ if and only if the measure $|\nabla f(z)|^2 y^p \, dx \, dy$ is a compact $p$-Carleson measure.

For $p = 1$, this result is well known and the proof is now standard (see for example [G, p. 239]). Modifying slightly the standard proof, one can get Theorem B for the case $0 < p < 1$. See also [ASX] for the unit circle version of this theorem.

Theorem B implies also that the condition

$$\sup_{z \in U} |\nabla f(z)|y < \infty$$

is necessary for $f \in Q_p$.

The following standard estimate can be found in, for example, [G, p. 238] which can be obtained directly by using the Green’s formula.

**Theorem C.** Suppose the measurable function $f$ on $\mathbb{R}$ satisfies $\int_{\mathbb{R}} \frac{|f(x)|^2}{1+x^2} \, dx < \infty$. Then

$$\int_U |\nabla f(z)|^2 \frac{yv}{|z - \bar{w}|^2} \, dx \, dy \approx \int_{\mathbb{R}} |f(t) - f(w)|^2 P_v(u - t) \, dt$$

where $w = u + iv \in U$. 
Schur’s Lemma. Suppose $K(w, z)$ is a positive function on $U \times U$. If there is a positive function $g$ on $U$ such that

$$
\int_U K(w, z) g^2(w) \, du \, dv \leq C g^2(z) \quad \text{and} \quad \int_U K(z, w) g^2(w) \, du \, dv \leq C g^2(z)
$$

hold for all $z \in U$, then the linear map given by

$$
h \mapsto \int_U K(w, z) h(z) \, dx \, dy
$$

is a bounded map on $L^2(U)$.

3. Proof of the main theorems

We first prove the following characterizations for bounded and compact $p$-Carleson measures.

Theorem 3.1. Suppose $\tau > 0$, $0 < p \leq 1$ and $\mu$ is a nonnegative measure on $U$. Then

(i) $\mu$ is a bounded $p$-Carleson measure if and only if

$$
M_\mu = \sup_{w = u + iv \in U} \int_U \frac{v^\tau}{|z - \bar{w}|^{\tau + p}} \, d\mu(z) < \infty.
$$

(ii) $\mu$ is a compact $p$-Carleson measure if and only if $M_\mu < \infty$ and

$$
\lim_{v \to 0^+} \int_U \frac{v^\tau}{|z - \bar{w}|^{\tau + p}} \, d\mu(z) = 0.
$$

Remark. The idea of the following proof is standard (see for example [G, p. 239]). When $\tau = p$, or $\tau + p = 2$, this result (part (i)) is also proved in [ASX] or [W], respectively.

Proof of Theorem 3.1. For any bounded interval $I$ with center $u$, let $v = |I|/2$ and write $w = u + iv$. We have $|z - \bar{w}| \leq 2|I|$ when $z \in S(I)$. Therefore,

$$
\frac{\mu(S(I))}{|I|^p} \leq C \int_{S(I)} \frac{v^\tau}{|z - \bar{w}|^{\tau + p}} \, d\mu(z) \leq C \int_U \frac{v^\tau}{|z - \bar{w}|^{\tau + p}} \, d\mu(z).
$$

This is enough to conclude the “if” parts of (i) and (ii).
Suppose \( \mu \) is a bounded \( p \)-Carleson measure. For fixed \( w = u + iv \in U \), let \( I \) be the interval with center \( u \) and length \( 2v \). By Theorem A(i), we have

\[
\int_U \frac{v^\tau}{|z - \bar{w}|^{\tau+p}} \, d\mu(z) = \int_{S(I)} \frac{v^\tau}{|z - \bar{w}|^{\tau+p}} \, d\mu + \sum_{n=0}^{\infty} \int_{S(I_n)} \frac{v^\tau}{|z - \bar{w}|^{\tau+p}} \, d\mu
\]

\[
\leq C \frac{v^\tau}{v^\tau + \mu(S(I))} + C \sum_{n=0}^{\infty} \frac{v^\tau}{(2^n v)^{\tau+p}} \mu(S(I_{n+1}))
\]

\[
\leq C + C \sum_{n=0}^{\infty} \frac{1}{(2^n)^\tau}
\]

\[
\leq C.
\]

This proves the “only if” part of (i).

If \( \mu \) is a compact \( p \)-Carleson measure, then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that the estimate

\[
\mu(S(I)) \leq \varepsilon |I|^p
\]

holds when \( |I| < \delta \). Let \( N \) be the largest integer satisfying \( N < \log_2 \frac{\delta}{|I|} \). We have \( 2^{-N} < 2|I|/\delta \) and \( |I_{n+1}| < \delta \), if \( n < N \). Hence, we can refine the previous estimate by

\[
\int_U \frac{v^\tau}{|z - \bar{w}|^{\tau+p}} \, d\mu(z) \leq C \frac{v^\tau}{v^\tau + \mu(S(I))} + C \sum_{n=0}^{N-1} \frac{v^\tau}{(2^n v)^{\tau+p}} \mu(S(I_{n+1}))
\]

\[
\leq C\varepsilon + C \sum_{n=0}^{N-1} \frac{\varepsilon}{2^m} + C \sum_{n=N}^{\infty} \frac{1}{2^m}
\]

\[
\leq C(\varepsilon + 2^{-N\varepsilon})
\]

\[
\leq C \left( \varepsilon + \left( \frac{2|I|}{\delta} \right)^{\varepsilon} \right).
\]

This is enough to conclude the desired result. The proof of Theorem 3.1 is completed. \( \square \)

Let \( g_I = \frac{1}{|I|} \int_I g(x) \, dx \) be the average of \( g \) on \( I \). It is clear that

\[
\int_I |g(x) - g_I|^2 \, dx = \inf_c \int_I |g(x) - c|^2 \, dx.
\]

Theorem 1 is included in the following theorem.
Theorem 3.2. Suppose $0 < p \leq 1$ and $g$ is a measurable function on $\mathbb{R}$. Denote

$$K(g) = \sup_I \frac{1}{|I|^p} \int_I |g(x) - g_I|^2 \, dx$$

and

$$L(g) = \sup_{w = u + iv \in U} v^{1-p} \int_\mathbb{R} |g(x) - g(w)|^2 P_v(u - x) \, dx.$$

(i) The following are equivalent:
   (ia) $K(g)$ is bounded;
   (ib) $L(g)$ is bounded; and
   (ic) the measure $|\nabla g(z)|^2 \, dy \, dx$ is a bounded $p$-Carleson measure.

(ii) The following are equivalent:
   (iia) $K(g)$ is bounded and $\lim_{|I| \to 0} \frac{1}{|I|^p} \int_I |g(x) - g_I|^2 \, dx = 0$;
   (iib) $L(g)$ is bounded and $\lim_{v \to 0} v^{1-p} \int_\mathbb{R} |g(x) - g(w)|^2 P_v(u - x) \, dx = 0$;
   (iic) the measure $|\nabla g(z)|^2 \, dy \, dx$ is a compact $p$-Carleson measure.

Proof. First we show that (ia) implies (ib). Fix $w = u + iv, \; v > 0$. Let $I$ be the interval with center $u$ and length $2v$. Since

$$\int_\mathbb{R} |g(x) - g(w)|^2 P_v(u - x) \, dx = \inf_c \int_\mathbb{R} |g(x) - c|^2 P_v(u - x) \, dx,$$

it is sufficient to show that

$$v^{1-p} \int_\mathbb{R} |g(x) - g_I|^2 P_v(u - x) \, dx \leq CK(g).$$

We consider the following estimate by using Theorem A(ii)

$$v^{1-p} \int_\mathbb{R} |g(x) - g_I|^2 P_v(u - x) \, dx$$

$$= v^{1-p} \int_I |g(x) - g_I|^2 P_v(u - x) \, dx + v^{1-p} \sum_{n=0}^{\infty} \int_{I_{n+1} \setminus I_n} |g(x) - g_I|^2 P_v(u - x) \, dx$$

$$\leq C v^{-p} \int_I |g(x) - g_I|^2 \, dx + C v^{-p} \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \int_{I_{n+1} \setminus I_n} |g(x) - g_I|^2 \, dx$$

$$\leq CK(g) + C v^{-p} \sum_{n=0}^{\infty} \frac{1}{2^{2n}} \int_{I_{n+1}} |g(x) - g_I|^2 \, dx.$$
Note that
\[
|g(x) - g_I|^2 \leq 2|g(x) - g_{I_{n+1}}|^2 + 2|g_{I_{n+1}} - g_I|^2 \\
\leq 2|g(x) - g_{I_{n+1}}|^2 + 2(n + 1) \sum_{k=0}^{n} |g_{I_{k+1}} - g_{I_k}|^2.
\]

By the assumption, we have
\[
\int_{I_{n+1}} |g(x) - g_{I_{n+1}}|^2 \, dx \leq (2^{n+1}|I|)^p K(g)
\]
and
\[
\int_{I_{k+1}} |g_{I_{k+1}} - g_{I_k}|^2 \, dx = |I_{n+1}||g_{I_{k+1}} - g_{I_k}|^2 \\
\leq \frac{|I_{n+1}|}{|I_k|} \int_{I_k} |g(x) - g_{I_{k+1}}|^2 \, dx \\
\leq \frac{|I_{n+1}|}{|I_k|} K(g)|I_{k+1}|^p \\
\leq 2^{n+2}|I|^p K(g).
\]

Therefore,
\[
\int_{I_{n+1}} |g(x) - g_{I}|^2 \, dx \leq 2(2^{n+1}|I|)^p K(g) + 2(n + 1) \sum_{k=0}^{n} 2^{n+2}|I|^p K(g) \\
\leq C(n + 1)^2 2^n |I|^p K(g).
\]

Hence,
\[
v^{1-p} \int_R |g(x) - g_I|^2 P_v(u - x) \, dx \leq CK(g) + Cv^{-p} \sum_{n=0}^{\infty} \frac{C(n + 1)^2 2^n |I|^p}{2^{2n}} K(g) \leq CK(g).
\]

We remark that (iia) implies (iib) can be obtained by refining above estimate, which is similar to what we did in Proof of Theorem 3.1.

We show next that (ib) implies (ia). For any bounded interval $I$ with center $u$, let $v = |I|/2$ and write $w = u + iv$. We have by Theorem A(ii)
\[
\frac{1}{|I|^p} \int_I |g(x) - g_I|^2 \, dx = \inf_c \frac{1}{|I|^p} \int_I |g(x) - c|^2 \, dx
\]
\[
\begin{align*}
\leq \frac{1}{(2a)^p} \int_I |g(x) - g(w)|^2 \, dx \\
\leq v^{1-p} \int_I |g(x) - g(w)|^2 P_v(u-x) \, dx \\
\leq v^{1-p} \int_R |g(x) - g(w)|^2 P_v(u-x) \, dx.
\end{align*}
\]

This is enough. Clearly (iib) implies (iia) is also a consequence of the above estimate.

Finally, we show the equivalence of (ib) and (ic), and the equivalence of (iib) and (iic). By Theorem C, we have

\[
v^{1-p} \int_U |g(t) - g(w)|^2 P_v(u-t) \, dt \leq \int_U \frac{v^{2-p}}{|z - \overline{w}|^2} |\nabla g(z)|^2 \, dx \, dy.
\]

We obtain the desired result by using Theorem 3.1 with \(\tau = 2 - p\). \(\Box\)

For fixed \(b > 1\), define the linear operator \(T_c\) by

\[
T_c \psi(z) = \int_U \frac{v^{b-1}}{|z - \overline{w}|^{b+\tau}} \psi(w) \, dw \, dv, \quad \tau > 0.
\]

Here \(\psi(z)\) is a measurable function on \(U\).

**Theorem 3.3.** Suppose \(0 < p \leq 1\), \(\tau > \frac{1-p}{2}\) and \(\psi\) is a measurable function on \(U\).

(i) If the measure \(|\psi(z)|^2 y^p \, dx \, dy\) is a bounded \(p\)-Carleson measure, then the measure \(|T_c \psi(z)|^2 y^{2\tau-2+p} \, dx \, dy\) is also a bounded \(p\)-Carleson measure.

(ii) If the measure \(|\psi(z)|^2 y^p \, dx \, dy\) is a compact \(p\)-Carleson measure, then the measure \(|T_c \psi(z)|^2 y^{2\tau-2+p} \, dx \, dy\) is also a compact \(p\)-Carleson measure.

**Remark.** (a) The integral which defines \(T_c \psi\) is convergent if the measure \(|\psi(z)|^2 y^p \, dx \, dy\) is a bounded \(p\)-Carleson measure. Indeed, for fixed \(z \in (0, 2b - 1 - p)\),
we have
\[
\left| \int_U \frac{v^{b-1}}{|z - \bar{w}|^{b+\tau}} \psi(w) \, du \, dv \right|^2 \leq \int_U \frac{v^2}{|z - \bar{w}|^{2\gamma + p}} |\psi(w)|^2 v^p \, du \, dv
\times \int_U \frac{v^{2b-2\gamma - p}}{|z - \bar{w}|^{2\gamma + p}} \, du \, dv.
\]

By Theorem 3.1, we know the first integral on the right side of above estimate is bounded. Direct computation yields that the second integral is bounded by $C_v^{-2\tau}$.

(b) The unit disc version of Theorem 3.3 is proved in [WX].

**Proof of Theorem 3.3.** For part (i), it is sufficient to show that the estimate
\[
\int_{S(I)} |T_\tau \psi(w)|^2 v^{2\tau - 2\gamma + p} \, du \, dv \leq C|I|^p
\]
holds for any bounded interval $I$ on $R$.

We start with the following estimate:
\[
\int_{S(I)} |T_\tau \psi(w)|^2 v^{2\tau - 2\gamma + p} \, du \, dv
\leq \int_{S(I)} \left( \int_U \frac{v^{b-1}}{|z - \bar{w}|^{b+\tau}} |\psi(z)| \, dx \, dy \right)^2 v^{2\tau - 2\gamma + p} \, du \, dv
= \int_{S(I)} \left( \left( \int_{S(I)} + \int_{U \setminus S(I)} \right) \frac{v^{b-1}}{|z - \bar{w}|^{b+\tau}} |\psi(z)| \, dx \, dy \right)^2 v^{2\tau - 2\gamma + p} \, du \, dv
\leq 2 \int_{S(I)} \left( \left( \int_{S(I)} \frac{v^{b-1} v^{\tau - 1 + \frac{p}{2}}}{|z - \bar{w}|^{b+\tau}} |\psi(z)| \, dx \, dy \right)^2 \, du \, dv
+ 2 \int_{S(I)} \left( \left( \int_{U \setminus S(I)} \frac{v^{b-1} v^{\tau - 1 + \frac{p}{2}}}{|z - \bar{w}|^{b+\tau}} |\psi(z)| \, dx \, dy \right)^2 \, du \, dv
= E_1 + E_2.
\]

Consider the linear operator $B : L^2(U) \to L^2(U)$ defined by
\[
h(w) \mapsto \int_U K(w, z)h(z) \, dx \, dy,
\]
where $K(w, z) = \frac{y^{b-1} p_{\frac{b-1+rac{p}{2}}{2}}}{|z-w|^{b-1+\frac{p}{2}}}$. It is easy to verify that for $g(z) = y^{-1/4}$ the estimates

$$
\int_U K(w, z)g^2(w) \, du \, dv \leq Cg^2(z) \quad \text{and} \quad \int_U K(z, w)g^2(w) \, du \, dv \leq Cg^2(z)
$$

hold for all $z \in U$. By Schur’s Lemma, we know that $B$ is a bounded operator.

Denoting

$$
h(z) = |\psi(z)||^{\rho/2} I_{S(I)}(z) \quad \forall z \in U,
$$

we have clearly that $h \in L^2(U)$ and $||h||_2^2 = \int_{S(I)} |\psi(z)|^2 y^\rho \, dx \, dy \leq C |I|^\rho$. Therefore, we can estimate $E_1$ by

$$
E_1 \leq 2 \int_U \left| \int_U K(w, z)h(z) \, dx \, dy \right|^2 \, du \, dv = 2 ||B(h)||_2^2 \leq C ||h||_2^2 \leq C |I|^\rho.
$$

To estimate $E_2$, we note first that for $n \geq 1$ the estimate $|z - \hat{w}| \geq C(2^n |I|)$ holds if $w \in S(I)$ and $z \in S(I_{n+1}) \setminus S(I_n)$. Direct computation yields also that for any fixed $a > 1$, we have the estimate

$$
\int_{S(I_n)} v^{a-2} \, du \, dv \leq C(2^n |I|)^a \quad \forall n \geq 0.
$$

Hence, rewriting the set $U \setminus S(I_1)$ as the disjoint union $\bigcup_{n \geq 1} S(I_{n+1}) \setminus S(I_n)$, we can estimate $E_2$ by

$$
E_2 = 2 \int_{S(I)} \left( \sum_{n \geq 1} \int_{S(I_{n+1}) \setminus S(I_n)} \frac{y^{b-1}}{|z - \hat{w}|^{b+\tau}} |\psi(z)| \, dx \, dy \right)^2 v^{2z-2+p} \, du \, dv
\leq C \int_{S(I)} \left( \sum_{n \geq 1} \frac{1}{(2^n |I|)^{b+\tau}} \int_{S(I_{n+1})} |\psi(z)| y^{b-1} \, dx \, dy \right)^2 v^{2z-2+p} \, du \, dv
\leq C |I|^{2\tau+p} \left( \sum_{n \geq 1} \frac{1}{(2^n |I|)^{b+\tau}} \int_{S(I_{n+1})} |\psi(z)| y^{b-1} \, dx \, dy \right)^2.
$$
By Hölder’s inequality, we have

\[
\int_{S(I_{n+1})} |\psi(z)| y^{b-1} \, dx \, dy \\
\leq \left( \int_{S(I_{n+1})} |\psi(z)|^2 y^p \, dx \, dy \right)^{1/2} \left( \int_{S(I_{n+1})} y^{2b-p-2} \, dx \, dy \right)^{1/2} \\
\leq \left( \int_{S(I_{n+1})} |\psi(z)|^2 y^p \, dx \, dy \right)^{1/2} (2^{n+1}|I|)^{b-p/2}.
\]

Thus, since the measure $|\psi(z)|^2 y^p \, dx \, dy$ is a bounded $p$-Carleson measure, we can continue the estimate of $E_2$ by

\[
E_2 \leq C|I|^p \left( \sum_{n \geq 1} \frac{1}{2n^2} \left( \frac{1}{(2^{n+1}|I|^p)} \int_{S(I_{n+1})} |\psi(z)|^2 y^p \, dx \, dy \right)^{1/2} \right)^2 \\
\leq C|I|^p \left( \sum_{n \geq 1} \frac{C}{2n^2} \right)^2 \\
\leq C|I|^p.
\]

These prove part (i).

As in the proof of Theorem 3.1, refine the above argument will lead to a proof of part (ii).

For fixed $b > 1$, consider the $t$-derivative $(t > 0)$ of an analytic function $f(z)$ on $U$ defined by

\[
f^{(t)}(z) = (-1)^t \frac{(2i)^{b-1}}{\pi} \frac{\Gamma(b + t)}{\Gamma(b)} \int_U \frac{f'(w)}{(z - \bar{w})^{b+t}} y^{b-1} \, du \, dv.
\]

Here $\Gamma(\cdot)$ is the Gamma function and, in order to have the integral convergent, we assume that $\sup_{z \in U}|f'(z)|y < \infty$ (recall that this is a necessary condition for $f \in Q_p$). We note that $f^{(t)}$ is just the usual $t$th-order derivative of $f$ if $t$ is a positive integer. It is not hard to check, by taking Fourier transformation, that,

\[
f^{(t)}(\cdot + iy)(\xi) = (i|\xi|)^t \hat{f}(\xi) e^{-y|\xi|} \quad \forall \xi \in \mathbb{R}.
\]

Therefore, we have

\[
f^{(t)}(x) = \mathcal{F}^{-1} f(x).
\]

By Theorem 3.3, we have the following corollary.
Corollary 3.4. Suppose $0 < p \leq 1$, $t > \frac{1-p}{2}$ and $f$ is analytic on $U$. Then

(i) the measure $|f'(z)|^2 y^p \, dx \, dy$ is a bounded $p$-Carleson measure if and only if the measure $|f^{(t)}(z)|^2 y^{2t-2+p} \, dx \, dy$ is a bounded $p$-Carleson measure.

(ii) the measure $|f'(z)|^2 y^p \, dx \, dy$ is a compact $p$-Carleson measure if and only if the measure $|f^{(t)}(z)|^2 y^{2t-2+p} \, dx \, dy$ is a compact $p$-Carleson measure.

Proof. The necessity follows directly from Theorem 3.3 with $\tau = t$. For sufficiency, we note first that

$$f'(z) = (-1)^{2-t} \frac{(2t)^{b+t-2}}{\pi} \frac{\Gamma(b+1)}{\Gamma(b+t-1)} \int_U \frac{f^{(t)}(w)}{(z-w)^{b+t+1}} w^{b+t-2} \, du \, dv$$

and this integral is convergent because of a similar reason as stated in the remark after Theorem 3.2. With this formula we can apply Theorem 3.3 again with $\tau = 1$ and $\psi(z) = f^{(t)}(z) y^{t-1}$ to get the desired results. □

For $f$, with the Fourier transformation $\hat{f}(\xi)$, consider the functions $f_+$ and $f_-$ defined by their Fourier transformation:

$$\hat{f}_+(\xi) = \chi_{(0,\infty)}(\xi) \hat{f}(\xi) \quad \text{and} \quad \hat{f}_-(\xi) = \chi_{(0,\infty)}(\xi) \overline{\hat{f}(-\xi)}.$$

It is easy to verify that both $f_+(z) = f_+ * P_y(x)$ and $f_-(z) = f_- * P_y(x)$ are analytic on $U$ and

$$f(z) = f_+(z) + f_-(z), \quad z \in U \quad \text{or} \quad R.$$

Moreover, for $t > 0$ we have

$$\mathcal{J}_t(f_\pm)(z) = (\mathcal{J}_t f)_\pm(z) = (f_\pm)^{(t)}(z).$$

Therefore,

$$\mathcal{J}_t f(z) = (f_+)^{(t)}(z) + (f_-)^{(t)}(z).$$

Theorem 2 is a consequence of Theorem 1 and the following theorem.

Theorem 3.5. Suppose $0 < p \leq 1$, $q = \frac{1-p}{2}$ and $f$ is a measurable function on $R$.

(i) The following are equivalent:

(a) $f$ is in $Q_p$;

(b) both $f_+$ and $f_-$ are in $Q_p$;

(c) the measure $|\nabla(\mathcal{J}_q f)(z)|^2 y \, dx \, dy$ is a bounded $p$-Carleson measure.
The following are equivalent:
(iia) \( f \) is in \( Q_{p,0} \);
(iiib) both \( f_+ \) and \( f_- \) are in \( Q_{p,0} \); and
(iic) the measure \( |\nabla (\mathcal{I}_{-q} f)(z)|^2 y \, dx \, dy \) is a compact \( p \)-Carleson measure.

Proof. The equivalence of (ia) and (ib), and the equivalence of (iia) and (iib) are clear. We prove the equivalence of (ib) and (ic), and the equivalence of (iib) and (iic).

Denote by \( \partial = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) \) and \( \bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) \). We have clearly that the identity \(|\nabla \varphi(z)|^2 = 2|\partial \varphi(z)|^2 + 2|\bar{\partial} \varphi(z)|^2\) holds for any \( C^1 \) function \( \varphi \) on \( U \). Hence, by Corollary 3.4 with \( t = q + 1 \), we get that condition (ic) is equivalent to that the measures \(|f_\pm|^{(q+1)}(z)|^2 y \, dx \, dy \) are bounded \( p \)-Carleson measures. This is equivalent to that \( f_\pm(z) \) are in \( Q_p \).

The proof of the equivalence of (iic) and (iib) is similar. \( \square \)

We now turn to several applications of our main results.

We note first that functions in Morrey space can be characterized by the boundedness of their fractional maximum functions. It is proved in [C] that, for \( 0 < \lambda < 1 \), \( g \) is in \( \mathcal{L}^{2,\lambda} \) if and only if \( \sup_I |I|^{-\lambda} \int_I |g(x)|^2 \, dx = M_g^2 < \infty \). Therefore, we can define the \( \mathcal{L}^{2,\lambda} \) norm of \( g \) by \( M_g \). We refer to this new norm for \( \mathcal{L}^{2,\lambda} \) in the rest of this paper.

Denote by \( \chi(x) \) the characteristic function of the interval \((0, \infty)\) on \( R \). Let \( \mathcal{M} \) be the set of all nonnegative Borel measures \( \sigma(z) \) on \( U \) with the normalized condition \( \sigma(U) = 1 \). For \( 0 < \lambda < 1 \) and \( \sigma \in \mathcal{M} \), let

\[
\omega_{\sigma,\lambda}(x) = \int_U v^{-\lambda} \chi(v - |x - u|) \, d\sigma(w).
\]

Denote by \( \mathcal{L}_{2,\lambda} \) the set of all functions \( h \) on \( R \) such that

\[
||h||_{\mathcal{L}_{2,\lambda}} = \inf_{\sigma \in \mathcal{M}} \left( \int_R |h(x)|^2 \frac{1}{\omega_{\sigma,\lambda}(x)} \, dx \right)^{1/2} < \infty.
\]

It is proved in [K] that under the pairing \( \int_R h(x) \bar{g}(x) \, dx \) the dual of \( \mathcal{L}_{2,\lambda} \) is \( \mathcal{L}^{2,\lambda} \). This result can be translated onto \( Q_p \) space by the use of Theorem 2, which is the following theorem.

**Theorem 3.6.** Suppose \( 0 < p < 1 \) and \( q = \frac{1-p}{2} \). Under the pairing \( \int_R h(x) \overline{f(x)} \, dx \) the dual of \( \mathcal{I}_{-q} \mathcal{L}_{2,p} \) is \( Q_p \).

Let \( H \) be the Hilbert transformation, i.e.

\[
\widehat{Hf}(\xi) = i \text{sign}(\xi) \hat{f}(\xi).
\]

It is easy to verify that \( Hf(z) = if_+(z) - if_-(z) \).

The following theorem is now obvious.
Theorem 3.7. Suppose $0 < p < 1$ and $g$ is a measurable function on $R$.

(i) The following are equivalent:
   (ia) $g$ is in $L^{2,p}$;
   (ib) both $g_+$ and $g_-$ are in $L^{2,p}$; and
   (ic) $Hg$ is in $L^{2,p}$.

(ii) The following are equivalent:
   (iia) $g$ is in $L^{2,p}_0$;
   (iib) both $g_+$ and $g_-$ are in $L^{2,p}_0$; and
   (iic) $Hg$ is in $L^{2,p}_0$.

Theorem 3.7 can be translated onto $Q_p$ by using Theorem 2.

Corollary 3.8. Suppose $0 < p < 1$ and $f$ is a measurable function on $R$. Then

(i) $f$ is in $Q_p$ if and only if $Hf$ is in $Q_p$;
(ii) $f$ is in $Q_{p,0}$ if and only if $Hf$ is in $Q_{p,0}$.

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References


