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Note

A Note on van der Waerden's Theorem

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INTRODUCTION

Furstenberg and Weiss proved in [2] that if $T_1,..., T_p$ are commuting homeomorphisms of a compact metric space X, then there is a point $x \in X$ and a sequence $n_k \to \infty$ so that $T_i^{n_k}x \to x$ simultaneously for $i = 1, 2,..., p$. From this, they easily derive the following celebrated result of van der Waerden:

THEOREM (B. L. van der Waerden [3]). *If the set N of nonnegative integers is partitioned into two classes, then at least one of the classes must contain arbitrarily long finite-arithmetic progressions.*

In this note, we give a combinatorial version of the Furstenberg–Weiss topological proof of van der Waerden's theorem. Prior to the Furstenberg–Weiss argument, all known proofs of van der Waerden's theorem proceeded inductively by verifying the stronger version obtained by first considering partitions of only some initial segment of N (originally a suggestion of Schreier), and secondly, considering partitions into r classes instead of just two (originally a suggestion of Artin). The combinatorial proof we give here avoids both of these strengthenings. (A combinatorial proof avoiding the second type of above-mentioned strengthening has recently been obtained by Deuber [1].)

The proof can be viewed as taking place in two steps. The first step involves defining an ordering on the set of partitions of N into two pieces and then noting that if there is a counterexample to van der Waerden's theorem, then there is one that is minimal with respect to this ordering. This seems to be the key novelty of the proof and it is easily seen to apply to any partition theorem that involves partitioning an infinite set and claiming the existence of a finite-homogeneous set. (It is somewhat of a tribute to

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topology to note that this idea occurs in [2] in the proof of Theorem 1.4 (last line on p. 66) where it is (accurately) asserted that "we can suppose without loss of generality that (X, G) is a homogeneous system.")

Some notation and terminology will simplify the following proof. Let $\mathscr{X}$ denote the set of all functions mapping N to $\{0, 1\}$, where we think of a function $f \in \mathscr{X}$ as an infinite sequence of 0's and 1's. (Hence, $\mathscr{X}$ is the set of partitions of N into two pieces). If $f \in \mathscr{X}$ and $n \in N$, then $f \mid [0, n]$ denotes the finite sequence $\langle f(0), f(1), ..., f(n) \rangle$. If s and f are sequences with at least s finite, then $s\hat{\ }f$ denotes the sequence obtained by concatenating s and f. If $f \in \mathscr{X}$ and $n \in N$, then the sequence

$$\langle f(n), f(n+1), ..., f(n+k) \rangle$$

will be called a "block of f (of length $k+1$)." If we also have $g \in \mathscr{X}$, then we shall say that the above block of f "occurs in g (at t)" if $f(n+i) = g(t+i)$ for $i = 0, 1, ..., k$.

The proof itself will occupy the next two sections. In Section 1 we shall consider the ordering on $\mathscr{X}$ referred to above. It should be noted that the observations there are in no way restricted to the present context of van der Waerden's theorem (e.g., there is no mention of arithmetic progressions). Finally, in Section 2 we shall rather quickly derive the desired result from the considerations of Section 1.

1. The Ordering

Define a relation $\lesssim$ on $\mathscr{X}$ by $f \lesssim g$ iff every block of f occurs in g. Clearly, $\lesssim$ is a preordering of $\mathscr{X}$ (i.e., it is reflexive and transitive). Let $\equiv$ be the equivalence relation induced by $\lesssim$ (i.e., $f \equiv g$ iff $f \lesssim g$ and $g \lesssim f$) and for $f \in \mathscr{X}$, let $[f]$ denote the equivalence class of f.

Lemma 1.1. *Every descending $\lesssim$-chain is bounded below, and hence, every element of $\mathscr{X}$ has a $\lesssim$-minimal element $\lesssim$-below it.*

Proof. If a finite sequence s occurs as blocks in cofinally many elements of the chain, then the same is true of either $s\hat{\ }\langle 0 \rangle$ or $s\hat{\ }\langle 1 \rangle$. Hence, we can easily construct $h \in \mathscr{X}$ inductively so that $h \lesssim f$ for every f in the chain. ∎

Lemma 1.2. *If f is $\lesssim$-minimal and $g \in [f]$, then* (a)–(c) *hold*:

(a) $\forall m, n \in N$ *($g \mid [0, n]$ occurs in f at some $t \geqslant m$).*

(b) *Either $\langle 0 \rangle\hat{\ }g \in [f]$ or $\langle 1 \rangle\hat{\ }g \in [f]$.*

(c) $\forall m, n \in N\, \exists s$ *($|s| = n$ and $s\hat{\ }g \in [f]$).*

Proof. (a) Let f' be the tail segment of f beginning at t. Then clearly, $f' \lesssim f$ and since f is minimal we have $g \equiv f \equiv f'$. Thus $g \mid [0, n]$ occurs in f' and so $g \mid [0, n]$ occurs in f at some $t \geqslant m$.

(b) If not, then we can choose some r large enough so that neither $\langle 0 \rangle^\frown g \mid [0, r]$ nor $\langle 1 \rangle^\frown g \mid [0, r]$ occurs in f. But by (a), we know that $g \mid [0, r]$ occurs in f at some $t \geqslant 2$; contradiction.

(c) This follows from n applications of (b). ∎

LEMMA 1.3. *If f is $\lesssim$-minimal, then there exists a function h_f so that for any $g \equiv f$, every block b of f occurs in $g \mid [0, h_f(b)]$.*

Proof. Supose not. Then here is some block b of f so that for every n we can choose $g_n \equiv f$ such that b does not occur in $g_n \mid [0, n]$. No single g can serve as g_n for infinitely many distinct n's (or else b does not occur in g at all and so $f \not\lesssim g$). Now construct g' inductively so that for every p we have infinitely many distinct g_n's such that $g' \mid [0, p] = g_n \mid [0, p]$. Then clearly, $g' \lesssim f$ but $g' \not\equiv f$ since b does not occur in g'. This contradicts the $\lesssim$-minimality of f. ∎

2. The Proof of van der Waerden's Theorem

With the notation and terminology that we are using, van der Waerden's theorem asserts that for every $f_0 \in \mathscr{X}$ and every $k \in N$ there is a sequence s of length one (i.e., $s = \langle 0 \rangle$ or $s = \langle 1 \rangle$) and $a, r \in N$ so that s occurs in f_0 at $a + r$, $a + 2r$,..., $a + kr$. Given such an f_0 we can choose (by Lemma 1.1) a $\lesssim$-minimal f so that $f \lesssim f_0$, and it clearly suffices to show that for some $g \equiv f$ we have an s as above that occurs in g at $a + r$,..., $a + kr$. The stronger inductive hypothesis that we shall employ asserts that we can do this not only for sequences of length one, but for any initial segment of f (and also that we can take $a = 0$). Thus, we inductively verify the following two assertions $\mathscr{S}(k)$ and $\mathscr{H}(k)$. The former is this stronger inductive statement; the latter is a stepping stone from $\mathscr{S}(k)$ to $\mathscr{S}(k+1)$.

$\mathscr{S}(k)$ For every $n \in N$ and every $\lesssim$-minimal f, there is some $g \equiv f$ and some $r > n$ so that $f \mid [0, n]$ occurs in g at r, $2r$,..., kr.

$\mathscr{H}(k)$ For every $n \in N$ and every $\lesssim$-minimal f, there is some $g \equiv f$ and some $r > n$ so that $g \mid [0, n]$ occurs in itself at 0, r,..., $(k-1)r$.

LEMMA 2.1. *$\mathscr{S}(0)$ holds.*

Proof. Lemma 1.2(a) shows that for a given n and f, any $g \in f$ works (e.g., we can take $g = f$; hence $\mathscr{H}(0)$ holds also). ∎

LEMMA 2.2. $\mathscr{G}(k) \Rightarrow \mathscr{H}(k+1)$.

Proof. Suppose that $\mathscr{G}(k)$ holds and that we are given $n \in N$ and a $\lesssim$-minimal element f of $\mathscr{X}$. Let $n_0 = n$ and let $g_0 = f$ and apply $\mathscr{G}(k)$ to the pair (n_0, g_0) to obtain $g_1 \equiv g_0$ and $r_1 > n_0$ so that

$$g_0 \mid [0, n_0] \text{ occurs in } g_1 \text{ at } r_1, 2r_1, \ldots, kr_1.$$

Let $n_1 = kr_1 + n_0$ and repeat the above procedure applying $\mathscr{G}(k)$ to the pair (n_1, g_1) to obtain $g_2 \equiv g_1$ and $r_2 > n_1$. Notice that we have

$$g_1 \mid [0, n_1] \qquad \text{occurs in } g_2 \quad \text{at} \quad r_2, 2r_2, \ldots, kr,$$

and so

$$g_0 \mid [0, n_0] \qquad \text{occurs in } g_2 \quad \text{at} \quad r_2 + r_1, 2r_2 + 2r_1, \ldots, kr_2 + kr_1.$$

We continue this procedure until we have obtained (g_i, n_i, r_i) for every $i \leqslant 2^{n_0+1}$. Now choose p and q so that $0 \leqslant p < q \leqslant 2^{n_0+1}$ and

$$g_p \mid [0, n_0] = g_q \mid [0, n_0].$$

Notice that our construction guarantees that $g_p \mid [\mathrm{p}, n_0]$ occurs in g_q at $r_q + \cdots + r_{p+1}, 2r_q + \cdots + 2r_{p+1}, \ldots, kr_q + \cdots + kr_{p+1}$. Hence, if we let $g = g_q$ and $r = r_q + \cdots + r_{p+1}$, then $g \equiv f$ (since $g = g_q \equiv g_0 = f$), $r > n = n_0$ and

$$g \mid [0, n] \qquad \text{occurs in itself at} \quad 0, r, \ldots, kr. \quad \blacksquare$$

LEMMA 2.3. $\mathscr{H}(k+1) \Rightarrow \mathscr{G}(k+1)$.

Proof. Given n and f, let $n' = h_f(b)$, where $b = f \mid [0, n]$ and h_f is as in Lemma 1.3. By $\mathscr{H}(k+1)$ there is some $g' \equiv f$ and some $r > n'$ so that

$$g' \mid [0, n'] \qquad \text{occurs in itself at} \quad 0, r, \ldots, kr.$$

Since $n' = h_f(b)$ and $b = f \mid [0, n]$, there is some a so that $f \mid [0, n]$ occurs in $g' \mid [0, n']$ at a. Thus

$$f \mid [0, n] \qquad \text{occurs in } g' \quad \text{at} \quad a, r + a, \ldots, kr + a.$$

By Lemma 1.2(c) we can choose s so that $|s| = r - a$ and so that if we let $g = s \hat{\ } g'$, then $g \equiv f$. But then

$$f \mid [0, n] \qquad \text{occurs in } g \quad \text{at} \quad r, 2r, \ldots, (k+1)r$$

as desired. $\blacksquare$

References

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3. B. L. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wisk. **15** (1927), 212–216.