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## Directed strongly regular graphs obtained from coherent algebras<sup>☆</sup>

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Dedicated to the memory of J.J. Seidel

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### Abstract

The notion of a directed strongly regular graph was introduced by A. Duval in 1988 as one of the possible generalizations of classical strongly regular graphs to the directed case. We investigate this generalization with the aid of coherent algebras in the sense of D.G. Higman. We show that the coherent algebra of a mixed directed strongly regular graph is a non-commutative algebra of rank at least 6. With this in mind, we examine the group algebras of dihedral groups, the flag algebras of a Steiner 2-designs, in search of directed strongly regular graphs. As a result, a few new infinite series of directed strongly regular graphs are constructed. In particular, this provides a positive answer to a question of Duval on the existence of a graph with certain parameter set having 20 vertices. One more open case

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<sup>☆</sup> This paper is a revised and shortened version of the preprint [29]. This preprint stimulated a new wave of interest in the investigations of d.s.r.g.'s, in particular results by L. Jorgensen, S. A. Hobart and T.J. Shaw, A. Duval and D. Iourinski. We refer to [11,21,25,30] where a part of recent investigations is discussed with more details. We also mention that the most of the current status of the theory of d.s.r.g.'s can be found in Andries Brouwer's home page <http://www.cwi.nl/~aeb/math/dsrg/>.

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with 14 vertices listed in Duval's paper is ruled out, while new interpretations in terms of coherent algebras are given for many of Duval's results.

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## 1. Introduction

The concept of a strongly regular graph is one of the central objects in modern algebraic graph theory. It was introduced by Bose in [2], although the concept itself had been known earlier under the notion "two-class association scheme". The interest in two-class association schemes in turn was stimulated by combinatorial designs of experiments as well as by finite geometries.

In 1967, Seidel published his first paper [37] on strongly regular graphs. Seidel's enthusiasm (cf. [4] for a survey) was one of the most important factors in the rapid development of the investigation of strongly regular graphs, however not the only one.

A little earlier, in 1964, Higman had already investigated finite permutation groups of rank 3; cf. [16]. Each such permutation group yields a pair of complementary strongly regular graphs. The significance of strongly regular graphs for the theory of finite groups became obvious a few years later, in Oxford, when Higman and Sims constructed (after a talk given by Hall on the sporadic simple group  $J_2$  and within the period of 24 h; see [14], Section 2.6) the new sporadic simple group  $HS$ .

Meanwhile, the strongly regular graphs which come from finite permutation groups of rank 3 have been classified via the classification of the finite simple groups; cf., e.g., [32]. A classification of all strongly regular graphs is certainly out of reach, but one might be tempted to explore to which extent the so-called adjacency algebra of a strongly regular graph might be able to replace the group-theoretical reasoning when considering strongly regular graphs in general.

By definition, the *adjacency algebra* of a graph is the matrix algebra generated by its adjacency matrix. Occasionally, the adjacency algebra of a graph reflects some of the graph-theoretical properties of the graph in question. For instance, the adjacency algebra of an undirected graph  $\Gamma$  contains the (all 1) matrix  $J$  if and only if  $\Gamma$  is connected and regular; cf. e.g., [22]. Similarly, an undirected graph is strongly regular if and only if its adjacency algebra contains  $J$  and has rank 3.

Looking at this algebraic characterization of strongly regular graphs it seems to be natural to look for directed graphs whose adjacency algebra has similar properties. Duval considered such graphs in 1988, and his paper [5] has motivated our present work. The notion which was introduced by Duval is called here a directed strongly regular graph. In our paper we mainly focus on "mixed" graphs, that is,

those which have undirected edges as well as directed ones. In such a graph  $\Gamma$ , the adjacency algebra is a proper subalgebra of the *coherent algebra generated by the adjacency matrix of  $\Gamma$* . The simultaneous use of both of these algebras will be our main methodological principle.

In Section 2, we introduce the basic concepts which we shall need in the present paper. We state Duval's main result on directed strongly regular graphs (d.s.r.g.'s). In Section 3, we define the coherent algebra of a graph. In later sections, we construct d.s.r.g.'s with certain prescribed coherent algebras. The constructions in Section 4 come from dihedral groups, those in Section 6 from dihedral schemes. Dihedral schemes generalize dihedral groups in the same way as buildings (in the sense of Tits) generalize Coxeter groups. In order to clarify this relationship, we give an introduction to dihedral schemes in Section 5. The last three sections are devoted to the discussion of small graphs having at most 20 vertices. In Section 7, new interpretations are given for a few sporadic examples found by Duval. In Section 8, we rule out one of the possibilities in Duval's list of small graphs. We include the proof in its full length, because the method seems to be interesting in its own right. The whole list itself is reviewed in Section 9. In this paper, we have not used computers for the enumeration of graphs. A few problems to which computers can be applied are briefly discussed in Section 9.

This paper originates from a talk by Klin at Oberwolfach, January 1994 (see [28]) who had presented a first portion of new results and posed questions some of which were solved by other authors soon after the meeting at Oberwolfach. Our initial intention was to submit a paper to the special volume of "Linear Algebra and its Applications" dedicated to Seidel on the occasion of his 75th birthday. Unfortunately, we spent too much time on polishing the initial draft of this paper, and missed the deadline for the submission to the volume. Sadly, we now dedicate this paper to his memory as he passed away in May, 2001.

## 2. Basic concepts and Duval's main result

Let  $n$  denote a positive integer, set  $\Omega := \Omega_n := \{1, 2, \dots, n\}$ , and let  $R$  denote a subset of the Cartesian product  $\Omega \times \Omega$ . Then  $\Gamma = (\Omega, R)$  is a (*directed*) graph with *vertex set*  $\Omega$  and *arc set*  $R$ . If  $R$  is an *antireflexive relation* over  $\Omega$ , that is, if  $R \cap \{(x, x) : x \in \Omega\} = \emptyset$ , then  $\Gamma$  is called a graph *without loops*.  $\Gamma$  is an *undirected graph* if  $R$  is antireflexive and symmetric, that is,  $R = R^t$ , where  $R^t = \{(y, x) : (x, y) \in R\}$ . Set  $\tilde{\Omega}^2 := \{(x, y) : x, y \in \Omega, x \neq y\}$ . A *complementary graph* to a directed graph  $\Gamma = (\Omega, R)$  without loops is a graph  $\bar{\Gamma} = (\Omega, \bar{R})$ , where  $\bar{R} = \tilde{\Omega}^2 \setminus R$ . For  $x \in \Omega$  the *input valency*  $iv(x)$  (resp. the *output valency*  $ov(x)$ ) is the cardinality of the set  $\{y : (y, x) \in R\}$  (resp.  $\{y : (x, y) \in R\}$ ).  $\Gamma$  is called *regular* of valency  $k$  if all its vertices have input valency  $k$  and output valency  $k$ . To each graph  $\Gamma = (\Omega, R)$  we associate its adjacency matrix  $A = A(\Gamma) = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ , where

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in R, \\ 0 & \text{if } (i, j) \notin R. \end{cases}$$

We exhibit two distinguished matrices, namely  $I = I_n$ , the *identity matrix* of order  $n$ , and  $J = J_n$ , the matrix whose entries are all equal to 1. For any two matrices  $A$  and  $B$  of order  $n$ , we denote by  $A \circ B$  the Schur–Hadamard (entrywise) product of  $A$  and  $B$ .

The classical notion of a *strongly regular graph* (briefly *s.r.g.*; see [2,4,15,23,37]) with parameters  $(n, k, \lambda, \mu)$ , is defined as an undirected graph  $\Gamma = (V, E)$  with  $n$  vertices whose adjacency matrix  $A = A(\Gamma)$  satisfies

$$A^2 = kI + \lambda A + \mu(J - I - A) \tag{2.1}$$

and

$$AJ = JA = kJ. \tag{2.2}$$

A tournament is a directed graph  $T$  whose adjacency matrix  $A = A(T)$  satisfies  $A + A^T + I = J$ . If, moreover,  $A$  satisfies (2.2) and

$$A^2 = \lambda A + \mu(J - I - A), \tag{2.3}$$

then  $T$  is called doubly regular [3].

In [5], Duval suggested the following generalization of the notion of an s.r.g. A directed graph without loops  $\Gamma = (V, R)$  with adjacency matrix  $A = A(\Gamma)$  is called a *directed strongly regular graph* (*d.s.r.g.*) with parameters  $(n, k, \mu, \lambda, t)$  or an  $(n, k, \mu, \lambda, t)$ -graph if  $A$  satisfies (2.2) and

$$A^2 = tI + \lambda A + \mu(J - I - A). \tag{2.4}$$

The notion of a d.s.r.g. is a generalization of both, s.r.g. and doubly regular tournaments. In fact, a d.s.r.g. with  $t = k$  is an s.r.g. and a d.s.r.g. with  $t = 0$  is a doubly regular tournament. But there exist also d.s.r.g.'s with

$$0 < t < k. \tag{2.5}$$

We shall call each d.s.r.g. which satisfies (2.5) a *mixed* (or *genuine*) d.s.r.g. The arc set  $R$  of every mixed d.s.r.g. can be uniquely decomposed as  $R = R_s \cup R_a$ ,  $R_s \cap R_a = \emptyset$ , where  $R_s$  is a symmetric and  $R_a$  is an antisymmetric binary relation. We will use the notation  $A_s = A(\Gamma_s)$ ,  $A_a = A(\Gamma_a)$ , where  $\Gamma_s = (V, R_s)$ ,  $\Gamma_a = (V, R_a)$ . We call a matrix  $A$  *antisymmetric* if  $A \circ A^t = 0$ . It is clear that the matrix  $A_a$  is antisymmetric. We will use the notation  $\overline{A}_s$  and  $\overline{A}_a$ , similarly to  $A_s$  and  $A_a$ , for the complementary graph  $\overline{\Gamma}$ .

Using fairly traditional tools from matrix theory, Duval has developed an excellent starting background for the theory of d.s.r.g.'s; see [5]. Firstly he proved that, for every  $(n, k, \mu, \lambda, t)$ -graph  $\Gamma$  with adjacency matrix  $A$ , its complementary graph  $\overline{\Gamma}$  with adjacency matrix  $\overline{A}$  is an  $(n, k', \mu', \lambda', t')$ -graph, where

$$\begin{aligned} k' &= (n - 2k) + k - 1, \\ \mu' &= (n - 2k) + \lambda, \\ \lambda' &= (n - 2k) + \mu - 2, \\ t' &= (n - 2k) + t - 1. \end{aligned}$$

Also, Duval established necessary conditions to the existence of a d.s.r.g.

**Theorem 2.1** (Duval’s Main Theorem). *Let  $\Gamma$  be a  $(n, k, \mu, \lambda, t)$ -graph. Then one of the following holds.*

- (i)  $\Gamma$  is an s.r.g. ( $t = k$ ).
- (ii)  $\Gamma$  is a doubly regular tournament ( $t = 0$ ).
- (iii)  $\Gamma$  is a genuine d.s.r.g. ( $0 < t < k$ ) and there exists some positive integer  $d$  for which the following requirements are satisfied:

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu, \tag{2.6}$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2, \tag{2.7}$$

$$d \mid (2k - (\mu - \lambda)(n - 1)), \tag{2.8}$$

$$\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod{2}, \tag{2.9}$$

$$\left\lfloor \frac{2k - (\mu - \lambda)(n - 1)}{d} \right\rfloor \leq n - 1. \tag{2.10}$$

Here, complete and empty graphs are considered as partial cases of s.r.g. ( $k = n - 1$  and  $k = 0$ , respectively). A lot of other theoretical results and helpful examples can be found in Duval’s paper, including a list of all feasible parameters with  $n \leq 20$  which will be discussed in Section 9. In our attempt to create a new vision of d.s.r.g.’s in terms of coherent algebras, we shall start from the consideration of certain types of matrix algebras.

Let  $W$  be an algebra of matrices of order  $n$  over  $\mathbb{C}$  such that:

- (i) there exists a basis  $\mathcal{A} = \{A_1, \dots, A_r\}$  of  $W$  (as a vector space) which consists of  $\{0, 1\}$ -matrices;
- (ii)  $A_i^t \in \mathcal{A}$  for  $1 \leq i \leq r$ , where  $A_i^t$  is the transpose of  $A_i$ ;
- (iii)  $\sum_{i=1}^r A_i = J$ ;
- (iv)  $I \in W$ .

Then  $W$  is called a *coherent algebra* of degree  $n$  and rank  $r$  with the *standard basis*  $\mathcal{A} = \{A_1, A_2, \dots, A_r\}$ . We may write  $W = \langle A_1, A_2, \dots, A_r \rangle$ .

It follows from the definition of a coherent algebra that there exist non-negative integers  $p_{ij}^k$  such that

$$A_i A_j = \sum_{k=1}^r p_{ij}^k A_k, \quad 1 \leq i, j \leq r.$$

The integers  $p_{ij}^k$  are called the *structure constants* of  $W$ .

Each *basis matrix*  $A_i$  of a coherent algebra  $W = \langle A_1, A_2, \dots, A_r \rangle$  can be regarded as the adjacency matrix  $A = A(\Gamma_i)$  of a graph  $\Gamma_i = (\Omega_n, R_i)$ . Then  $\Gamma_i$  and  $R_i$  are called a *basis graph* and a *basis relation*, respectively, of the coherent algebra  $W$ . The basis relations of a coherent algebra give rise to a *coherent configuration* in the sense of [18].

If all basis graphs of a coherent configuration are regular, then it is called an *association scheme*. The coherent algebra which corresponds to an association scheme is called a *homogeneous coherent algebra* or *Bose–Mesner algebra* (or briefly *BM-algebra*) of the association scheme. The centralizer algebra of a permutation group  $G$  is the algebra of matrices which commute with the permutation matrices corresponding to permutations in  $G$ . It is easy to see that the centralizer algebra of a transitive permutation group  $G$  is a homogeneous coherent algebra.

Let  $G$  denote a finite group, and let  $X$  denote a non-empty subset of  $G$ . We denote by  $X^{(-1)}$  the set  $\{x^{-1} : x \in X\}$ . The element  $\sum_{x \in X} x$  in the *group ring*  $\mathbb{Z}(G)$  will be called a *simple quantity*. It will be denoted as  $\underline{X}$ . Assume now that  $e \notin X$ . Then the graph  $\Gamma = \Gamma(G, X) = (G, \{(x, y) : x, y \in G, yx^{-1} \in X\})$  is called the *Cayley graph* over  $G$  with respect to  $X$ .

*Historical and bibliographical remarks.* The notion of a coherent algebra appeared in Moscow in 1968 in [43] under the name “cellular algebra”. More accessible is a collective volume [42]. The notion of coherent configuration was introduced by Higman in 1970 in [17], the term “coherent algebra” is adopted from [19]. The textbook [1] as well as the lecture notes [45] are standard references for the foundations of the theory of association schemes. A more detailed exposition of the notions considered in this section can be found in [8,9].

### 3. The coherent algebra of a d.s.r.g.

Let  $W_1, W_2$  be two coherent algebras of degree  $n$ . Then  $W_1 \cap W_2$  is a coherent algebra again (cf. [20,40]). This result implies that there exists a *unique minimal coherent algebra*  $W = W(A_1, A_2, \dots, A_k) = \langle\langle A_1, A_2, \dots, A_k \rangle\rangle$  which contains given  $n$  by  $n$  matrices  $A_1, A_2, \dots, A_k$ . This algebra  $W$  will be called the *coherent algebra generated by*  $A_1, A_2, \dots, A_k$  (or the *coherent closure* of  $A_1, A_2, \dots, A_k$ ).

To every d.s.r.g.  $\Gamma$  with adjacency matrix  $A$  we associate a coherent algebra  $W(\Gamma) = \langle\langle A \rangle\rangle$ . We shall say that  $W(\Gamma)$  is the coherent algebra of  $\Gamma$ . If  $\Gamma$  is a genuine d.s.r.g., then obviously  $I, A_s, A_a, \overline{A}_s, \overline{A}_a \in W(\Gamma)$ . These matrices are linearly independent. Hence we obtain  $\text{rank}(W(\Gamma)) \geq 5$ . We shall prove that the equality cannot occur.

**Lemma 3.1** (Higman [18]). *Let  $W$  be a homogeneous coherent algebra of rank  $\leq 5$ . Then  $W$  is commutative.*

*Recall that we use a non-traditional definition of an antisymmetric matrix.*

**Lemma 3.2.** *Let  $\Gamma$  be a regular non-empty directed graph without undirected edges (i.e.,  $A(\Gamma)$  is an antisymmetric matrix). Then  $A = A(\Gamma)$  has at least one non-real eigenvalue.*

**Proof.** Suppose that  $\Gamma$  is regular of valency  $k$ . Then  $A^2$  has an eigenvalue  $k^2$ . Since  $A$  is antisymmetric, the trace of  $A^2$  is zero. Therefore, at least one of the eigenvalues of  $A$  is non-real.  $\square$

**Theorem 3.3.** *Let  $\Gamma$  be a genuine d.s.r.g. Then:*

- (a)  $W(\Gamma)$  is non-commutative;
- (b)  $\text{rank}(W(\Gamma)) \geq 6$ .

**Proof.** Set  $A = A(\Gamma)$ . Then  $A = A_s + A_a$ , with  $A_s, A_a \in W(\Gamma)$ . Suppose that  $W(\Gamma)$  is commutative. Then  $A_s$  commutes with  $A_a$ , and, therefore the eigenvalues of  $A$  are the sums of the corresponding eigenvalues of  $A_s$  and  $A_a$ . All eigenvalues of a symmetric matrix  $A_s$  are real. By Lemma 3.2, at least one of the eigenvalues of  $A_a$  is non-real. It follows that at least one of the eigenvalues of  $A$  is also non-real. This contradicts the part of the proof of Theorem 2.2 of [5], in which it was shown that all eigenvalues of a genuine d.s.r.g. are integers. Thus  $W(\Gamma)$  is non-commutative.

To prove (b), we only have to exclude the case  $\text{rank}(W(\Gamma)) = 5$ . It follows from (a) that  $W(\Gamma)$  is non-commutative. By Lemma 3.1,  $\mathcal{A}(\Gamma)$  is non-homogeneous. The only non-homogeneous coherent algebra of rank 5 is the centralizer algebra of the intransitive permutation representation of the symmetric group  $S_n$  on  $n + 1$  points (one of the points is fixed by every element of  $S_n$ ). One can easily check that the adjacency matrix of a genuine d.s.r.g. can not belong to this algebra.  $\square$

We conclude this section with the following simple helpful observation due to Pech. We shall use it in Section 6.

**Proposition 3.4.** *Let  $\Gamma$  be a d.s.r.g.,  $A = A(\Gamma)$ , let  $\Gamma^t$  be a graph such that  $A(\Gamma^t) = A^t$ . Then  $\Gamma^t$  is also a d.s.r.g. with the same parameters as  $\Gamma$ .*

**Proof.** The assertion follows immediately by taking the transpose of the both sides of the Eq. (2.4).  $\square$

#### 4. D.s.r.g.'s arising from dihedral groups

Let  $n$  denote an integer with  $n \geq 3$ , let  $D_n$  denote the dihedral group of order  $2n$ , and let  $C_n = \langle c \rangle$  denote the cyclic normal subgroup of  $D_n$  of order  $n$ . In this section, we shall construct Cayley graphs over  $D_n$  which are d.s.r.g.'s. We shall denote by

$\mathbb{C}(D_n)$  the group ring of  $D_n$  over  $\mathbb{C}$ . If  $X \subset D_n$ , then the Cayley graph  $\Gamma(D_n, X)$  is a d.s.r.g. if and only if  $\gamma = \underline{X}$  satisfies the equation analogous to (2.4):

$$\gamma^2 = t\underline{e} + \lambda\gamma + \mu(\underline{D_n} - \underline{e} - \gamma). \tag{4.1}$$

If (4.1) is satisfied, then  $\Gamma(D_n, X)$  is a  $(2n, |X|, \mu, \lambda, t)$ -graph.

**Example 4.1.** Let  $n = 4, d \in D_4 \setminus C_4$  be an arbitrary involution. Then the Cayley graph  $\Gamma(D_4, X)$  with  $X = \{c, d, c^3d\}$  is a  $(8, 3, 1, 1, 2)$ -graph.

**Lemma 4.1.** Let  $X, Y \subseteq C_n$ , where  $n$  is odd, satisfy the following conditions

- (i)  $\underline{X} + \underline{X}^{(-1)} = \underline{C_n} - \underline{e}$ ,
- (ii)  $\underline{Y} \cdot \underline{Y}^{(-1)} - \underline{X} \cdot \underline{X}^{(-1)} = \varepsilon \underline{C_n}, \varepsilon \in \{0, 1\}$ .

Let  $a \in D_n \setminus C_n$ . Then the Cayley graph  $\Gamma = \Gamma(D_n, X \cup aY)$  is a d.s.r.g. with parameters  $(2n, n - 1 + \varepsilon, \frac{n-1}{2} + \varepsilon, \frac{n-3}{2} + \varepsilon, \frac{n-1}{2} + \varepsilon)$ . In particular, if  $X$  satisfies (i) and  $Y = Xg$  or  $X^{(-1)}g$  for some  $g \in C_n$ , then  $\Gamma$  is a d.s.r.g. with parameters  $(2n, n - 1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$ .

**Proof.** Let  $\gamma = \underline{X \cup aY} = \underline{X} + \underline{aY}$ . Then

$$\begin{aligned} \gamma^2 &= (\underline{X} + \underline{aY})^2 \\ &= \underline{X}^2 + \underline{aX}^{(-1)} \cdot \underline{Y} + \underline{aY} \cdot \underline{X} + \underline{Y}^{(-1)} \cdot \underline{Y} \\ &= \underline{X}^2 + \underline{Y} \cdot \underline{Y}^{(-1)} + \underline{aY}(\underline{C_n} - \underline{e}) \\ &= \underline{X}^2 + \varepsilon \underline{C_n} + \underline{X} \cdot \underline{X}^{(-1)} + \underline{aY}(\underline{C_n} - \underline{e}) \\ &= \underline{X} \cdot (\underline{C_n} - \underline{e}) + \varepsilon \underline{C_n} + \underline{a}(|Y|\underline{C_n}) - \underline{aY} \\ &= (|X| + \varepsilon) \cdot \underline{C_n} + |Y|\underline{aC_n} - (\underline{X} + \underline{aY}). \end{aligned}$$

To finish the proof it is enough to check  $|X| + \varepsilon = |Y|$ . It follows from (i) that  $|X| = \frac{n-1}{2}$ . From (ii) we obtain  $|Y|^2 = |X|^2 + \varepsilon n = (|X| + \varepsilon)^2$ , and the proof is complete. □

**Example 4.2.** Let  $n$  be odd. If  $X = Y = \{c, c^2, \dots, c^{(n-1)/2}\}$ , then the conditions of Lemma 4.1 are satisfied, and we obtain a  $(2n, n - 1, \frac{n-1}{2}, \frac{n-3}{2}, \frac{n-1}{2})$ -graph. If  $n = 3$ , then the resulting graph is the  $(6, 2, 1, 0, 1)$ -graph in [5, p. 72].

**Lemma 4.2.** Let  $X, Y \subseteq C_n$ , where  $n$  is odd, and let  $a \in D_n \setminus C_n$ . Suppose that the Cayley graph  $\Gamma(D_n, X \cup aY)$  is a d.s.r.g. If  $\underline{X} + \underline{X}^{(-1)} = \underline{C_n} - \underline{e}$ , then  $\underline{Y} \cdot \underline{Y}^{(-1)} - \underline{X} \cdot \underline{X}^{(-1)} = \varepsilon \underline{C_n}, \varepsilon \in \{0, 1\}$ .

**Proof.** Let  $(2n, k, \mu, \lambda, t)$  be the parameters of the graph. Then

$$(\underline{X} + \underline{aY})^2 = t\underline{e} + \lambda(\underline{X} + \underline{aY}) + \mu(\underline{C_n} + \underline{aC_n} - \underline{e} - \underline{X} - \underline{aY}). \tag{4.2}$$



On the other hand

$$(\underline{X} + \underline{aY})^2 = \underline{X}^2 + \underline{Y} \cdot \underline{Y}^{(-1)} + |Y|\underline{aC}_n - \underline{aY}. \tag{4.3}$$

Comparing (4.2) and (4.3) we obtain

$$\underline{X}^2 + \underline{Y} \cdot \underline{Y}^{(-1)} = t\underline{e} + \lambda\underline{X} + \mu(\underline{C}_n - \underline{e} - \underline{X}), \tag{4.4}$$

$$|Y|\underline{C}_n - \underline{Y} = \lambda\underline{Y} + \mu(\underline{C}_n - \underline{Y}). \tag{4.5}$$

If  $Y = C_n$ , then (4.5) implies  $\lambda = n - 1$ . By the assumption, the coefficient of  $e$  in  $\underline{X}^2$  is zero, hence (4.4) implies  $t = n$ . Then by (2.6), taking into account that  $k = \frac{3n-1}{2}$ , we find  $\mu = \frac{3n+1}{2} > n = t$ , which is impossible by [5, Theorem 2.3].

Thus  $Y \neq C_n$  and (4.5) implies

$$\begin{cases} \lambda - \mu = -1, \\ \mu = |Y|. \end{cases} \tag{4.6}$$

Substituting (4.6) into (4.4), we get

$$\begin{aligned} \underline{Y} \cdot \underline{Y}^{(-1)} &= -\underline{X}^2 + t\underline{e} + |Y|(\underline{C}_n - \underline{e}) - \underline{X} \\ &= -\underline{X}(\underline{C}_n - \underline{e} - \underline{X}^{(-1)}) + t\underline{e} + |Y|(\underline{C}_n - \underline{e}) - \underline{X} \\ &= \underline{X} \cdot \underline{X}^{(-1)} + (|Y| - |X|)\underline{C}_n + (t - |Y|)\underline{e}. \end{aligned}$$

This implies  $|Y| = t$  and

$$|Y|^2 = |X|^2 + (|Y| - |X|)n. \tag{4.7}$$

Then (4.7) implies either  $|Y| = |X|$ , or  $|Y| + |X| = n$ . In the latter case, since  $|X| = \frac{n-1}{2}$ , we find  $|Y| = \frac{n+1}{2}$ . Hence  $|Y| - |X| \in \{0, 1\}$ .  $\square$

**Lemma 4.3.** *Let  $n$  be an odd prime, and let  $a \in D_n \setminus C_n$ . Suppose that the Cayley graph  $\Gamma = \Gamma(D_n, X \cup aY)$  is a genuine d.s.r.g. Then*

$$\underline{X} + \underline{X}^{(-1)} = \underline{C}_n - \underline{e}. \tag{4.8}$$

**Proof.** Let  $X = \{c^{i_1}, \dots, c^{i_m}\}$ ,  $m = |X|$ ,  $0 < i_1 < \dots < i_m < n$ . It is well-known (see, e.g. [31, p. 66]), that  $D_n$  has an irreducible character  $\chi$  such that  $\chi(h) = 0$ ,  $h \notin C_n$ , and  $\chi(c^j) = w^j + w^{-j}$ , where  $w$  is a primitive  $n$ th root of unity. Since the eigenvalues of  $\Gamma$  are integers,

$$\chi(\underline{X} + \underline{aY}) \in \mathbb{Z}. \tag{4.9}$$

On the other hand,

$$\chi(\underline{X} + \underline{aY}) = w^{i_1} + \dots + w^{i_m} + w^{n-i_m} + \dots + w^{n-i_1}.$$

Note that the element  $\sum_{j=1}^{n-1} d_j w^j$ , where  $d_j$ 's are rational, is rational if and only if  $d_1 = \dots = d_{n-1}$ . Hence (4.9) implies either

- (i)  $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, n - 1\}$ , or
- (ii)  $\{i_1, \dots, i_m\} \cup \{n - i_m, \dots, n - i_1\} = \{1, 2, \dots, n - 1\}$ .

In the first case we obtain  $X = X^{(-1)}$ , hence  $(X \cup aY)^{(-1)} = X \cup aY$ . This implies  $t = k$ , contradicting the assumption that  $\Gamma$  is genuine. The second case gives the assertion of the lemma.  $\square$

**Theorem 4.4.** *Let  $n$  be an odd prime,  $X, Y \subseteq C_n$ ,  $a \in D_n \setminus C_n$ . Then the Cayley graph  $\Gamma(D_n, X \cup aY)$  is a genuine d.s.r.g. if and only if  $X, Y$  satisfy the conditions of Lemma 4.1.*

**Proof.** The theorem is a consequence of all lemmata which were proved in this section.  $\square$

**Example 4.3.** Let  $\Gamma_1$  be the graph defined in Example 4.2 with  $n = 7$ . Let  $\Gamma_2 = \Gamma(D_7, \{c, c^2, c^4, ac, ac^2, ac^4\})$ . Then it follows from Lemma 4.1 that both  $\Gamma_1$  and  $\Gamma_2$  are genuine d.s.r.g.'s with parameters  $(14, 6, 3, 2, 3)$ . It is easy to find the coherent algebras of these graphs:  $W(\Gamma_1) = \mathbb{C}(D_7)$ , while  $W(\Gamma_2) = \langle \underline{e}, \underline{c}, \underline{c^2}, \underline{c^4}, \underline{c^3}, \underline{c^5}, \underline{c^6}, \underline{a}, \underline{ac}, \underline{ac^2}, \underline{ac^4}, \underline{ac^3}, \underline{ac^5}, \underline{ac^6} \rangle$ . Hence  $\Gamma_1$  and  $\Gamma_2$  are not isomorphic. We can show that  $\text{Aut}(\Gamma_1) = D_7$ , while  $\text{Aut}(\Gamma_2)$  has order 42.

**Example 4.4.** Let  $\Gamma = \Gamma(D_9, \{c, c^3, c^4, c^7, ac, ac^3, ac^4, ac^7\})$ . Then it follows from Lemma 4.1 that  $\Gamma$  is an  $(18, 8, 4, 3, 4)$ -graph. A similar argument as in the previous example shows that  $\Gamma$  is not isomorphic to the graph in Example 4.2 with  $n = 9$ . We have  $\text{Aut}(\Gamma) = (\mathbb{Z}_3 \wr \mathbb{Z}_3) \times S_2$ .

**Remark 1.** On the basis of Theorem 4.4, a method is described for finding a complete list of all pairwise non-isomorphic genuine d.s.r.g.'s over  $D_n$ , when  $n$  is an odd prime, like in [9]. Here results on S-rings over dihedral groups [36,38,41] can be helpful. However, we attract the reader's attention to the mistake in [38]. The S-ring  $W(\Gamma_2)$  considered in Example 4.3 is missing in the "complete list" of all S-rings over  $D_7$  which is presented in [38].

**Remark 2.** We do not consider here the problem of classifying all genuine d.s.r.g.'s over  $D_n$  for arbitrary  $n$ . There exist cases which can not be covered by means of the construction given in Lemma 4.1. In particular, other solutions can be found using pairs of inequivalent Hadamard difference sets over cyclic group of order  $n$  (see for details survey paper [26] by Jungnickel).

### 5. The flag algebra of BIBD with $\lambda = 1$

A *balanced incomplete block design* (BIBD) is an incidence structure  $S = (\mathcal{P}, \mathcal{B})$ , where  $\mathcal{P}$  is a finite set of points,  $\mathcal{B}$  is a family of  $k$ -element subsets (called blocks)

of  $\mathcal{P}$ , such that every pair of distinct points is contained in exactly  $\lambda$  blocks. In a BIBD, every point is contained in exactly  $r$  blocks, where  $r = \lambda(v - 1)/(k - 1)$ . The set  $(v, b, r, k, \lambda)$ , where  $b = |\mathcal{B}|$ , is called the parameters of the BIBD.

In what follows we shall consider the special case where  $\lambda = 1$ . For a BIBD with  $\lambda = 1$ ,  $v$  and  $b$  can be computed from  $k$  and  $r$ : set  $x = k - 1$ ,  $y = r - 1$ . Then we have

$$\begin{cases} v = 1 + x + xy, \\ b = \frac{(1+x+xy)(y+1)}{x+1}, \\ r = y + 1, \\ k = x + 1. \end{cases} \tag{5.1}$$

Let  $\mathcal{F}$  denote the set of incident point-block pairs. The elements of  $\mathcal{F}$  are called flags. Set  $n := |\mathcal{F}|$ . Then

$$n = (1 + x + xy)(1 + y). \tag{5.2}$$

The well-known Fisher’s inequality says that, for every BIBD,  $b \geq v$ . A BIBD is called symmetric if  $b = v$ .

We define the following binary relations on  $\mathcal{F}$  (here  $f = (p, C)$  and  $g := (q, D)$  are two flags):

- $R_0 = \{(f, f) : f \in \mathcal{F}\}$ , (the diagonal of  $\mathcal{F}$ ),
- $R_1 = \{(f, g) : p \neq q, C = D\}$ , coLinear flags,
- $R_2 = \{(f, g) : p = q, C \neq D\}$ , coNcurrent flags,
- $R_3 = \{(f, g) : p \neq q, C \neq D, (q, C) \in \mathcal{F}\}$ ,
- $R_4 = \{(f, g) : p \neq q, C \neq D, (p, D) \in \mathcal{F}\}$ ,
- $R_5 = \{(f, g) : p \neq q, C \neq D, (q, C) \notin \mathcal{F}, (p, D) \notin \mathcal{F}, C \cap D \neq \emptyset\}$ ,
- $R_6 = \{(f, g) : p \neq q, C \neq D, (q, C) \notin \mathcal{F}, (p, D) \notin \mathcal{F}, C \cap D = \emptyset\}$ .

The following lemma is an easy consequence of the definition of BIBD with  $\lambda = 1$ .

**Lemma 5.1.** *Assume that  $S$  is a BIBD with  $\lambda = 1$ . Then, for each  $i \in \{0, \dots, 6\}$ ,  $R_i$  defines a regular graph  $\Gamma_i = (\mathcal{F}, R_i)$ . The valencies  $n_i$  of  $\Gamma_i$  are:*

$$\begin{aligned} n_0 = 1, \quad n_1 = x, \quad n_2 = y, \quad n_3 = n_4 = xy, \\ n_5 = x^2y, \quad n_6 = xy(y - x). \end{aligned}$$

$R_1, R_2, R_5$ , and  $R_6$  are symmetric relations;  $R_3$  and  $R_4$  are paired antisymmetric relations.  $\square$

Note that  $R_6 = \emptyset$  if  $S$  is symmetric. We shall mainly restrict our attention to the case  $y > x$ . However, we note that our results remain valid even for the case  $y = x$  after the trivial modifications.

If  $A = (a_{ij})$  and  $B = (b_{ij})$  are integer matrices we shall write  $B \geq A$  ( $B$  covers  $A$ ) if  $b_{ij} \geq a_{ij}$  for all  $i, j$ . Let us now prove that  $M = (\mathcal{F}, R)$  is an association scheme with six classes. In order to prove this, we shall consider the adjacency matrices  $A_i$  of the graphs  $\Gamma_i = (\mathcal{F}, R_i)$ ,  $i \in \{0, 1, \dots, 6\}$ . We need to prove that  $W = \langle A_0, A_1, A_2, A_3, A_4, A_5, A_6 \rangle$  is an algebra of dimension 7.

A matrix  $A$  is called *doubly stochastic* if the sum of the elements in each row and each column of  $A$  is equal to a constant. This constant is called the valency of  $A$ . By Lemma 5.1, each of the matrices  $A_0, \dots, A_6$  is doubly stochastic, hence every element of the algebra  $W$  is also doubly stochastic. Let us use the following mnemonic notation for the matrices  $A_1$  and  $A_2$ :  $A_1 =: L$ ,  $A_2 =: N$ . (See the definition of  $R_1$  and  $R_2$ .)

**Lemma 5.2.**  $A_3 = LN$ ,  $A_4 = NL$ ,  $A_5 = LNL$ ,  $A_6 = NLN - LNL$ .

**Proof.**  $A_3$  has valency  $xy$ , and is covered by  $LN$ . The valency of  $LN$  is also  $xy$ . Therefore,  $LN = A_3$ . A similar reasoning shows that  $A_4 = NL$  and  $A_5 = LNL$ . Now let us consider  $NLN$ . It is easy to see that  $NLN$  covers  $A_5 + A_6$ . Then we again compare the valencies, and we obtain that  $A_6 = NLN - LNL$ .  $\square$

**Lemma 5.3.** *We have*

$$LNLN = xNL + (x - 1)LNL + x(NLN - LNL) \quad (5.3)$$

and

$$NLNL = xLN + (x - 1)LNL + x(NLN - LNL). \quad (5.4)$$

**Proof.** Let us first prove (5.3). Let us start with a flag  $(p, C)$  and let us find all paths of length 4 which begin at  $(p, C)$  and which alternately go through arcs of the relations  $R_1$  and  $R_2$ . Let  $(p, C), (q, C), (q, D), (r, D), (r, E)$  be such a path. First of all, we claim that  $p \neq r$ , and  $C \neq E$ . Indeed, if  $p = r$ , then  $p, q \in C \cap D$ . But  $p \neq q$  and  $C \neq D$ , contrary to  $\lambda = 1$ . Similarly,  $q \neq r$  and  $C \neq D$  implies that  $C \neq E$ .

Assume first that  $(p, E) \in \mathcal{F}$ . Then the above claim shows that there exist at least  $x$  paths to  $(r, E)$ .

Now assume that  $(p, E) \notin \mathcal{F}$ . Then there exist at least  $x - 1$  paths to  $(r, E)$  in the case where  $E \cap C \neq \emptyset$ , and there exist at least  $x$  paths to  $(r, E)$  if  $E \cap C = \emptyset$ .

Altogether we have constructed  $x \cdot yx + (x - 1) \cdot x^2y + x(y^2x - x^2y) = x^2y^2$  distinct paths of length 4. However altogether there are exactly  $x^2y^2$  paths. Hence all possibilities are accounted and (5.3) has been proved. The proof of (5.4) is similar.  $\square$

**Theorem 5.4.**  $W = \langle A_0, A_1, A_2, A_3, A_4, A_5, A_6 \rangle$  is a coherent algebra.

**Proof.** It is easy to see that  $R_1$  defines a graph with vertex set  $\mathcal{F}$  which is isomorphic to a disjoint union of complete graphs each of which has  $x + 1$  vertices. From this it follows that

$$L^2 = xI + (x - 1)L. \quad (5.5)$$

Similarly,

$$N^2 = yI + (y - 1)N. \quad (5.6)$$

Together with (5.3) and (5.4) we now have a set of defining relations for the matrix algebra  $W$ . It is evident that the set of these relations is sufficient to express each product of matrices  $A_i, A_j, 0 \leq i, j \leq 6$ , as a linear combination of the seven matrices  $A_0, \dots, A_6$ . Therefore,  $W$  is closed under multiplication. The axioms (i)–(iv) for a coherent algebra are satisfied evidently, hence  $W$  is a coherent algebra.  $\square$

Table 1 gives the intersection numbers of the coherent algebra  $W$ . In a cell of Table 1 which is the intersection of row  $i$  and column  $j$ , the intersection numbers  $p_{ij}^k, 0 \leq k \leq 6$  are given, read from the top to the bottom.

### Remarks

1. In the case  $x = y$  the relation  $R_6$  is empty. One can use Table 1 to obtain the intersection numbers of the rank 6 coherent algebra on the flags of a projective plane, if the last row and column as well as the last entry in each cell of Table 1 are deleted. Note that these intersection numbers appeared already in [9] where they were computed by direct combinatorial considerations without the use of defining relations.
2. In the case of a non-symmetric BIBD with  $\lambda = 1$ , the rank 7 coherent algebra defined on flags (the flag algebra of BIBD) was first given in [46]. In [44] it is indicated how to obtain the corresponding coherent algebra for finite Moore geometries.

It follows from the proof of Theorem 5.4 that the coherent algebra  $W$  of rank 7 (or of rank 6 in the case  $x = y$ ) is generated (as a usual matrix algebra) by  $L$  and  $N$ . The matrices  $L$  and  $N$  are adjacency matrices of disjoint unions of complete graphs. Following [44] an element  $A$  of the standard basis of a coherent algebra  $W$  is called a (*generalized*) *involution* if the following condition holds:

$$A^2 = xI + (x - 1)A \quad \text{for some integer } x. \quad (5.7)$$

Note that  $A$  is symmetric. Moreover, if  $x = 1$ , then (5.7) is equivalent to the fact that  $A$  is a permutation matrix of an involution (thereby justifying the use of the word “(generalized) involution”).

Let  $W$  be a coherent algebra, and let  $A_1, A_2$  be two involutions of  $W$ . Let us call  $W$  *dihedral* if  $W = \langle\langle A_1, A_2 \rangle\rangle$  and  $A_1 + A_2$  is the adjacency matrix of a connected graph. In [46], it was proved that

Table 1  
Structure constants of a rank 7 flag algebra

$i \setminus j$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
	0	1	0	0	0	0	0
	0	0	1	0	0	0	0
	0	0	0	1	0	0	0
	0	0	0	0	1	0	0
	0	0	0	0	0	1	0
	0	0	0	0	0	0	1
1	0	$x$	0	0	0	0	0
	1	$x-1$	0	0	0	0	0
	0	0	0	$x$	0	0	0
	0	0	1	$x-1$	0	0	0
	0	0	0	0	0	$x$	0
	0	0	0	0	1	$x-1$	0
	0	0	0	0	0	0	$x$
2	0	0	$y$	0	0	0	0
	0	0	0	0	$y$	0	0
	1	0	$y-1$	0	0	0	0
	0	0	0	0	0	$x$	$y-x$
	0	1	0	0	$y-1$	0	0
	0	0	0	1	0	$x-1$	$y-x$
	0	0	0	1	0	$x$	$y-x-1$
	0	0	0	0	$xy$	0	0
	0	0	$y$	0	$(x-1)y$	0	0
	0	0	0	0	0	$x^2$	$x(y-x)$

3	1	0	$y - 1$	0	0	$(x - 1)x$	$(x - 1)(y - x)$
	0	0	0	$x$	0	$(x - 1)x$	$x(y - x)$
	0	1	0	$x - 1$	$y - 1$	$(x - 1)^2$	$(x - 1)(y - x)$
	0	0	0	$x$	0	$x^2$	$x(y - x - 1)$
	0	0	0	$xy$	0	0	0
	0	0	0	0	0	$xy$	0
	0	$x$	0	$x(y - 1)$	0	0	0
4	0	0	0	0	$x$	$(x - 1)x$	$x(y - x)$
	1	$x - 1$	0	0	0	$(y - 1)x$	0
	0	0	1	$x - 1$	$x - 1$	$(x - 1)^2$	$x(y - x)$
	0	0	1	$x - 1$	$x$	$(x - 1)x$	$x(y - x - 1)$
	0	0	0	0	0	$x^2y$	0
	0	0	0	$xy$	0	$(x - 1)xy$	0
	0	0	0	0	$x^2$	$(x - 1)x^2$	$x^2(y - x)$
5	0	$x$	0	$x(y - 1)$	$x(x - 1)$	$(x - 1)^2x$	$(x - 1)x(y - x)$
	0	0	$x$	$x(x - 1)$	$x(x - 1)$	$(x - 1)^2x$	$x^2(y - x)$
	1	$x - 1$	$x - 1$	$(x - 1)^2$	$(x - 1)^2$	$x^3 - 3x^2 + 2x + xy - 1$	$(x - 1)x(y - x)$
	0	0	$x$	$x(x - 1)$	$x^2$	$(x - 1)x^2$	$x^2(y - x - 1)$
	0	0	0	0	0	0	$xy(y - x)$
	0	0	0	0	0	0	$xy(y - x)$
	0	0	0	0	$x(y - x)$	$x^2(y - x)$	$x(y - x)(y - x - 1)$
6	0	0	0	0	$x(y - x)$	$x^2(y - x)$	$x(y - x)(y - x - 1)$
	0	0	$y - x$	$x(y - x)$	$(x - 1)(y - x)$	$(x - 1)x(y - x)$	$x(y - x)(y - x - 1)$
	0	0	$y - x$	$x(y - x)$	$(x - 1)(y - x)$	$(x - 1)x(y - x)$	$x(y - x)(y - x - 1)$
	1	$x$	$y - x - 1$	$x(y - x - 1)$	$x(y - x - 1)$	$x^2(y - x - 1)$	$x^3 + 3x^2 - 2x^2y - 2xy + xy^2 + 2x - y$

- (a) the flag algebras of projective planes are the only non-commutative dihedral coherent algebras of rank 6;
- (b) let  $W$  be a non-commutative dihedral coherent algebra of rank 7 generated by the involutions  $A_1$  and  $A_2$ , and assume that the valencies of  $A_1$  and  $A_2$  are distinct, then  $W$  is the flag algebra of a non-symmetric BIBD with  $\lambda = 1$ .

It turns out that dihedral coherent configurations are a natural tool for the investigation not only of BIBD's with  $\lambda = 1$  but also of generalized polygons and amalgamated products of finite groups. In particular, the famous theorem of Feit–Higman [10] can be seen naturally as a first result on dihedral coherent algebras; see [44]. We believe that a deeper investigation of the relationship between dihedral coherent algebras and d.s.r.g.'s is a worthwhile subject of interest.

We would like to mention in conclusion of this section that the idea of the use of defining relations for the description of the flag algebra of block designs and Moore geometries goes back to [27,33–35] and especially [39]. In a more general setting this idea can be even traced to [24]. In another context, defining relations on involutions were considered in [13].

### 6. D.s.r.g.'s arising from dihedral coherent algebras

In this section, we construct new d.s.r.g.'s from dihedral coherent algebras. We keep the notation of the adjacency matrices defined in the previous section.

**Proposition 6.1.** *Let  $W$  be the flag algebra of a projective plane of order  $q$ ,  $W = \langle A_0, A_1, A_2, A_3, A_4, A_5 \rangle$ , where  $A_3$  and  $A_4$  are a pair of antisymmetric matrices. Let  $i \in \{3, 4\}$ ,  $j \in \{1, 2, 5\}$ . Then  $A_i + A_j$  is the adjacency matrix of a genuine d.s.r.g. whose parameters are*

$$((q + 1)(q^2 + q + 1), q^2 + 2q, 2q - 1, q + 1, 2q)$$

for the cases  $(i, j) = (3, 5)$  or  $(4, 5)$  and

$$((q + 1)(q^2 + q + 1), q^2 + q, q, q - 1, q)$$

in all other cases.

The proof consists of routine computations which are based on the use of the structure constants of  $W$ . They can be read off from Table 1. Here we set  $q := x = y$ . However, Table 2.1.1 in [9] is more convenient for the computations, because it represents the rank 6 flag algebra in an evident form.

**Proposition 6.2.** *Let  $W = \langle A_0, A_1, A_2, A_3, A_4, A_5, A_6 \rangle$  be a rank 7 flag algebra of a non-symmetric BIBD with  $\lambda = 1$ , let  $n = (1 + x + xy)(1 + y)$ . Then  $A \in W$  is*



an adjacency matrix of a genuine d.s.r.g. if and only if one of the following holds (up to the complementation of a graph):

- (a)  $A = A_1 + A_i$ , and  $A$  is the adjacency matrix of an  $(n, x + xy, x, x - 1, x)$ -graph;
- (b)  $A = A_1 + A_2 + A_i$ ,  $(n, x + y + xy, x + 1, x + y - 1, x + y)$ -graph;
- (c)  $A = A_1 + A_i + A_5$ ,  $x = 1$ ,  $((y + 1)(y + 2), 2y + 1, 2, y, y + 1)$ -graph;

where  $i \in \{3, 4\}$ .

**Proof.** First we assume  $i = 3$  and compute  $A^2$  by using Table 1. In case (a) we have

$$A^2 = xA_0 + (x - 1)(A_1 + A_3) + x(A_2 + A_4 + A_5 + A_6).$$

In case (b), we have

$$A^2 = (x + y)A_0 + (x + y - 1)(A_1 + A_2 + A_3) + (x + 1)(A_4 + A_5 + A_6).$$

In case (c), the coefficient of  $A_4$  (resp.  $A_6$ ) in  $A^2$  is  $x^3 + x$  (resp.  $x^3 + x^2$ ). Thus  $x = 1$  and we have

$$A^2 = (y + 1)A_0 + y(A_1 + A_3 + A_5) + 2(A_2 + A_4 + A_6).$$

Therefore, if  $i = 3$ , then the cases (a)–(c) give a d.s.r.g. as described. By Proposition 3.4, the same conclusion holds if  $i = 4$ .

In order to finish the proof we have to examine all other partitions of the set  $\{1, 2, 3, 4, 5, 6\}$  of indices to two parts which can possibly give a genuine d.s.r.g. There are altogether 14 suitable partitions, 6 of them being currently considered. Certain routine computations show that in the remaining 8 cases no d.s.r.g. can be obtained.  $\square$

### Remarks

1. Two particular cases of Propositions 6.1 and 6.2 were announced in [28]. One of these two cases was formulated in [28] in different terms; namely, the vertices of a d.s.r.g. were the ordered pairs of different elements of an  $m$ -element set. One can easily see that the flags of a trivial BIBD (with  $x = 1$ ) can be identified with the ordered pairs of different points of this BIBD.
2. Let us set  $x = 1$  and  $y = 3$  in (b) and (c) of Proposition 6.2. Then we obtain a  $(20, 7, 2, 3, 4)$ -graph, which gives a positive answer to one of the open questions in Duval's list [5].
3. In the same way all coherent subalgebras of the dihedral coherent algebra defined by the flags of certain classes of designs can be described (see [30]).

## 7. New interpretations of known graphs

In this section, we consider a few constructions due to Duval from the point of view of coherent algebras.

Let  $W_1$  be a homogeneous coherent algebra of degree  $n$  and rank  $d + 1$  with basic relations  $R_0, R_1, \dots, R_d$  defined on an  $n$ -element set  $\Omega_1$ . Let  $W_2$  be a coherent algebra of degree  $m$  and rank  $d' + 1$  with basic relations  $S_0, S_1, \dots, S_{d'}$  defined on an  $m$ -element set  $\Omega_2$ . Let  $\Omega = \Omega_1 \times \Omega_2$  and  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_d, \bar{S}_0, \bar{S}_1, \dots, \bar{S}_{d'}$  be the relations on  $\Omega$  defined as follows:

$$\bar{R}_i = \{((a, b), (c, d)) \in \Omega \times \Omega : (a, c) \in R_i\},$$

$$\bar{S}_j = \{((a, b), (a, d)) \in \Omega \times \Omega : (b, d) \in S_j\}.$$

Then the relations  $\bar{R}_1, \bar{R}_2, \dots, \bar{R}_d, \bar{S}_0, \bar{S}_1, \dots, \bar{S}_{d'}$  define a coherent configuration on the set  $\Omega$  with coherent algebra  $W$  of rank  $d + d' + 1$ . This coherent algebra is called the *wreath product* of algebras  $W_1$  and  $W_2$  (see [42]), and is denoted by  $W_1 \wr W_2$ .

**Proposition 7.1.** *Let  $\Gamma$  be a genuine  $(n, k, \mu, \lambda, t)$ -graph with adjacency matrix  $A$ , coherent algebra  $W_1$  and the automorphism group  $G_1$  acting on the vertex set  $\Omega_1$ . Then*

- $B' = A \otimes J_m$  is the adjacency matrix of a genuine d.s.r.g.  $\Gamma^*$  if and only if  $t = \mu$ ; in this case  $\Gamma^*$  is an  $(nm, km, \mu m, \lambda m, tm)$ -graph;
- $B'' = A \otimes J_m + I_n \otimes (J_m - I_m)$  is the adjacency matrix of a genuine d.s.r.g.  $\Gamma^{**}$  if and only if  $\lambda = t - 1$ ; in this case  $\Gamma^{**}$  is a  $(nm, (k + 1)m - 1, \mu m, (t + 1)m - 2, (t + 1)m - 1)$ -graph;
- $\Gamma^*$  and  $\Gamma^{**}$  are invariant under the wreath product  $G_1 \wr S_m$ .
- The coherent algebras of the graphs  $\Gamma^*$  and  $\Gamma^{**}$  are both the wreath product  $W_1 \wr W_2$ , where  $W_2 = \langle I_m, J_m - I_m \rangle$ .

The proof of (a) and (b) was in [5]. The proof of (c) and (d) follows easily from the definitions.

Two sporadic constructions given below illustrate the symmetry of other graphs in Duval's list.

**Example 7.1.** We take, as the starting point the pentagon. The vertices and arcs (altogether 15 elements) of the pentagon form the vertex set of a new graph  $\Gamma$ . There are the following directed arcs in  $\Gamma$ : (here and below different letters denote different elements of  $\Omega_1 = \{1, 2, 3, 4, 5\}$ )

- from an old vertex  $a$  to an old arc  $(a, b)$ ;
- from an old arc  $(a, b)$  to an old vertex  $b$ ;
- from an old arc  $(a, b)$  to an old arc  $(c, d)$  if and only if there exist an old arc  $(b, c)$ .

There are the following undirected edges in  $\Gamma$ :

- old edges in the pentagon;
- two old arcs  $(a, b)$  and  $(c, d)$  with  $a = c$  or  $b = d$ .

This graph  $\Gamma$  is a  $(15, 4, 1, 1, 2)$ -graph given in Section 9 of [5]. It follows from our construction that  $\Gamma$  is invariant under the intransitive action of the dihedral group  $D_5$  on the set of vertices and arcs of the pentagon.

The centralizer algebra of this action has rank 23. One can show that  $W(\Gamma)$  coincides with a non-homogeneous coherent algebra of rank 23. Therefore  $\text{Aut}(\Gamma) = D_5$ .

**Example 7.2.** The graph depicted in Fig. 1 has adjacency matrix given in Section 4 of [5]. This graph is a  $(18, 4, 1, 0, 3)$ -graph.

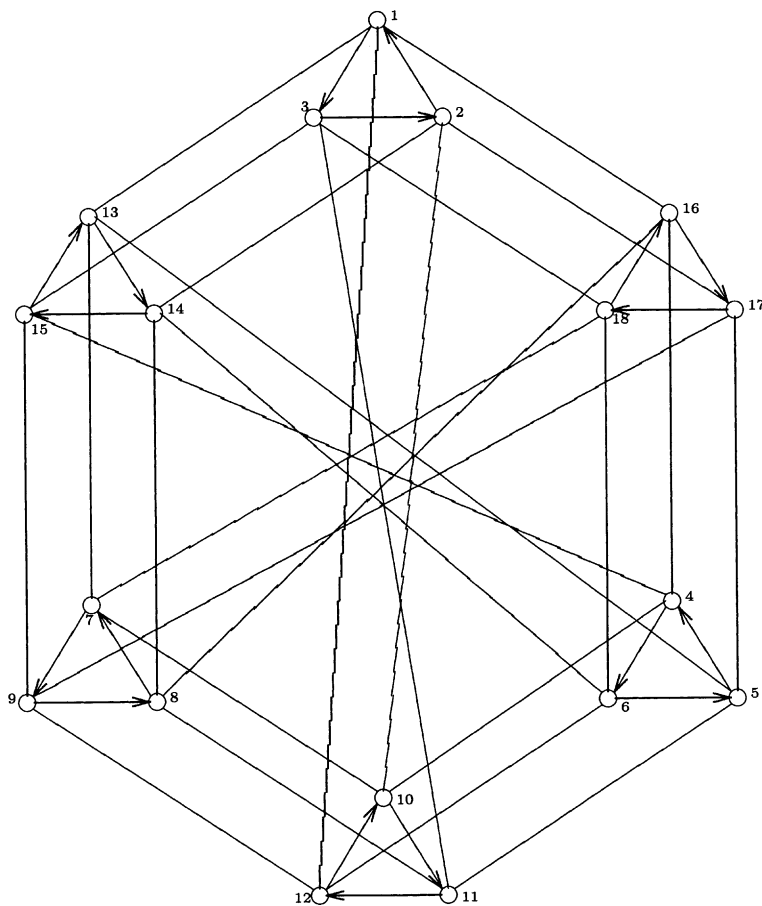


Fig. 1. A diagram of a  $(18, 4, 1, 0, 3)$ -graph.

Set  $G := \text{Aut}(\Gamma)$ . Let  $H < G$  be the subgroup which stabilizes (as a set) each of the directed triangles in  $\Gamma$ . Then  $H$  is a normal subgroup of  $G$ . One can easily check that  $H = \langle g_1 \rangle$ , where

$$g_1 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)(13, 14, 15)(16, 17, 18).$$

Hence  $G = H \cdot \bar{G}$ , where  $\bar{G} \cong G/H \cdot \bar{G}$  can be considered as a subgroup of  $\text{Aut}(\bar{\Gamma}) = S_2 \wr S_3$ , where  $\bar{\Gamma}$  is a quotient graph of  $\Gamma$ , namely  $\bar{\Gamma}$  is the 6-vertex complete bipartite regular graph. Now let us examine the permutations  $g_2, g_3, g_4, g_5$  and  $g_6$  where

$$g_2 = (13, 16)(14, 17)(15, 18)(4, 7)(5, 8)(6, 9),$$

$$g_3 = (1, 4)(2, 5)(3, 6)(10, 13)(11, 14)(12, 15),$$

$$g_4 = (16, 12, 13)(17, 10, 14)(18, 11, 15)(4, 6, 5)(8, 9, 7),$$

$$g_5 = (1, 16)(2, 18)(13, 4)(14, 6)(5, 15)(8, 12)(9, 11)(7, 10)(3, 17),$$

$$g_6 = (13, 16)(14, 17)(15, 18).$$

One can easily check that  $g_2, g_3, g_4, g_5 \in G$  while  $g_6 \notin G$ . This implies that  $|\bar{G}| = 36$ ,  $\bar{G} = (S_2 \wr S_3)^{\text{pos}}$ , where  $(S_2 \wr S_3)^{\text{pos}}$  denotes the subgroup of even permutations in  $(S_2 \wr S_3)$ . Now we obtain that  $G = \mathbb{Z}_3 \cdot (S_2 \wr S_3)^{\text{pos}}$ ,  $|G| = 108$ . It is easy to see that the centralizer algebra of the action of  $G$  on the vertex set of  $\Gamma$  has rank 7. This algebra coincides with the coherent algebra of  $\Gamma$ .

**Remark.** The last example may be of independent interest because it demonstrates the existence of the central extension of  $\mathbb{Z}_3$  by the group  $(S_2 \wr S_3)^{\text{pos}}$ .

## 8. Non-existence of a (14, 5, 2, 1, 4)-graph

This section is the only one in the paper which concerns with a question of non-existence. A set of parameters of a d.s.r.g. is called *feasible* if it satisfies all known “standard” necessary conditions for the existence of a d.s.r.g. A feasible set of parameters is called *realizable* if there exists at least one graph with such parameters.

The parameter set (14, 5, 2, 1, 4) is feasible in the sense that all the necessary conditions (2.6)–(2.10) are satisfied. The goal of this section is to show that this parameter set is not realizable.

**Theorem 8.1.** *There does not exist a (14, 5, 2, 1, 4)-graph.*

The remainder of this section is devoted to the proof of the above theorem. Let  $\Gamma = (\Omega, E)$  be a (14, 5, 2, 1, 4)-graph. Let  $A = A(\Gamma)$  be its adjacency matrix. Let

$S = A(\Gamma_s)$ ,  $N = A(\Gamma_a)$ , so that  $A = S + N$ , where  $S$  is a symmetric and  $N$  an anti-symmetric permutation matrix. Then we have

$$A^2 = 4I + A + 2(J - I - A). \tag{8.1}$$

We shall consider the cycle structure of the permutation  $N$ , but first we consider  $N^2$ . Let  $O_1, \dots, O_l$  denote the orbits of the cyclic permutation group  $\langle N^2 \rangle$  on  $\Omega$ . It is easy to see that  $N \circ N^t = 0$  implies  $|O_j| \geq 2$  for  $1 \leq j \leq l$ . We write  $L := \{1, 2, \dots, l\}$ .

In order to avoid the use of additional notation we shall identify  $N$  and  $S$  with the arc set of  $\Gamma_a$  and  $\Gamma_s$ , respectively.

**Lemma 8.2.** *The numbers*

$$m_{ij} = |\{v \in O_j | (u, v) \in S\}| \tag{8.2}$$

are independent of the choice of  $u \in O_i$ . Moreover, we have

$$|O_i| m_{ij} = |O_j| m_{ji} \quad (i, j \in L), \tag{8.3}$$

$$\sum_{j=1}^l m_{ij} = 4 \quad (i \in L). \tag{8.4}$$

**Proof.** Taking the transpose of (8.1), and subtracting it from (8.1), we find

$$(N - N^{-1})S = -(S + N + N^{-1} + I)(N - N^{-1}).$$

This implies that  $K := \text{Ker}(N - N^{-1}) = \text{Ker}(N^2 - I)$  is invariant under  $S$ . Since the characteristic vectors  $\mathbf{O}_i$  of  $O_i$  ( $i \in L$ ) form a basis of  $K$ , we see that  $S\mathbf{O}_j$  is a linear combination of  $\mathbf{O}_i$ 's, say  $S\mathbf{O}_j = \sum_{i=1}^l m_{ij} \mathbf{O}_i$ . Then it is easy to see that the coefficient  $m_{ij}$  coincides with the number defined in (8.2) for any  $u \in O_i$ . The equality (8.3) is an immediate consequence of the definition of  $m_{ij}$ , while (8.4) follows from the fact that our graph has parameter  $t = 4$ .  $\square$

Let us call a subset  $I$  of  $L$  *balanced* if for any  $i \in I$  and for any  $j \in L$  the equality  $|O_j| = |O_i|$  implies  $j \in I$ .

**Lemma 8.3.** *Let  $I$  be a non-empty proper balanced subset of  $L$ . Then there exist  $i \in I$  and  $j \notin I$  such that  $m_{ij} \neq 0$ .*

**Proof.** Since  $\Gamma$  is a connected graph, there exists an edge  $(u, v)$  of  $\Gamma$  with  $u \in O_i$  and  $v \in O_j$  for suitable  $i \in I$  and  $j \notin I$ . If  $(u, v) \in N$ , then  $N\mathbf{O}_i = \mathbf{O}_j$ . This implies  $|O_i| = |O_j|$ , contrary to  $I$  being balanced. Therefore,  $(u, v) \in S$ , hence  $m_{ij} \neq 0$ .  $\square$

**Lemma 8.4.** *The permutation group  $\langle N^2 \rangle$  has two orbits of length 7 on  $\Omega$ .*

**Proof.** For a prime  $p$ , let  $I_p$  (resp.  $I'_p$ ) denote the set  $\{i \in L \mid (p, |O_i|) = p\}$  (resp.  $\{i \in L \mid (p, |O_i|) = 1\}$ ). Suppose  $I_p \neq \emptyset$  and  $I'_p \neq \emptyset$ . Since  $I_p$  is balanced, Lemma 8.3 implies that  $|O_i|m_{ij} = |O_j|m_{ji} \neq 0$  for some  $i \in I_p$  and  $j \in I'_p$ .

If  $p \geq 5$ , then (8.4) gives a contradiction. Thus, in particular,  $I_7 \neq \emptyset$  implies  $I'_7 = \emptyset$ . Since  $N^2$  cannot be a 14-cycle, this gives the assertion. It remains to show that  $I_p = \emptyset$  for any prime  $p \neq 7$ .

If  $p = 5$  or  $11$ , then  $I_p \neq \emptyset$  automatically implies  $I'_p \neq \emptyset$ , so we must have  $I_5 = I_{11} = \emptyset$ .

If  $p = 3$ , then (8.4) forces  $m_{ji} = 3$ . Note that  $|O_i| \neq 6$  since  $N$  cannot have a 12-cycle. Also,  $|O_i| \neq 9$  since we have shown  $I_5 = \emptyset$ . Thus  $|O_i| = 3$ . Then for any  $v, v' \in O_j$  such that  $v \neq v'$ , all the three points of  $O_i$  are common neighbors of  $v, v'$ , contrary to  $\max(\lambda, \mu) = 2$ . Therefore we conclude  $I_3 = \emptyset$ .

Now it is easy to see that not all  $|O_i|$  are powers of 2. Consequently, we have  $L = I_7$  and the assertion follows.  $\square$

By Lemma 8.4, we have either

- (i)  $N$  is a 14-cycle, or
- (ii)  $N$  is a product of two 7-cycles.

In any case,  $\langle N^2 \rangle$  has two orbits  $O_1, O_2$  of length 7, and  $K$  has a basis  $\{\mathbf{O}_1, \mathbf{O}_2\}$ .

Let

$$\tilde{S} := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

be the matrix representation of  $S$  on  $K$  with respect to this basis. The diagonal entries of  $\tilde{S}$  must be even since there is no regular 7-vertex graph of odd valency. Hence we have

$$\tilde{S} \in \left\{ \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \right\},$$

while the matrix representation  $\tilde{N}$  of  $N$  on  $K$  is

$$\tilde{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

according as (i) or (ii). The eigenvalues of  $\tilde{A} := \tilde{S} + \tilde{N}$  must belong to the set  $\{5, 1, -2\}$  of eigenvalues of  $A$ . This rules out the case (i) completely. The case (ii) requires more careful consideration.

**Lemma 8.5.** For each  $u \in \Omega$ , exactly one of the pairs  $(u, N^2(u)), (N^{-1}(u), N(u))$  belongs to  $S$ .

**Proof.** Since  $\lambda = 1$  and  $(u, N(u)) \in E$ , there exists a unique point  $w$  such that  $(u, w) \in E$  and  $(w, N(u)) \in E$ . Since  $N$  is a permutation, neither pairs belong to

$N$  and, therefore,  $(u, w) \in S$  and  $(w, N(u)) \in S$ . Since  $\mu = 2$ , there exists a unique point  $v$  other than  $w$  such that  $(N(u), v) \in E$  and  $(v, u) \in E$ .

If  $(N(u), v) \in S$  and  $(v, u) \in S$ , then  $\lambda = 1$  forces  $v = w$ , which is absurd. Thus at least one of  $(N(u), v)$ ,  $(v, u)$  belongs to  $N$ .

If  $(N(u), v) \in N$ , then  $v = N^2(u)$ . Since  $N$  is a product of two 7-cycles, we have  $N^3(u) \neq u$ , hence  $(N^2(u), u) \notin N$ . Thus  $(v, u) = (N^2(u), u) \in S$ .

If  $(N(u), v) \notin N$ , then  $(v, u) \in N$ , so that  $v = N^{-1}(u)$  and, as before,  $(N(u), N^{-1}(u)) \notin N$ . Since  $(N(u), v) \notin N$ ,  $(N(u), v) \in S$ . Now  $(N(u), N^{-1}(u)) \in S$  and  $(N^{-1}(u), N(u)) \in S$ .

Suppose that both  $(u, N^2(u))$  and  $(N^{-1}(u), N(u))$  belong to  $S$ . Then the preceding argument shows that  $v = N^2(u) = N^{-1}(u)$ . This is a contradiction since  $N^3$  has no fixed point.  $\square$

With all the preparations, we can complete the proof of Theorem 8.1 as follows. According to Lemma 8.5, let us assume, without loss of generality, that  $(u, N^2(u)) \in S$  and  $(N^{-1}(u), N(u)) \notin S$ . Now apply Lemma 8.5 to  $N(u)$ . We find  $(N(u), N^3(u)) \notin S$ . Applying the same argument to  $N^2(u)$  we get that  $(N^2(u), N^4(u)) \in S$ . Continuing this process, we find  $(N^6(u), N^8(u)) \in S$ . This is a contradiction since  $(N^6(u), N^8(u)) = (N^{-1}(u), N(u)) \notin S$ .  $\square$

**Remark.** It might be possible to find a shorter proof for the non-existence by means of the standard strategy of constructive enumeration [7], that is, trying to construct the canonical adjacency matrix of a  $(14, 5, 2, 1, 4)$ -graph.

We believe that the idea of our proof which is based on equitable partitions of a regular graph (in the sense of [12]) as well as on arguments from linear algebra may be helpful in more general situations, namely, whenever  $k - t = 1$ .

## 9. Duval's list of small graphs revisited

Investigation of genuine d.s.r.g.'s is still in the opening stage, when the creating of a catalogue of small graphs plays a stimulating role in the development of the whole theory.

The first small catalogue was compiled by Duval who, with the aid of a computer, has generated a list of all possible parameter sets for genuine d.s.r.g.'s with  $n \leq 20$ . In his list Duval mentioned all cases of the existence known to him; the problem of the complete enumeration of graphs (up to isomorphism) was not considered.

In this section we briefly summarize information known to us about the Duval's list (see Table 2). Together with a reference to the construction of a graph  $T$  we give also the following information:

Table 2  
 Duval's list of small graphs revisited

Number	$n$	$k$	$\mu$	$\lambda$	$t$	Existence/constr.	Rank	$\text{Aut}(\Gamma)$	$ \text{Aut}(\Gamma) $	Remarks
1	6	2	1	0	1	Example 4.2	6	$D_3$	6	
2	8	3	1	1	2	Example 4.1	8	$D_4$	8	
3a	10	4	2	1	2	Example 4.2	10	$D_5$	10	
3b	10	4	2	1	2	[5] Section 5	6	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	20	
4	12	3	1	0	1	Proposition 6.2(a)	7	$S_4$	24	
5	12	4	2	0	2	Proposition 7.1(a)	7	$D_3 \wr S_2$	384	1
6a	12	5	2	2	3	Proposition 6.2(b)	7	$S_4$	24	
6b	12	5	2	2	3	Proposition 7.1(b)	7	$D_3 \wr S_2$	384	1
7	14	5	2	1	4	No, see Section 8	—	—	—	
8a	14	6	3	2	3	Example 4.3, $\Gamma_1$	14	$D_7$	14	
8b	14	6	3	2	3	Example 4.3, $\Gamma_2$	6	$\mathbb{Z}_7 \rtimes \mathbb{Z}_6$	42	
9	15	4	1	1	2	Example 7.1	23	$D_5$	10	Intrans.
10	15	5	2	1	2	?				
11	16	6	3	1	3	?				
12	16	7	2	4	5	Proposition 7.1(b)	9	$D_4 \wr S_2$	$2^{11}$	2
13	16	7	3	3	4	?				
14	18	4	1	0	3	Example 7.2	7	See Ex.	108	
15	18	5	1	2	3	?				
16	18	6	3	0	3	Proposition 7.1(a)	7	$D_3 \wr S_3$	$6^7$	1
17	18	7	3	2	5	?				
18	18	8	3	4	5	Proposition 7.1(b)	7	$D_3 \wr S_3$	$6^7$	1
19a	18	8	4	3	4	Example 4.2	18	$D_9$	18	
19b	18	8	4	3	4	Example 4.4	10	$(\mathbb{Z}_3 \wr \mathbb{Z}_3) \times S_2$	162	
20	20	4	1	0	1	Proposition 6.2(a)	7	$S_5$	120	
21a	20	7	2	3	4	Proposition 6.2(b)	7	$S_5$	120	
21b	20	7	2	3	4	Proposition 6.2(d)	7	$S_5$	120	
22a	20	8	4	2	4	Proposition 7.1(a)	11	$D_5 \wr S_2$	$5 \times 2^{11}$	3a
22b	20	8	4	2	4	Proposition 7.1(a)	7	$F_5^4 \wr S_2$	$5 \times 2^{12}$	3b
23a	20	9	4	4	5	Proposition 7.1(b)	11	$D_5 \wr S_2$	$5 \times 2^{11}$	3a
23b	20	9	4	4	5	Proposition 7.1(b)	7	$F_5^4 \wr S_2$	$5 \times 2^{12}$	3b



- rank of  $W(\Gamma)$ ,
- description and order of  $\text{Aut}(\Gamma)$ ,
- additional remarks about the group  $\text{Aut}(\Gamma)$  and the coherent algebra  $W(\Gamma)$ .

In cases, when we are able to deliver more than one graph with a given set of parameters, only two such graphs are included in the table (even if there exist more than two graphs).

The problem of the complete constructive enumeration of d.s.r.g.'s with the parameters from the Duval's list requires the use of a computer based on the techniques of a constructive enumeration of combinatorial objects (see [6,7]).

### Remarks

1. In all our examples the rank of the automorphism group  $\text{Aut}(\Gamma)$  coincides with the rank of the coherent algebra generated by  $A(\Gamma)$ .
2. A number in column "Remarks" means the number of the initial graph which is used for the construction via Proposition 7.1.
3. All graphs besides the graph in line 9 have a transitive automorphism group.
4.  $F_n^m$  denotes the Frobenius group of order  $mn$  which is isomorphic to the semi-direct product  $\mathbb{Z}_n \rtimes \mathbb{Z}_m$ .

We believe that the following questions also appear worthy of investigation:

- continuation of Duval's list up to 50–100 vertices;
- complete interpretation of all Duval's results in terms of coherent algebras;
- determination of the smallest genuine d.s.r.g.'s  $\Gamma$  for which rank of  $W(\Gamma)$  is smaller than rank of  $\text{Aut}(\Gamma)$  (the existence of such examples follows from the existence of BIBD's with  $\lambda = 1$  which do not have flag-transitive automorphism group);
- classification of d.s.r.g.'s with small value of  $k - t$ , especially with  $k - t = 1$ ;
- description of new infinite series of d.s.r.g.'s;
- Duval's question about the existence of a (486, 22, 1, 0, 21)-graph;
- classification of d.s.r.g.'s with the small values of rank of  $W(\Gamma)$ , especially with the rank equal to 6 and 7.

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J.J. Seidel's methods in the investigation of strongly regular graphs have foreshadowed techniques which nowadays are successfully employed in the theory of association schemes. The present paper may be understood as verification of the fact that generalizations in other directions seem also to be worthy of investigation. We here fondly recall Seidel's longstanding contributions to the theory of strongly regular graphs, in particular his development of the methods alluded to above, and the vast encouragement his commitment to the area has provided to all who have had the pleasure of knowing him.

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