Localized gluing of Riemannian metrics in interpolating their scalar curvature

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We show that two smooth nearby Riemannian metrics can be glued interpolating their scalar curvature. The resulting smooth metric is the same as the starting ones outside the gluing region and has scalar curvature interpolating between the original ones. One can then glue metrics while maintaining inequalities satisfied by the scalar curvature. We also glue asymptotically Euclidean metrics to Schwarzschild ones and the same for asymptotically Delaunay metrics, keeping bounds on the scalar curvature, if any. This extends the Corvino gluing near infinity to non-constant scalar curvature metrics.

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1. Introduction

The Corvino–Schoen method enables gluing near infinity constant scalar curvature metrics (or more generally relativistic initial data) to a Schwarzschild (or Kerr) type model. This method was used in many contexts and has a lot of very nice applications [12,13,3,4,8,11,6,9], etc.

It is now natural to see how far this approach can be extended. In [14] the author shows that the method works for a large class of underdetermined elliptic operators of any order. In the present study, we will see that the gluing can be done if we substitute the assumption that the scalar curvature interpolates between the starting metric and a model, for the constant scalar curvature assumption. We then keep the bound on the scalar curvature if there is one. We also recover a constant scalar curvature metric if we start with such a metric.

Let (M, g) be a smooth Riemannian manifold. We do not assume that (M, g) is connected nor complete nor compact. Let Ω_i, $i = 1, 2, 3$, be open subsets of M with smooth boundary and such that $\overline{\Omega}_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_3$. We set $\Omega = \Omega_2 \setminus \overline{\Omega}_1$ and we assume that $\overline{\Omega}$ is compact. Let $\bar{g}$ be another smooth Riemannian metric on $\Omega_3 \setminus \overline{\Omega}_1$.

We will use a perturbation argument to glue $\bar{g}$ with g on Ω, so we are interested in P_g, the linearized scalar curvature operator:

$$P_g h := DR(g)h = \operatorname{div}_g \operatorname{div}_g h + \Delta_g \operatorname{Tr}_g h - \langle \operatorname{Ric}(g), h \rangle_g,$$

where our Laplacian $\Delta = \nabla^*\nabla$ is positive. The surjectivity of P_g is, at least formally, related to the injectivity of its L^2 formal adjoint:

$$P_g^*u = \mathrm{Hess}_g\, u + \Delta_g u\, g - u\, \mathrm{Ric}(g).$$

We will say that the metric g is *non-degenerate* on Ω if the kernel $\mathcal{K}$ of P_g^* is trivial on this set. This condition is generic [1].

We now state the local deformation:

Theorem 1.1. *Let χ be a smooth cutoff function equal to 1 near $\overline{\Omega}_1$ and to 0 near the complementary of Ω_2. If g is non-degenerate and $\overline{g}$ is close to g on $\overline{\Omega}$ then there exists a symmetric covariant two tensor $h \in C^\infty(\Omega_3)$, supported in $\overline{\Omega}$ such that the metric*

$$\tilde{g} := \chi g + (1-\chi)\overline{g} + h,$$

solves

$$R(\tilde{g}) = \chi R(g) + (1-\chi)R(\overline{g}).$$

The closeness of the metrics used is made precise in Proposition 2.5, proximity in $C^m(\overline{\Omega})$ with $m \geqslant [n/2] + 6$ is sufficient for instance.

This first result is very closely related to Theorem 1 in [12], and the proof is almost the same: it is given in Section 2.

Probably the most important consequence of Theorem 1.1 is that eventual inequality verified by the scalar curvature are maintained after gluing. For instance for positive mass Theorems type results (see e.g. [15,7,2]) one assumes that the scalar curvature is bounded below by the scalar curvature of the model. Our result can then be used at a first step to prove positive mass theorems for non-spin manifolds of large dimensions (see also comments after Theorem 1.2).

We are now interested in asymptotically Euclidean (AE for short) metrics (see Section 3 for the precise definition). We will use the procedure to glue these metrics with a Schwarzschild model (slice), defined on $\mathbb{R}^n\backslash\{c\}$ by:

$$g_S = g_{m,c} = \left(1 + \frac{m}{(n-1)|x-c|^{n-2}}\right)^{\frac{4}{n-2}}\delta,$$

where δ is the Euclidean metric. The gluing will occur on an annulus

$$A_{\lambda,4\lambda} = \left\{x \in \mathbb{R}^n,\ \lambda < r = |x| < 4\lambda\right\}.$$

The proximity of the metrics will be guaranteed for λ large.

Theorem 1.2. *Let g be an asymptotically Euclidean metric of order $\alpha > n/2 - 1$. Assume the mass m_g of g is not zero, assume also $R(g) = O(r^{-\beta})$ with $\beta > n$, and finally assume that g satisfies the asymptotic parity condition. Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ the metric g can be glued with a Schwarzschild metric on the annulus $A_{\lambda,4\lambda}$. The resulting metric g_λ has scalar curvature interpolating between $R(g)$ and 0 on $A_{\lambda,4\lambda}$. In particular, if g has non-negative scalar curvature then so has g_λ.*

The asymptotic parity condition is defined in Section 3, it ensures the center of mass to be well defined on an AE chart.

Letting $\lambda \to \infty$ in Theorem 1.2 proves in particular that the set of non-negative scalar curvature AE metrics which are exactly Schwarzschild outside of a compact set is dense in the set of non-negative scalar curvature AE metrics. This show that the positive mass conjecture for non-spin AE manifolds of large dimensions need only be proved for manifolds exactly Schwarzschild outside of a compact set.

Finally, we show in Section 4 that the method can be adapted to asymptotically Delaunay metrics giving Theorem 4.1.

There also exists such a constant scalar curvature gluing in an asymptotically hyperbolic setting [6]. The AH metrics are then glued to a Schwarzschild AdS (a particular Kottler) metric on an annulus near the infinity. This result can certainly be adapted to non-constant scalar curvature metrics as before. However the result there is probably not sharp, so it seems sensible to wait for a sharper (and if possible simpler) version before extending it to the non-constant scalar curvature case.

The approach developed here can certainly be adapted to the full constraint map [13,4] and to other operators [14], but the utility is less clear for the moment.

2. Gluing on a fixed set

In this section we give the proof of Theorem 1.1. This proof is almost the same as in [12] or [4] for instance. We just recall the different steps for completeness.

2.1. Weighted spaces

We will use the spaces already introduced in the appendix of [4] in the special case of a compact boundary. We keep the general notation of [4] for easy comparison with that paper.

Let $x \in C^\infty(\overline{\Omega})$ be a (non-negative) defining function of the boundary $\partial \Omega = x^{-1}(\{0\})$.

Let $a \in \mathbb{N}$, $s \in \mathbb{R}$, $s > 0$ and let us define

$$\phi = x^2, \qquad \psi = x^{2(a-n/2)} e^{-s/x} \quad \text{and} \quad \varphi = x^{2a} e^{-s/x}.$$

For $k \in \mathbb{N}$ let $H^k_{\phi,\psi}$ be the space of H^k_{loc} functions or tensor fields such that the norm

$$\|u\|_{H^k_{\phi,\psi}} := \left(\int\limits_M \left(\sum_{i=0}^k \phi^{2i} |\nabla^{(i)} u|_g^2 \right) \psi^2 \, d\mu_g \right)^{\frac{1}{2}} \tag{2.1}$$

is finite, where $\nabla^{(i)}$ stands for the tensor $\underbrace{\nabla \cdots \nabla}_{i \text{ times}} u$, with ∇ – the Levi-Civita covariant derivative of g; For $k \in \mathbb{N}$ we denote by $\mathring{H}^k_{\phi,\psi}$ the closure in $H^k_{\phi,\psi}$ of the space of H^k functions or tensors which are compactly (up to a negligible set) supported in Ω, with the norm induced from $H^k_{\phi,\psi}$. The $\mathring{H}^k_{\phi,\psi}$'s are Hilbert spaces with the obvious scalar product associated to the norm (2.1). We will also use the following notation

$$\mathring{H}^k := \mathring{H}^k_{1,1}, \qquad L^2_\psi := \mathring{H}^0_{1,\psi} = H^0_{1,\psi},$$

so that $L^2 \equiv \mathring{H}^0 := \mathring{H}^0_{1,1}$. We set

$$W^{k,\infty}_\phi := \left\{ u \in W^{k,\infty}_{loc} \text{ such that } \phi^i |\nabla^{(i)} u|_g \in L^\infty \right\},$$

with the obvious norm, and with $\nabla^{(i)} u$ – the distributional covariant derivatives of u.

For $k \in \mathbb{N}$ and $\alpha \in [0,1]$, we define $C^{k,\alpha}_{\phi,\varphi}$ the space of $C^{k,\alpha}$ functions or tensor fields for which the norm

$$\|u\|_{C^{k,\alpha}_{\phi,\varphi}} = \sup_{x \in M} \sum_{i=0}^k \left(\left\| \varphi \phi^i \nabla^{(i)} u(x) \right\|_g + \sup_{0 \neq d_g(x,y) \leqslant \phi(x)/2} \varphi(x) \phi^{i+\alpha}(x) \frac{\|\nabla^{(i)} u(x) - \nabla^{(i)} u(y)\|_g}{d_g^\alpha(x,y)} \right)$$

is finite.

Remark 2.1. In the context of compact boundary, it is more usual to use $\phi = x$ and for ψ and φ a power of x which can be done here also as long as we work with finite differentiability. We choose to take the exponential weight to treat all the cases in the same way.

2.2. The gluing

We give the proof of Theorem 1.1 by three propositions, giving the different steps needed. Let g be a smooth fixed Riemannian metric on $\overline{\Omega}$. We denote by $\mathcal{K}$ the kernel of P_g^*. For any metric $\tilde{g}$ on $\overline{\Omega}$, we set

$$\mathcal{L}_{\phi,\psi}(\tilde{g}) := \psi^{-2} P_{\tilde{g}} \psi^2 \phi^4 P_{\tilde{g}}^*.$$

We denote by $\mathcal{K}^\perp$ the $L^2_\psi(\tilde{g})$ orthogonal to (the fixed set) $\mathcal{K}$, and $\pi_{\mathcal{K}^\perp}$ the $L^2_\psi(\tilde{g})$ projection onto $\mathcal{K}^\perp$. We have (see [4] for instance)

Proposition 2.2. *For $k \geqslant 0$, the map*

$$\pi_{\mathcal{K}^\perp} \mathcal{L}_{\phi,\psi}(\tilde{g}) : \mathcal{K}^\perp \cap \mathring{H}^{k+4}_{\phi,\psi}(\tilde{g}) \longrightarrow \mathcal{K}^\perp \cap \mathring{H}^k_{\phi,\psi}(\tilde{g}) \tag{2.2}$$

is an isomorphism such that the norm of its inverse is bounded independently of $\tilde{g}$ close to g in $W^{k+4,\infty}_\phi$.

We can now state

Proposition 2.3. *Let $k > n/2$. Let χ be a smooth cutoff function equal to 1 near $\overline{\Omega}_1$ and to 0 near the complementary of Ω_2. Let us define*

$$g_\chi := \chi g + (1-\chi)\overline{g} \quad \text{and} \quad R_\chi := \chi R(g) + (1-\chi)R(\overline{g}).$$

If $\bar{g}$ is close to g in $C^{k+4}(\overline{\Omega})$ then there exists a unique $h = \psi^2 \phi^4 P^*_{g_\chi} u$, with $u \in \mathring{H}^{k+4}_{\phi,\psi}(g_\chi)$ such that

$$\pi_{\mathcal{K}^\perp} \psi^{-2} \big[R(g_\chi + h) - R_\chi \big] = 0. \tag{2.3}$$

Proof. Apply [4] Theorem 5.9, with $K = Y = J = 0$, $N = u$ and $\delta\rho = R_\chi - R(g_\chi)$. In fact we can solve

$$\pi_{\mathcal{K}^\perp} \psi^{-2} \big[R(g_\chi + h) - R(g_\chi) \big] = \pi_{\mathcal{K}^\perp} \psi^{-2} \big[R_\chi - R(g_\chi) \big]$$

for g_χ close to g by a uniform inverse function theorem using Proposition 2.2. Note that $R_\chi - R(g_\chi)$ vanishes near the boundary and tends to zero together with $g_\chi - g$ when $\bar{g}$ approaches g. $\quad\square$

Remark 2.4. The cutoff function χ used to interpolate the scalar curvatures can be chosen to be different from the one interpolating the metrics.

From Proposition 5.10 and Corollary 5.11 of [4] we have more regularity.

Proposition 2.5. Under the conditions of Proposition 2.3, assume moreover that $k \geqslant [\frac{n}{2}] + 1$, and $\bar{g}$ is close to g in $C^{k+4,\alpha}(\overline{\Omega})$. Then the solution h of Proposition 2.3 is in $\phi^2 \psi^2 C^{k+2,\alpha}_{\phi,\varphi}(\Omega)$. Moreover, if $\bar{g} \in C^\infty$, then $h \in \phi^2 \psi^2 C^\infty_{\phi,\varphi}(\Omega) \subset C^\infty(\overline{\Omega})$ and can be smoothly extended by zero across the boundary.

This proposition then concludes the proof of Theorem 1.1 which assumes that the metric g is non-degenerate. We will see in the next sections that the kernel projection can be removed in some circumstances when a kernel is present.

2.3. Remark on a regularity improvement

In the Corvino–Schoen construction, there is a loss of regularity in the following way. Assume g has regularity of order $k + 4$ and $\bar{g}$ has regularity $k + 2$, the final glued metrics can only have a regularity of order k because of the Ricci term in $P^*_{g_\chi}$. In [5] we regularize g_χ in order to recover a better regularity. There is a simpler way that can be used for many applications such as the ones that follow. We define $\widehat{\mathcal{L}}(g_\chi)$ as $\mathcal{L}(g_\chi)$ but by replacing just the $P^*_{g_\chi}$ term by P^*_g. Clearly, this operator has the same isomorphism properties as $\mathcal{L}(g)$ for $\bar{g}$ close to g with $k + 2$ derivatives, with also a uniform inverse bound. Using this, we then improve the regularity of the final glued metric. We do not use this for the applications, in order to avoid unnecessary complications of the comprehension.

3. Exactly Schwarzschild end

We show that the gluing construction of Corvino [12] for scalar flat AE metrics, can be done without the scalar flat assumption, by interpolating the scalar curvature of g and zero. This just needs a fast decay of the scalar curvature (the natural decay for $R(g) \in L^1$).

An asymptotically Euclidean manifold (M, g) of order $\alpha > 0$ is for us a Riemannian manifold with (at least) an end $E \approx [A, +\infty[\times \mathbb{S}^{n-1}$, $A > 0$, such that there exists a chart of this end on which the components of g satisfy, for multi-indices β,

$$\partial^\beta (g_{ij} - \delta_{ij}) = O(r^{-\alpha - |\beta|}),$$

near infinity, where $r = |x|$.

In order to guarantee existence of a center of mass, we impose an asymptotic parity condition. That is, we assume there exist an AE chart in which

$$|g^-_{ij}| + r|\partial_k(g^-_{ij})| \leqslant C(1+r)^{-\alpha_-}, \quad \alpha_- > \alpha, \; \alpha + \alpha_- > (n-1), \tag{3.1}$$

and

$$|R(g)^-| \leqslant C(1+r)^{-\beta_-}, \quad \beta_- > \beta, \; \beta_- > n+1, \tag{3.2}$$

where $f^-(x) = \frac{1}{2}[f(x) - f(-x)]$. Note that this condition may probably be relaxed [10].

We can now give the proof of Theorem 1.2.

Proof. Let $\phi_\lambda : A_{1,4} \longmapsto A_{\lambda,4\lambda}$ be defined by

$$\phi_\lambda(x) = \lambda x.$$

For any α-asymptotically Euclidean metric g we define on $A_{1,4}$

$$g_\lambda = \lambda^{-2}\phi_\lambda^* g.$$

In particular one has $g_\lambda = \delta + O(\lambda^{-\alpha})$ on $A_{1,4}$. Now let g as in Theorem 1.2. Let χ be a smooth non-negative function equal to 1 on $A_{1,2}$ and 0 on $A_{3,4}$. Let

$$\overline{g}_{\lambda,S} = \chi g_\lambda + (1 - \chi) g_{\lambda,S}.$$

The equation we will solve is

$$R(\overline{g}_{\lambda,S} + h) = \chi R(g_\lambda), \tag{3.3}$$

where h and its derivatives vanish on $\partial A_{1,4}$. Let

$$\mathcal{K} = \mathrm{span}\{1, x^1, \ldots, x^n\},$$

the kernel of P_δ^*. We can use the preceding procedure to solve

$$\pi_{\mathcal{K}^\perp} \psi^{-2}\big[R(\overline{g}_{\lambda,S} + h) - R(\overline{g}_{\lambda,S})\big] = \pi_{\mathcal{K}^\perp} \psi^{-2}\big[\chi R(g_\lambda) - R(\overline{g}_{\lambda,S})\big],$$

for any λ large enough. Note that the correction is of order the error, that is $h = O(\lambda^{-\gamma})$, $\gamma = \min(\alpha, n - 2, \beta - 2) = \min(\alpha, n - 2)$.

We will now prove that we can choose $S = (m, c)$ to kill the kernel projection. Let $\tilde{g}_{\lambda,S} = \overline{g}_{\lambda,S} + h = \delta + O(\lambda^{-\gamma})$. The components of the projections onto the kernel are

$$q_\lambda^0(S) := \big\langle 1, \psi^{-2}\big[R(\tilde{g}_{\lambda,S}) - \chi R(g_\lambda)\big]\big\rangle_{L_\psi^2} = \int_{A_{1,4}} \big[R(\tilde{g}_{\lambda,S}) - \chi R(g_\lambda)\big] = \int_{A_{1,4}} R(\tilde{g}_{\lambda,S}) + O(\lambda^{2-\beta}),$$

$$q_\lambda^l(S) := \big\langle x^l, \psi^{-2}\big[R(\tilde{g}_{\lambda,S}) - \chi R(g_\lambda)\big]\big\rangle_{L_\psi^2} = \int_{A_{1,4}} x^l\big[R(\tilde{g}_{\lambda,S}) - \chi R(g_\lambda)\big] = \int_{A_{1,4}} x^l R(\tilde{g}_{\lambda,S}) + O(\lambda^{2-\beta}).$$

We can deduce (see Appendix A for details, also recall that $\beta > n$)

$$q_\lambda^0(S) = 4\omega_{n-1}\lambda^{2-n}\big[(m - m_g) + o(1)\big] + O(\lambda^{2-\beta}) = 4\omega_{n-1}\lambda^{2-n}\big[(m - m_g) + o(1)\big],$$

$$q_\lambda^l(S) = 4\omega_{n-1}\lambda^{2-n}\big[C_{g_S}^l - C_g^l + o(1)\big] + O(\lambda^{2-\beta}) = 4\omega_{n-1}\lambda^{2-n}\big[C_{g_S}^l - C_g^l + o(1)\big],$$

where the $o(1)$'s are uniform relatively to (m, C) when (m, C) stay in a compact set where $m \neq 0$, say a small closed ball around (m_g, C_g). From an application to the Brouwer fixed point theorem (see e.g. Lemma 3.18 in [4] for details), for any λ large enough, there exist (m, C) (so an $S = (m, c)$) such that $q_\lambda(S) = 0$. $\square$

4. Exactly Delaunay end

We show that the gluing of asymptotically Delaunay metrics of [8,11] can be extended to non-constant scalar curvature metrics.

Let N be a smooth compact $(n - 1)$-dimensional manifold without boundary. A (generalized) Delaunay metrics on $\mathbb{R} \times N$ is of the form

$$\mathring{g} = u^{\frac{4}{n-2}}(dy^2 + \mathring{h}),$$

where $\mathring{h}$ is the Einstein metric on N with scalar curvature $(n-1)(n-2)$, and $u = u(y) > 0$ is a periodic solution of

$$u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0.$$

The metric depends on two parameters, the period of u and the neck size ε which is the minimum of u (we do not allow the critical cases of the sphere and the cylinder):

$$0 < \varepsilon < \left(\frac{n-2}{2}\right)^{\frac{n-2}{4}}.$$

Let (M, g) be a smooth n-dimensional Riemannian manifold. We will say that M is asymptotically Delaunay, if there exists an end $E \approx [0, +\infty) \times N$, on which g is asymptotic to a Delaunay metric $\mathring{g} = \mathring{g}_\varepsilon$ together with derivatives up to order four.

On the end E the Delaunay metric can be written of the form

$$\mathring{g}_\varepsilon = dx^2 + e^{2f(x)}\mathring{h},$$

where f has a period $T = T(\varepsilon)$. Let χ be a smooth cut off function on $[0, T]$ equal to 1 near zero and zero near T, and let χ_i be its translated on $[iT + \sigma, (i+1)T + \sigma]$ for a fixed σ to be made precise later. Let $\Omega_i = [iT + \sigma, (i+1)T + \sigma] \times N$.

We then have (compare [8], Theorem 3.1)

Theorem 4.1. *Assume that* (M, g) *is asymptotic to a Delaunay metric* $\mathring{g}_\varepsilon$ *on an end* E. *Then for* i *large enough, there exist* ε' *and a metric* g_i *on* E *which coincide with* g *before* Ω_i *and with a Delaunay* $\mathring{g}_{\varepsilon'}$ *after* Ω_i *and such that*

$$R(g_i) = \chi_i R(g) + (1 - \chi_i) n(n - 1).$$

Proof. We just mention the changes needed in the proof of Theorem 3.1 in [8]. We define $g_{\varepsilon'} = \chi_i g + (1 - \chi_i)\mathring{g}_{\varepsilon'}$ and $R_{\varepsilon'} = \chi_i R(g) + (1 - \chi_i) n(n - 1)$. We can then solve on Ω_i, for i large enough the equation

$$\pi_{\mathcal{K}^\perp} \psi_i^{-2} \big[R(g_{\varepsilon'} + h_i) - R(g_{\varepsilon'}) \big] = \pi_{\mathcal{K}^\perp} \psi_i^{-2} \big[R_{\varepsilon'} - R(g_{\varepsilon'}) \big],$$

where ψ_i is the translation on Ω_i of ψ defined on $[0, T]$ as before. Here $\mathcal{K}$ is one-dimensional and spanned by the $\mathring{N}$ of [8]. We also call the resulting metric $\tilde{g}_{\varepsilon'} := g_{\varepsilon'} + h_i$. Here also the correction is of order the perturbation introduced $h_i = O(|\varepsilon' - \varepsilon|) + o_i(1)$, where $o_i(1)$ tends to zero uniformly relatively to ε' bounded when i tends to infinity. The projection onto the kernel becomes (see Eq. (3.11) in [8])[1]

$$q_i(\varepsilon') := \int_{\Omega_i} \mathring{N} \big[R(\tilde{g}_{\varepsilon'}) - R_{\varepsilon'} \big] d\mu_{g_{\varepsilon'}}.$$

This projection can be rewritten as

$$
\begin{aligned}
q_i(\varepsilon') &= \int_{\Omega_i} \mathring{N} \big[R(\tilde{g}_{\varepsilon'}) - n(n-1) \big] d\mu_{g_{\varepsilon'}} + \int_{\Omega_i} \mathring{N} \big[n(n-1) - R_{\varepsilon'} \big] d\mu_{g_{\varepsilon'}} \\
&= \int_{\Omega_i} \mathring{N} \big[R(\tilde{g}_{\varepsilon'}) - n(n-1) \big] d\mu_{g_{\varepsilon'}} + o_i(1) \\
&= \int_{\Omega_i} \mathring{N} \big[R(\tilde{g}_{\varepsilon'}) - n(n-1) \big] d\mu_{\tilde{g}_{\varepsilon'}} + O\big((|\varepsilon' - \varepsilon| + o_i(1))^2 \big) + o_i(1).
\end{aligned}
$$

We can then proceed as in Eq. (3.19) in [8] (with the misprints of the boundary and the measure corrected):

$$
\begin{aligned}
q_i(\varepsilon') &= \int_{\{(i+1)T+\sigma\} \times N} \mathbb{U}^i \, dS_i - \int_{\{iT+\sigma\} \times N} \mathbb{U}^i \, dS_i + O\big((|\varepsilon' - \varepsilon| + o_i(1))^2 \big) + o_i(1) \\
&= \lambda \big(m' - \mathring{m} \big) + O\big((\varepsilon' - \varepsilon)^2 \big) + O\big(|\varepsilon' - \varepsilon| \big) o_i(1) + o_i(1),
\end{aligned}
$$

where we recall that σ has been chosen such that

$$\lambda := 4\omega_{n-1}(n-1) \mathring{N}_{|x=\sigma} \neq 0.$$

By the intermediate value theorem, for any i large enough, we can choose m' (equivalently ε') such that $q_i(\varepsilon') = 0$. $\quad\square$

Acknowledgements

Appendix A. Scalar curvature, mass and center of mass

Here we recall the basic definitions and relations between the objects of the title in the AE setting.

Let g be an asymptotically Euclidean metric of order $\alpha > n/2 - 1$. Assume also that the asymptotic parity condition (3.1) and (3.2) holds. In this section, we will lower or raise indices with the Euclidean metric δ and its inverse. Let us define

$$V_{ij} := g_{ij} - (\mathrm{Tr}_\delta g) \delta_{ij}.$$

The mass of g is defined by

$$m_g := \lim_{\lambda \to \infty} m_g(\lambda), \quad m_g(\lambda) := \frac{1}{4\omega_{n-1}} \int_{\mathbb{S}_\lambda} \big(\partial^i V_{ij} \big) \nu^j ds,$$

---

[1] The measure used in [8] Eq. (3.11) appears to be the wrong one when using the theorem of [4] as described there, this does not affect their proof, just a small change in Eq. (3.19) there is needed.

where $\nu^j = x^j/|x|$ is the unit normal, ds is the standard measure on the sphere $\mathbb{S}_\lambda$ of radius λ and ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. The center of mass is defined by

$$C_g^l = (mc)_g^l := \lim_{\lambda \to \infty} C_g^l(\lambda), \quad C_g^l(\lambda) := \frac{1}{4\omega_{n-1}} \int_{\mathbb{S}_\lambda} [x^l \partial^i V_{ij} - V_j^l] \nu^j \, ds.$$

The scalar curvature satisfies near infinity,

$$R(g)\sqrt{|g|} = \partial^j \partial^i V_{ij} + Q,$$

where $|g|$ is the determinant of g, $e = g - \delta$, and Q is quadratic in $\partial e = O(r^{-(\alpha+1)})$. In particular we see that,

$$\frac{1}{4\omega_{n-1}} \int_{A_{\lambda,\nu}} R(g) \, d\mu_g = m_g(\lambda) - m_g(\nu) + O\left(\lambda^{-2\alpha+n-2}\right),$$

and when $R(g) = O(r^{-\beta})$, $\beta > n$, we have (for $\nu = +\infty$):

$$m_g(\lambda) = m_g + O\left(\lambda^{-2\alpha+n-2}\right) + O\left(\lambda^{-\beta+n}\right) = m_g + o(1).$$

In the same way, and using parity considerations (just write $e = e^+ + e^-$), we see that,

$$\frac{1}{4\omega_{n-1}} \int_{A_{\lambda,\nu}} x^l R(g) \, d\mu_g = C_g^l(\lambda) - C_g^l(\nu) + O\left(\lambda^{n-1-\alpha-\alpha_-}\right).$$

When $R(g)^- = O(r^{-\beta_-})$, $\beta_- > n+1$, we have (for $\nu = +\infty$):

$$C_g^l(\lambda) = C_g^l + O\left(\lambda^{n-1-\alpha-\alpha_-}\right) + O\left(\lambda^{n+1-\beta_-}\right) = C_g^l + o(1).$$

As the computations needed are made on a rescaled annulus we give the simple relations between the scalar curvatures and the measures. Let g be α-AE and g_λ as in Section 3, for $x \in A_{1,4}$ let $y = \phi_\lambda(x)$ so

$$R(g_\lambda)(x) = \lambda^2 R(g)(y),$$

and

$$\sqrt{|g_\lambda|}(x) \, dx = \lambda^{-n} \sqrt{|g|}(y) \, dy = \left(1 + O\left(\lambda^{-\alpha}\right)\right)\lambda^{-n} \, dy.$$

Thus one obtains

$$R(g_\lambda)(x)\sqrt{|g_\lambda|}(x) \, dx = \lambda^{2-n} R(g)(y)\sqrt{|g|}(y) \, dy. \tag{A.1}$$

References

[1] R. Beig, P.T. Chruściel, R. Schoen, Kids are non-generic, Ann. Henri Poincaré 6 (2005) 155–194.
[2] S. Brendle, F.C. Marques, Scalar curvature rigidity of geodesic balls in S^n, arXiv:1005.2782, 2010.
[3] P.T. Chruściel, E. Delay, Existence of non-trivial, vacuum, asymptotically simple space–times, Class. Quantum Grav. (2002) 71–79.
[4] P.T. Chruściel, E. Delay, On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications, Mémoires de la S.M.F. (2003), 103 p.
[5] P.T. Chruściel, E. Delay, Manifold structures for the set of solutions of the general relativistic constraints equations, Journal of Geometry and Physics 51 (2004) 442–472, arXiv:gr-qc/0309001.
[6] A. Chruściel, E. Delay, Gluing constructions for asymptotically hyperbolic manifolds with constant scalar curvature, Comm. Anal. Geom. 17 (2) (2009) 343–381.
[7] P.T. Chruściel, M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, Pacific J. Math. 212 (2003) 231–264, arXiv:dg-ga/0110035, MR MR2038048 (2005d:53052).
[8] P.T. Chruściel, D. Pollack, Singular Yamabe metrics and initial data with exactly Kottler–Schwarzschild–de Sitter ends, Ann. Henri Poincaré 9 (4) (2008) 639–654.
[9] P.T. Chruściel, J. Corvino, J. Isenberg, Construction of the N-body time symmetric initial data sets in general relativity, arXiv:0909.1101v1 [gr-qc], 2009.
[10] P.T. Chruściel, J. Corvino, J. Isenberg, Construction of the N-body initial data sets in general relativity, arXiv:1004.1355 [gr-qc], 2010.
[11] P.T. Chruściel, F. Pacard, D. Pollack, Singular Yamabe metrics and initial data with exactly Kottler–Schwarzschild–de Sitter ends II, Math. Res. Lett. 16 (1) (2009) 157–164.
[12] J. Corvino, Scalar curvature deformation and a gluing construction for the Einstein constraint equations, Commun. Math. Phys. 214 (2000) 137–189.
[13] J. Corvino, R. Schoen, On the asymptotics for the vacuum Einstein constraint equations, J. Differential Geom. 73 (2) (2006) 185–217.
[14] E. Delay, Smooth compactly supported solutions of some underdetermined elliptic PDE, with gluing applications, arXiv:1003.0535, 2010.
[15] R. Shoen, S.T. Yau, On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. 65 (1) (1979) 45–76.