The Denjoy–Wolff Theorem in the Open Unit Ball of a Strictly Convex Banach Space

Jaroslaw Kapeluszny and Tadeusz Kuczumow

Instytut Matematyki UMCS, 20-031 Lublin, Poland
E-mail: tadek@golem.umcs.lublin.pl

and

Simeon Reich

Department of Mathematics, The Technion–Israel Institute of Technology,
32000 Haifa, Israel
E-mail: sreich@techunix.technion.ac.il

Received March 19, 1998; accepted September 19, 1998

Let $X$ be a complex strictly convex Banach space with an open unit ball $B$. For each compact, holomorphic and fixed-point-free mapping $f: B \rightarrow B$ there exists $\zeta \in \partial B$ such that the sequence $\{f^n\}$ of iterates of $f$ converges locally uniformly on $B$ to the constant map taking the value $\zeta$.

Key Words: the Denjoy–Wolff theorem; horospheres; iterates of a holomorphic mapping; the Kobayashi distance.

1. INTRODUCTION

In 1926 Denjoy [8] and Wolff [33] proved that the iterates $f^n$ of a fixed-point free holomorphic map $f: A \rightarrow A$ converge locally uniformly to a unique boundary point $\zeta$. This convergence result was extended to the open unit ball of $\mathbb{C}^n$ [19, 21, 24] and to some other domains in $\mathbb{C}^n$ [1–3]. In the case of infinite dimensional Hilbert balls, Stachura [28] gave an example which shows that the Denjoy–Wolff theorem fails even for biholomorphic self-maps. But, if a fixed-point-free self-mapping $f$ of the Hilbert ball is either firmly $k_B$-nonexpansive or an averaged mapping of the first or second kind, then for each point $x$ the sequence of iterates $\{f^n(x)\}$ converges to a unique boundary point [15, 16, 25]. More information can be found in [31] and [32]. Recently, Chu and Mellon [7] proved that the Denjoy–Wolff theorem is still valid for a compact holomorphic fixed-point-free self-map of the open unit ball of a Hilbert space, and Kryczka and Kuczumow extended this result to $k_B$-nonexpansive self-mappings of the...
Hilbert ball \cite{20}. In this paper we prove that such results are valid for holomorphic mappings on the open unit ball of a strictly convex Banach space and for \(k_B\)-nonexpansive mappings on the open unit ball of a uniformly convex Banach space. As a matter of fact, our results are new even in the finite-dimensional case.

2. PRELIMINARIES

All Banach spaces will be complex. \(B\) will always denote the open unit ball of a Banach space \(X\) and \(B(z, r)\) will be the open ball of center \(z\) and radius \(r > 0\) in \(X\). If \(K\) is a subset of the Banach space \(X\), then \(\overline{K}\) denotes the norm closure of \(K\) in \(X\). Let \(D\) be a bounded domain in a Banach space \(X\). The Kobayashi distance in \(D\) is denoted by \(\kappa_D\). It is known that \(\kappa_D\) is locally equivalent to the norm \(\| \cdot \|\) \cite{12, 16, 18}. We remark in passing that all distances assigned to a convex bounded domain \(D\) by Schwarz-Pick systems of pseudometrics \cite{12, 16, 18} coincide \cite{9, 23}.

If \(D\) is also convex, then directly from the definition of \(\kappa_D\) we obtain that for \(z_1, z_2, w_1, w_2 \in D, 0 \leq t \leq 1, \) and \(r > 0\) the inequalities \(\kappa_D(z_1, z_2) \leq r\) and \(\kappa_D(w_1, w_2) \leq r\) imply
\[
k_D((1-t)z_1 + tw_1, (1-t)z_2 + tw_2) \leq r \tag{2.1}
\]
\cite{22}.

If \(B\) is the open unit Hilbert ball, then \(k_B\) is given by the formula
\[
k_B(w, z) = \arg \tanh \left( 1 - \frac{(1-\|w\|^2)(1-\|z\|^2)^{1/2}}{1-\langle w, z \rangle} \right) \tag{2.2}
\]
where \(w, z \in B \cite{16, 17}\).

Remark 2.1. We will use the following notion in the Banach space \(X\). If \(Y\) is the one dimensional subspace generated by \(0 \neq w \in X\) and \(z_1, z_2 \in Y\), let \((z_1, z_2)\) denote the standard scalar product in \(Y\) generated by \(w/\|w\|\). Now, if we consider \(k_B\) restricted to \((Y \times Y) \cap (B \times B)\), then \(k_B\) coincides with \(k_A\) \cite{18}, where \(A\) is the unit disc, and therefore we can use the formula (2.2).

We will also need the following property of \(k_B\).

For \(z \in B\) the index \(\hat{R}(z)\) of \(z\) is the supremum of the radii of all affine discs in \(B\) with center \(z\). The following inequality is valid for all \(w, z \in B\):
\[
\arg \tanh \left( \frac{\|z-w\|}{\|z-w\| + \hat{R}(z)} \right) \leq k_B(z, w) \tag{2.3}
\]
where
\[ \varphi(z) = \mathcal{R}(z) \left[ 1 + \frac{\|z - \tilde{z}\|}{1 - \|z\|} \right] \]
(2.4)

Let \( D \) be a bounded domain in a Banach space \( X \). A subset \( K \) of \( D \) is said to lie strictly inside \( D \) if \( \text{dist}(K, \partial D) > 0 \). A mapping \( f: D \to D \) is said to map \( D \) strictly inside \( D \) if \( f(D) \) lies strictly inside \( D \).

We say that \( f: D \to D \) is \( k_D \)-nonexpansive if
\[ k_D(f(w), f(z)) \leq k_D(w, z) \]
for all \( w, z \in D \). Each holomorphic \( f: D \to D \) is \( k_D \)-nonexpansive [12, 16, 18].

\( \text{Aut}(D) \cap C_c(D) \) will denote the group of all those biholomorphisms \( \varphi \) of \( D \) onto itself such that \( \varphi \) and \( \varphi^{-1} \) have 1-1 continuous extensions to the boundary \( \partial D \).

Now we recall a few facts about holomorphic mappings.

**Theorem 2.1** (Earle–Hamilton) [10]. Let \( D \) be a bounded domain in a Banach space \( X \). If a holomorphic \( f: D \to D \) maps \( D \) strictly inside itself, then there exists \( 0 \leq s < 1 \) such that
\[ k_D(f(w), f(z)) \leq s \cdot k_D(w, z) \]
for all \( w, z \in D \). Moreover, for any \( z \in D \) the sequence of iterates \( \{f^n(z)\} \) converges to the unique fixed point of \( f \).

Hence, if \( D \) is a bounded convex domain, then by Theorem 2.1 the mapping \( g_{t, z} = (1 - t) z + tf(\cdot) : D \to D \) is a \( k_D \)-contraction for every \( z \in D \) and \( 0 \leq t < 1 \). Therefore for each \( k_D \)-nonexpansive mapping \( f: D \to D \) the mapping \( f_{t, z} = g_{t, z} \cdot f = (1 - t) z + tf : D \to D \) is a \( k_D \)-contraction and has exactly one fixed point which we denote by \( h(t, z) \). Let us fix \( z_0 \in D \) and \( 0 \leq t < 1 \). The mapping \( h(t, \cdot) : D \to D \) is \( k_D \)-nonexpansive (holomorphic if \( f \) is holomorphic \([6, 12]\)) as a limit of the sequence \( \{f^n_{t, z_0}\} \).

**Theorem 2.2** [7]. Let \( B \) be the open unit ball of a Banach space \( X \) and let \( f: B \to B \) be a compact holomorphic map. Then every subsequence of the sequence \( \{f^n\} \) of iterates of \( f \) has a subsequence converging locally uniformly to a holomorphic function \( g: B \to B \).

**Theorem 2.3** (The strong maximum principle) [29]. Let \( X \) be a strictly convex Banach space and let \( D \) be a domain in a Banach space \( Y \). If \( B \) is the open unit ball in \( X \) and \( f: D \to B \) is holomorphic, then either \( f(D) \subset B \) or \( f \) is constant.
In [1] Abate introduced the following notion of horospheres in a bounded domain $D \subset X$ (see also [34]). For $z_0 \in D$, $\xi \in \partial D$, and $R > 0$ the small horosphere $E_{z_0}(\xi, R)$ and the big horosphere $F_{z_0}(\xi, R)$ of center $\xi$ and radius $R$ are defined by

\[
E_{z_0}(\xi, R) = \{ z \in D : \limsup_{w \to \xi} [k_D(z, w) - k_D(z_0, w)] < \frac{1}{2}\log R \}
\]

and

\[
F_{z_0}(\xi, R) = \{ z \in D : \liminf_{w \to \xi} [k_D(z, w) - k_D(z_0, w)] < \frac{1}{2}\log R \}.
\]

Horospheres are useful tools in investigations of holomorphic mappings [1, 4, 7, 13, 16, 21, 24, 26, 32, 34]. For our goal we introduce new horospheres in $B$ by using sequences with limits on $\partial B$.

Let $z_0 \in B$, $\xi \in \partial B$, $R > 0$, $w_n \in B$, $n = 1, 2, \ldots$, and $\lim_{n} w_n = \xi$. Let us assume in addition that the limit

\[
\lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)]
\]

exists for each $z \in B$. The new horosphere $G(\xi, R, z_0, \{w_n\})$ in $B$ is then defined as follows:

\[
G(\xi, R, z_0, \{w_n\}) = \{ z \in B : \lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)] < \frac{1}{2}\log R \}.
\]

Now we recall a result on nonexpansive mappings in a metric space. Recall that a metric space $(X, d)$ is called finitely compact if each bounded, closed and nonempty subset of $X$ is compact. We say that $f: X \to X$ is nonexpansive if

\[
d(f(x), f(y)) \leq d(x, y)
\]

for all $x, y \in X$. If we consider the behavior of a sequence of iterates of a nonexpansive mapping on a finitely compact space $X$, then the basic result is due to Calka.

Theorem 2.4 [5]. Let $f$ be a nonexpansive mapping of a finitely compact metric space $X$ into itself. If for some $x_0 \in X$ the sequence $\{f^n(x_0)\}$ contains a bounded subsequence, then for every $x \in X$ the sequence $\{f^n(x)\}$ is bounded.

After applying the Calka theorem, the method of asymptotic centers [11, 14, 16], and either the Schauder theorem [27] or the Earle–Hamilton theorem, one can obtain the following result.
Theorem 2.5 [20]. Let $X$ be a Banach space with the open unit ball $B$. If $f: B \to B$ is a compact $k_B$-nonexpansive mapping, then the following conditions are equivalent:

(i) $f$ has a fixed point;
(ii) there exists $z \in B$ and a subsequence of its iterates $\{f^n(z)\}$ such that $\sup_n \|f^n(z)\| < 1$;
(iii) there exists $z \in B$ such that $\sup_n \|f^n(z)\| < 1$;
(iv) for each $z \in B$ we have $\sup_n \|f^n(z)\| < 1$;
(v) there exists a nonempty, closed, convex and $f$-invariant subset $A$ of $B$ such that $\sup_{z \in A} \|z\| < 1$;
(vi) there exists a nonempty $f$-invariant subset $A$ of $B$ such that $\sup_{z \in A} \|z\| < 1$;
(vii) there exists a sequence $\{z_n\}$ such that $z_n - f(z_n) \to 0$ and $\sup_n \|z_n\| < 1$.

3. HOROSPHERES

We begin this section by proving an auxiliary lemma.

Lemma 3.1. Let $X$ be a Banach space. Let $\{w_n\}$ be an arbitrary sequence in $B$ with $\lim_n w_n = \zeta \in \partial B$ and $w_n \neq 0$ for $n = 1, 2, \ldots$. Let $0 < \alpha < 1$, and let $\text{Proj}_n$ denote, for each $n$, the projection of $X$ on the one dimensional space $\text{lin}(w_n)$ generated by $w_n$ given by

$$\text{Proj}_n(x) = f_n(x) \frac{w_n}{\|w_n\|}, \quad x \in X,$$

where $f_n$ is a linear functional with $\|f_n\| = 1$ and $f_n(w_n) = \|w_n\|$. If

$$z_n = \text{Proj}_n(\alpha \zeta),$$

then

$$z_n \in B, \quad \lim_{n \to \infty} z_n = \alpha \zeta, \quad \lim_{n \to \infty} k_B(z_n, \alpha \zeta) = 0 \quad \text{and} \quad \lim_{n \to \infty} \langle z_n, w_n \rangle = \alpha.$$  
(3.1)
Proof. It is obvious that $z_n \in B$. We also have

$$\|z_n - \alpha \xi\| = \left\| \alpha f_n(\xi) \frac{w_n}{\|w_n\|} - \alpha \xi \right\| \to 0.$$  

Hence we get

$$\lim_{n \to \infty} k_B(z_n, \alpha \xi) = 0.$$  

Next, taking $\alpha_n = \alpha f_n(\xi)$ we obtain

$$\|\alpha_n - \alpha\| = \left\| \alpha f_n(\xi) - \alpha f_n \left( \frac{w_n}{\|w_n\|} \right) \right\| \leq \left\| \xi - \frac{w_n}{\|w_n\|} \right\| \to 0.$$  

Hence

$$(z_n, w_n) = \alpha_n \|w_n\| \to \alpha.$$  

This lemma enables us to establish the following result.

**Theorem 3.2.** For every $\xi \in \partial B$ and $0 < \alpha < 1$ we have

$$\lim_{w \to \xi} \left[ k_B(\alpha \xi, w) - k_B(0, w) \right] = -k_B(0, \alpha \xi). \quad (3.2)$$  

**Proof.** Let us observe (see (2.2)) that

$$k_B(0, w) = \text{arg tanh} \left( \frac{w}{\|w\|} \right) = \frac{1}{2} \log \frac{1 + \|w\|}{1 - \|w\|} = \frac{1}{2} \log \frac{(1 + \|w\|)^2}{(1 - \|w\|)^2}. \quad (3.3)$$  

for $w \in B$, and that

$$k_B(0, \alpha \xi) = \text{arg tanh} \left( \alpha \xi \right) = \frac{1}{2} \log \frac{1 - \alpha^2}{(1 - \alpha)^2}. \quad (3.4)$$  

Now let us take an arbitrary sequence $\{w_n\}$ in $B$ with $w_n \neq 0$ for all $n$ and $\lim_n w_n = \xi$. For each $n$, let $\text{Proj}_n$ denote the projection of $X$ on the one dimensional space $\text{lin}(w_n)$ given in Lemma 3.1. If $z_n = \text{Proj}_n(\alpha \xi)$, then by
(3.1) we get \( z_n \in B \), \( \lim_{n \to \infty} z_n = \pi \xi \), and \( \lim_{n \to \infty} k_B(z_n, \pi \xi) = 0 \). Therefore we need to prove that
\[
\lim_{n \to \infty} [k_B(z_n, w_n) - k_B(0, w_n)] = -k_B(0, \pi \xi), \tag{3.5}
\]
in place of (3.2). Since \( w_n, z_n \in \text{lin}(w_n) \) for every \( n \) we have (see (2.2))
\[
k_B(z_n, w_n) = \arg \tanh \left( 1 - \frac{(1 - \|z_n\|^2)(1 - \|w_n\|^2)}{|1 - (z_n, w_n)|^2} \right)^{1/2} - \frac{|1 - (z_n, w_n)| + \|1 - (z_n, w_n)|^2}{(1 - \|z_n\|^2)(1 - \|w_n\|^2)} \tag{3.6}
\]
and by (3.1) and (3.4) this implies that
\[
\lim_{n \to \infty} [k_B(z_n, w_n) - k_B(0, w_n)] = \frac{1}{2} \log \frac{(1 - \|z\|^2)}{1 - \|z\|^2} = -k_B(0, \pi \xi)
\]
Thus the formula (3.5) is valid.

**Remark 3.1.** If \( X \) is a Hilbert space, then the limit
\[
\lim_{w \to \xi} [k_B(z, w) - k_B(0, w)] = \frac{1}{2} \log \frac{|1 - (z, \xi)|^2}{(1 - \|z\|^2)}
\]
exists for every \( z \in B \) and \( \xi \in \partial B \) and the ellipsoid \([13, 16, 26]\)
\[
E(\xi, R) = \left\{ z \in B : \frac{|1 - (z, \xi)|^2}{(1 - \|z\|^2)} < R \right\},
\]
where \((\cdot, \cdot)\) is the scalar product in \( X \), is equal to the horospheres
\[
E_0(\xi, R) = G(\xi, R, 0, \{ w_n \}) = F_0(\xi, R),
\]
where \( \lim_n w_n = \xi \). [34]
Remark 3.2. If $X$ is a separable Banach space, $\xi \in \partial B$, $w_n \in B$ for $n = 1, 2, \ldots$, and $\lim_{n} w_n = \xi$, then by a standard diagonalization procedure there exists a subsequence $\{w_{n_k}\}$ such that all the limits

$$\lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})], \quad z, z_0 \in B,$$

exist, and therefore all the horospheres $G(\xi, R, z_0, \{w_{n_k}\})$ are well defined.

Theorem 3.3. Let $X$ be a Banach space with the open unit ball $B$. Let $\xi$ and $\{w_n\}$ be fixed and let $G(\xi, R, z_0, \{w_n\})$ exist for each $z_0 \in B$ and $R > 0$. Then the horospheres $G(\xi, R, z_0, \{w_n\})$ have the following properties:

(i) $\emptyset \neq E_{\xi}(\xi, R) \subset G(\xi, R, z_0, \{w_n\}) \subset F_{\xi}(\xi, R)$ for every $z_0 \in B$, $\xi \in \partial B$ and $R > 0$;

(ii) $G(\xi, R, z_0, \{w_n\})$ is convex;

(iii) for every $0 < R_1 < R_2$ we have $[G(\xi, R_1, z_0, \{w_n\}) \cap B] \subset G(\xi, R_2, z_0, \{w_n\})$;

(iv) for every $R > 1$ we have $B(z_0, \frac{1}{2} \log R) \subset G(\xi, R, z_0, \{w_n\})$;

(v) for every $R < 1$ we have $G(\xi, R, z_0, \{w_n\}) \cap B(z_0, -\frac{1}{2} \log R) = \emptyset$;

(vi) $\bigcup_{R > 0} G(\xi, R, z_0, \{w_n\}) = B$ and $\bigcap_{R > 0} G(\xi, R, z_0, \{w_n\}) = \emptyset$;

(vii) $\xi \in \bigcap_{R > 0} G(\xi, R, z_0, \{w_n\}) \cap \partial B$;

(viii) if $X$ is strictly convex, then $\bigcap_{R > 0} G(\xi, R, z_0, \{w_n\}) \cap \partial B = \{\xi\}$;

(ix) if $\varphi, \varphi^{-1} \in \text{Aut}(B) \cap C^0(\overline{B})$, then $\varphi(G(\xi, R, z_0, \{w_n\})) = G(\varphi(\xi), R, \varphi(z_0), \{\varphi(w_n)\})$;

(x) if $z_1 \in B$ and $\lim_{n \to \infty} [k_B(z_1, w_n) - k_B(z_0, w_n)] = \frac{1}{2} \log L$, then $G(\xi, z_1, \{w_n\}) \subset G(\xi, LR, z_0, \{w_n\})$.

Proof. (i) The inclusions $E_{\xi}(\xi, R) \subset G(\xi, R, z_0, \{w_n\}) \subset F_{\xi}(\xi, R)$ for every $z_0 \in B$, $\xi \in \partial B$ and $R > 0$ are obvious. Next, by Theorem 3.2 the horosphere $G(\xi, R, 0, \{w_n\})$ is nonempty for every $R > 0$. Now it is sufficient to observe that by (3.2) we get

$$\lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)] = \lim_{n \to \infty} [k_B(z, w_n) - k_B(0, w_n)] + \lim_{n \to \infty} [k_B(0, w_n) - k_B(z_0, w_n)]$$

$$= -k_B(0, z_0) + k_B(0, z_0)$$

for each $0 < \alpha < 1$. 
ii) It is sufficient to apply (2.1).

(iii)-(vi) are obvious.

(vii) Apply Theorem 3.2.

(vii) See (ii), (iii) and (vii).

(ix) Obvious.

(x) By the equality

\[
\lim_{n \to \infty} [k_B(z, w_n) - k_B(z_0, w_n)] = \lim_{n \to \infty} [k_B(z, w_n) - k_B(z_1, w_n)] + \lim_{n \to \infty} [k_B(z_1, w_n) - k_B(z_0, w_n)]
\]

we get

\[
G(\xi, R, z_1, \{w_n\}) \subset G(\xi, LR, z_0, \{w_n\}),
\]

as claimed.

4. THE DENJOY–WOLFF THEOREM

**Theorem 4.1.** Let \( X \) be a strictly convex Banach space with the open unit ball \( B \). Let \( f: B \to B \) be a compact holomorphic map with no fixed point in \( B \). Then there exists \( \xi \in \partial B \) such that the sequence \( \{f^n\} \) of iterates of \( f \) converges locally uniformly on \( B \) to the constant map taking the value \( \xi \).

**Proof.** Since \( f \) is fixed-point free, the limit \( \xi \) of every convergent sequence \( \{w_n\} = \{h(t_n, z_n)\} \) (\( t_n \to 1, \ 0 < t_n < 1, \ z_n \in B \) for \( n = 1, 2, \ldots \)) belongs to \( \partial B \). Let \( X_0 \) denote a separable closed subspace of \( X \) which contains \( f(B) \) and set \( B_0 = B \cap X_0 \). Then \( f|_{B_0} \) is holomorphic [4]. By Remark 3.2 we can assume without loss of generality that

\[
\lim_{n \to \infty} [k_{B_0}(z, w_n) - k_{B_0}(z_0, w_n)]
\]

exists for all \( z, z_0 \in B_0 \). Now, for \( z_0 \in B_0 \) let \( G(\xi, R, z_0, \{w_n\}) \) denote the horosphere in \( X_0 \). Let us observe that for \( z \in B_0 \) we have
The above inequality implies that
\[
\lim_{n \to \infty} [k_B(f(z), w_n) - k_B(0, w_n)]
\]
\[
= \lim_{n \to \infty} [k_B(f(z), f_{w_n}(z)) + k_B(f_{w_n}(z), f_{w_n}(w_n)) - k_B(0, w_n)]
\]
\[
\leq \lim_{n \to \infty} [k_B(z, w_n) - k_B(0, w_n)].
\]

The above inequality implies that
\[
f(G(\xi, R, 0, \{w_n\})) \subset G(\xi, R, 0, \{w_n\})
\]
for arbitrary \( R > 0 \). Let us take \( z \in B_0 \). By Theorem 2.5 we have
\[
\lim_{n \to \infty} \|f^n(z)\| = 1.
\]
If \( A \subset \partial B \) denotes the set of all accumulation points of the sequence \( \{f^n(z)\} \), then by the compactness of \( f \) the set \( A \) is nonempty and by Theorems 2.2 and 2.3, \( A \) is independent of the choice of \( z \) and therefore by Theorem 3.3(viii)
\[
\emptyset \neq A \subset \bigcap_{R > 0} \overline{G(\xi, R, 0, \{w_n\})} \cap \partial B = \{\xi\}.
\]
Hence \( \lim_n f^n(z) = \xi \) and by Theorems 2.2 and 2.3 the sequence \( \{f^n\} \) is locally uniformly convergent on \( B \) to the constant map \( \xi \).

Directly from the proof of Theorem 4.1 we also deduce the following result.

**Corollary 4.2.** Let \( X \) be a strictly convex Banach space with the open unit ball \( B \). Let \( f : B \to B \) be a compact holomorphic map with no fixed point in \( B \). Then there exists \( \xi \in \partial B \) such that \( \{h(t, \cdot)\} \) tends uniformly on \( B \) to the constant map \( \xi \) and the sequence \( \{f^n\} \) is locally uniformly convergent to the same constant map \( \xi \).

Before we consider \( k_B \)-nonexpansive mappings in uniformly convex Banach spaces we prove the following lemma.

**Lemma 4.3.** Let \( X \) be a uniformly convex Banach space with the open unit ball \( B \). If \( w_n, z_n \in B \) for \( n = 1, 2, \ldots \), \( \lim_n z_n = \xi \in \partial B \) and \( \sup_n k_B(z_n, w_n) < \infty \), then \( \|z_n - w_n\| \to 0 \).

**Proof.** Let us observe that by the uniform convexity of \( X \) the index \( \tilde{R}(\tilde{z}) \) of \( \tilde{z} \) tends to 0 when \( \|\tilde{z}\| \to 1 \). Taking \( 0 < \alpha < 1 \) and \( \tilde{z} = \alpha \tilde{z} \) we get (see (2.4))
\[ \varphi(z_n) = \hat{R}(\hat{z}) \left[ 1 + \frac{\|z_n - \hat{z}\|}{1 - \|\hat{z}\|} \right] \]

\[ = \hat{R}(\hat{z}) \left[ 1 + \frac{\|z_n - \alpha z\|}{1 - \alpha} \right] \to 2\hat{R}(\hat{z}), \]

and hence \( \varphi(z_n) \) is arbitrarily small for \( \alpha \) sufficiently close to 1 and all \( n \) sufficiently large. Therefore the inequalities (see (2.3))

\[ \arg \tanh \left( \frac{\|z_n - w_n\|}{\|z_n - w_n\| + 2\varphi(z_n)} \right) \leq k_B(z_n, w_n) \leq \sup_n k_B(z_n, w_n) < \infty \]

yield

\[ \lim_{n \to \infty} \|z_n - w_n\| = 0. \]

Next we need the following observation.

**Remark 4.1.** Let \( X \) be a Banach space with the open unit ball \( B \) and the Kobayashi distance \( k_B \). Let \( X_0 \) be a closed separable subspace of \( X \) with the open unit ball \( B_0 = B \cap X_0 \), \( \xi \in \partial B_0 \), \( w_n \in B_0 \) for \( n = 1, 2, ..., \) and \( \lim_{n} w_n = \xi \). Then by Remark 3.2 there exists a subsequence \( \{w_{n_k}\} \) such that all the limits

\[ \lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})], \quad z, z_0 \in B_0, \]

exist. Therefore in place of the horospheres \( G \) with respect to the Kobayashi distance \( k_B \), we can analogously define the horospheres

\[ G_{B_0, k_B}(\xi, R, z_0, \{w_{n_k}\}) = \{z \in B_0 : \lim_{k \to \infty} [k_B(z, w_{n_k}) - k_B(z_0, w_{n_k})] < \frac{1}{2} \log R\} \]

in \( B_0 \). The horospheres \( G_{B_0, k_B} \) have the properties (ii), (iii), (vi), (vii), (viii) (with \( B \) replaced by \( B_0 \)) given in Theorem 3.3.

Now we are ready to prove the following theorem.

**Theorem 4.4.** Let \( X \) be a uniformly convex Banach space with the open unit ball \( B \). Let \( f : B \to B \) be a compact \( k_B \)-nonexpansive map with no fixed point in \( B \). Then there exists \( \xi \in \partial B \) such that the sequence \( \{f^n\} \) of iterates of \( f \) converges locally uniformly on \( B \) to the constant map taking the value \( \xi \).

**Proof.** Applying Lemma 4.3 instead of Theorems 2.2 and 2.3, and replacing in \( B_0 \) the horospheres with respect to \( k_B \) with the horospheres \( G_{B_0, k_B} \) (see Remark 4.1), we can repeat the proof of Theorem 4.1 to obtain
the convergence of \( \{ f^n(z) \} \) to \( \xi \) for each \( z \in B_0 \). The \( k_B \)-nonexpansiveness, the compactness of \( f \) and Lemma 4.3 imply the locally uniform convergence of \( \{ f^n \} \) on \( B \).

**Corollary 4.5.** Let \( X \) be a uniformly convex Banach space with the open unit ball \( B \). Let \( f: B \to B \) be a compact \( k_B \)-nonexpansive map with no fixed point in \( B \). Then there exists \( \zeta \in \partial B \) such that \( \{ h(t, \cdot) \} \) tends uniformly on \( B \) to the constant map \( \zeta \) and the sequence \( \{ f^n \} \) is locally uniformly convergent to the same constant map \( \zeta \).

**ACKNOWLEDGMENTS**

The research of the third author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund-M. and M.L. Bank Mathematics Research Fund. Part of this work was done when the second author visited the Department of Mathematics at the Technion. He thanks the Department for its hospitality.

**REFERENCES**