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Eigenvalue Asymptotics of the Neumann Laplacian of Regions and Manifolds with Cusps

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We study the eigenvalue asymptotics of a Neumann Laplacian $-\Delta_N^\Omega$ in unbounded regions Ω of \mathbf{R}^2 with cusps at infinity (a typical example is $\Omega = \{(x, y) \in \mathbf{R}^2 : x > 1, |y| < e^{-x^2}\}$) and prove that $N_E(-\Delta_N^\Omega) \sim N_E(H_V) + E/2 \text{Vol}(\Omega)$, where H_V is the canonical one-dimensional Schrödinger operator associated to the problem. We establish a similar formula for manifolds with cusps and derive the eigenvalue asymptotics of a Dirichlet Laplacian $-\Delta_D^\Omega$ for a class of cusp-type regions of infinite volume. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let Ω be a region in \mathbf{R}^d . We recall that the Neumann Laplacian $H_N = -\Delta_N^\Omega$ is the unique self-adjoint operator whose quadratic form is

$$g(f, f) = \int_{\Omega} |\nabla f|^2 dx \quad (1.1)$$

on the domain $H^1(\Omega) = \{f \in L^2(\Omega) \mid \nabla f \in L^2(\Omega)\}$, where the gradient is

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taken in the distributional sense. One similarly defines the Dirichlet Laplacian $H_D = -\Delta_D^\Omega$ as the unique self-adjoint operator whose quadratic form is given by the closure of (1.1) on the domain $C_0^\infty(\Omega)$. If Ω is a bounded region with a smooth boundary it is well known that both H_N and H_D have compact resolvents and that their eigenvalue distributions are given by Weyl's law

$$N_E(H_N) \sim N_E(H_D) \sim \frac{\tau_d}{(2\pi)^d} \text{Vol}(\Omega) E^{d/2}, \quad (1.2)$$

where by $f(E) \sim g(E)$ we mean $\lim_{E \rightarrow \infty} f(E)/g(E) = 1$. We denote by τ_d the volume of a unit ball in \mathbf{R}^d , by $\text{Vol}(\Omega)$ the Lebesgue measure of Ω , and by $N_E(A)$ the number of eigenvalues of the operator A which are less than E . If one drops the condition that Ω have a smooth boundary, nothing dramatic happens with the Dirichlet Laplacian H_D . As long as $\text{Vol}(\Omega) < \infty$, H_D will have a compact resolvent and (1.2) remains true [17]. On the other hand, the spectrum of H_N can undergo rather spectacular changes. The following theorem was proved in [10]:

THEOREM. *Let S be a closed subset of the positive real axis. Then there exists a bounded domain Ω for which*

$$\sigma_{\text{ess}}(H_N) = S.$$

In the theorem, Ω can be chosen in such a way that its boundary has a singularity at exactly one point. We will be interested in the other extreme, namely when the domain Ω retains a nice boundary, but is unbounded, and in particular is of the form

$$\Omega = \{(x, y) \in \mathbf{R}^2 : x > 1, |y| < f(x)\}. \quad (1.3)$$

Through the rest of the paper we will suppose that f is $C^\infty[1, \infty)$ and strictly positive, and that its first three derivatives are bounded (although less regularity could be required). If $f(x) \rightarrow 0$, the Dirichlet Laplacian still has a compact resolvent [13, 16], but if $f(x) = x^{-1}$, or even if $f(x) = \exp(-x)$, (so $\text{Vol}(\Omega) < \infty$), Davies and Simon [6] showed that $\sigma_{\text{ac}}(H_N)$ is nonempty. The difference in the spectral behavior is again striking, and one feels that a rather rapid decay of f should be required to ensure compactness of the resolvent of H_N . The following beautiful theorem was proven in [7]:

THEOREM. *If Ω is given by (1.3), H_N has a compact resolvent if and only if*

$$\lim_{x \rightarrow \infty} \left(\int_1^x \frac{1}{f(t)} dt \right) \left(\int_x^\infty f(t) dt \right) = 0. \quad (1.4)$$

In this paper we study the large E asymptotics of the eigenvalue distribution of H_N in the regions (1.3). As in [6], the main role is played by the one-dimensional Schrödinger operator

$$H_V = -\frac{d^2}{dx^2} + V(x), \quad V(x) = \frac{1}{4} \left(\frac{f'}{f} \right)^2 + \frac{1}{2} \left(\frac{f'}{f} \right)', \quad (1.5)$$

acting on $L^2[1, \infty)$, and with the Dirichlet boundary condition at 1. We make the following two hypotheses:

$$V(x) \rightarrow \infty, \quad f''(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty; \quad (\text{H1})$$

$$\text{if } 0 < \varepsilon < 1, \quad N_E((1 \pm \varepsilon) H_V) = N_E(H_V)(1 + O(\varepsilon)). \quad (\text{H2})$$

Our main result is

THEOREM 1.1. *If (H1) and (H2) are satisfied, we have*

$$N_E(H_N) \sim N_E(H_V) + \frac{E}{2} \text{Vol}(\Omega). \quad (1.6)$$

Remark 1. Hypothesis (H1) implies that

$$f(x) + f'(x)^2/f(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty \quad (1.7)$$

(see Section 2.1). In turn, Davies and Simon [6] showed that if (1.7) is satisfied, H_N will have a compact resolvent if and only if H_V does. Consequently, both sides in (1.6) are finite, and in particular (H1) implies that $\text{Vol}(\Omega) < \infty$. Hypothesis (H2) prevents $N_E(H_V)$ from growing too rapidly (e.g., exponentially), which is needed to make our perturbation argument work. For example, it is satisfied if V is a convex function [21] or if $V(x) \sim x^\alpha (\ln x)^\beta$, $\alpha > 0$. On the other hand, if $V(x) \sim \ln x$ (e.g., $f(x) = \exp(-x \ln x)$), it is not, and our argument does not apply.

Remark 2. The fact that Ω is symmetric is irrelevant. If $\Omega = \{(x, y) : x > 1, -f_1(x) < y < f_2(x)\}$, (1.6) remains valid provided that $f_1'' \rightarrow 0$, $f_2'' \rightarrow 0$, and H_V , defined with $f = (f_1 + f_2)/2$, satisfies (H1), (H2). Also, if (H1) is replaced by a more involved hypothesis, the result extends (as usual [6, 17]) to the case when \mathbf{R}^2 is replaced by \mathbf{R}^{d+1} , $(x_1, x_\perp) \in \mathbf{R}^{d+1}$, $x_\perp \in \mathbf{R}^d$, $\Omega = \{x \mid x_\perp/f(x_1) \in G, 1 \leq x_1 \leq \infty\}$, where G is a bounded connected set. The asymptotics is given by (2.6) (replacing M with Ω).

EXAMPLE 1. Let $f(x) = \exp(-x^\alpha)$. H_N has a compact resolvent if and only if $\alpha > 1$. One calculates

$$V(x) = \frac{\alpha^2}{4} x^{2(\alpha-1)} - \frac{\alpha(\alpha-1)}{2} x^{(\alpha-2)}.$$

The semiclassical formula [21] yields

$$N_E(H_\nu) \sim \frac{1}{4(\alpha-1)\sqrt{\pi}} \left(\frac{\alpha}{2}\right)^{1/(1-\alpha)} \frac{\Gamma(1/(2(\alpha-1)))}{\Gamma(\frac{3}{2} + 1/(2(\alpha-1)))} E^{1/2 + 1/(2(\alpha-1))}, \quad (1.8)$$

and thus both (H1) and (H2) are satisfied. Formulas (1.6) and (1.8) imply that the asymptotics of $N_E(H_N)$ satisfies Weyl's law if $\alpha > 2$; it is given by (1.8) if $1 < \alpha < 2$. If $\alpha = 2$ we have $N_E(H_N) \sim E/2(\text{Vol}(\Omega) + \frac{1}{2})$. The leading order is the same as that in Weyl's law but the constant is larger. We observe a phase transition in the eigenvalue asymptotics for the value $\alpha_c = 2$. In [6] it was shown that for $\alpha = 1$, $\sigma_{ac}(H_N) = [\frac{1}{4}, \infty)$, and for $0 < \alpha < 1$, $\sigma_{ac}(H_N) = [0, \infty)$. In both cases $\sigma_{sing}(H_N) = \emptyset$, and $\sigma_{pp}(H_N)$ consists of a discrete set $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty$ of embedded eigenvalues of finite multiplicity (see also [12]).

EXAMPLE 2. Let $f(x) = \exp(-x^2g(x))$, $g(x) = 1 + \cos^2(\sqrt{\ln(1+x)})$. We have $V(x) \sim x^2g(x)^2$ and the semiclassical formula yields

$$N_E(H_\nu) \sim \frac{E}{4g(E)}.$$

Hypotheses (H1), (H2) are satisfied, and we observe that $N_E(H_N)/E$ stays bounded above and below but $\lim_{E \rightarrow \infty} N_E(H_N)/E$ does not exist.

The simplest way to understand the result of Theorem 1.1 is to consider a subspace P of $L^2(\Omega)$ consisting of functions u which depend on the x variable only. On $C_0^2(\bar{\Omega}) \cap P$ the form (1.1) acts as

$$\int_1^\infty \left| \frac{d}{dx} u(x) \right|^2 2f(x) dx,$$

and viewed as a form on $L^2([1, \infty), 2f(x) dx)$ yields an operator which is (up to a change of boundary condition at $x = 1$) unitarily equivalent to H_ν . It is now immediate that $N_E(-\Delta_N^\Omega) \geq N_E(H_\nu)$ ($f(E) \geq g(E)$ means that $\liminf_{E \rightarrow \infty} f(E)/g(E) \geq 1$), but in an equally simple way we can say even more. Denote $\Omega_L = \{(x, y) : 1 < x < L, |y| < f(x)\}$ and put an additional Dirichlet boundary condition along the line $x = L$. Dirichlet-Neumann bracketing yields

$$N_E(-\Delta_N^\Omega) \geq N_E(H_\nu) + \frac{E}{2} \text{Vol}(\Omega_L).$$

Letting $L \rightarrow \infty$, we obtain the one-sided inequality in (1.6), which is obviously true under the sole condition that f is a $C^2[1, \infty)$ function. It is the other, nontrivial direction of (1.6) which forces us to place conditions on f and V , and which could be proven using techniques developed in [6, 18]. The main technical point in such an approach is to obtain control of H_N on the subspace orthogonal to P . Here we will adopt a different strategy which, we believe, sheds some new light on the problem. Let $M = (-1, 1) \times (1, \infty)$ be a strip with the metric

$$ds_M^2 = dx^2 + f(x)^2 dy^2, \quad (1.9)$$

and denote by H_N the Laplace–Beltrami operator on M with the Neumann boundary condition. Separating the variables we obtain that H_N is unitarily equivalent to the operator $\bigoplus_{n \geq 0} H_n$ acting on $\bigoplus_{n \geq 0} L^2([1, \infty), dx)$, where

$$H_n = H_V + \left(\frac{n\pi}{2f(x)} \right)^2$$

with a boundary condition $\psi'(1) = (f'(1)/2f(1))\psi(1)$. The main technical ingredient in this approach is to show that $N_E(\bigoplus_{n \geq 1} H_n)$ satisfies Weyl's law. Then the analog of Theorem 1.1 for H_N is immediate. After a suitable coordinate change, region Ω is transformed into the strip M , with a metric which is (under the conditions of the Theorem 1.1) asymptotically of the form (1.9). At this point, a relatively easy perturbation argument will yield (1.6).

Finally, we remark that the above approach appears useful in studying eigenvalue asymptotics of a Dirichlet Laplacian in a region Ω given by (1.3), with $f(x) \rightarrow 0$ and $\text{Vol}(\Omega) = \infty$. While we can recover most of the known results on the asymptotics of $N_E(H_D)$ in such regions (but not all; e.g., we cannot treat the case $f(x) = (\ln(1+x))^{-1}$, see [1, 17]), here we restrict ourselves to giving a new proof of the well-known [1, 17, 18, 20])

THEOREM 1.2. *Let $\Omega = \{(x, y) : |x|^\alpha |y| \leq 1\}$. Then*

$$N_E(H_D) \sim \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi} \right)^{1/\alpha} \zeta(1/\alpha) \frac{\Gamma(1+1/2\alpha)}{\Gamma(\frac{3}{2}+1/2\alpha)} E^{1/2+1/2\alpha}, \quad \text{if } 0 < \alpha < 1,$$

$$N_E(H_D) \sim \frac{1}{\pi} E \ln E, \quad \text{if } \alpha = 1,$$

where ζ is the standard zeta function. The case $\alpha > 1$ follows by symmetry.

2. NEUMANN LAPLACIANS ON MANIFOLDS AND REGIONS WITH CUSPS

We begin by studying the eigenvalue distribution of a Laplace–Beltrami operator on a Riemannian manifold of the form $M = N \times [1, \infty)$, with a metric ds_M^2 , $\text{Vol}(M) < \infty$. Here, N is a compact, oriented Riemannian manifold (with or without boundary), $\dim(N) = d$, with a metric ds_N^2 , and a volume element dm_N . We remark that the boundary of M does not have to be C^∞ , but it is certainly piecewise C^∞ and therefore causes no problem in the discussion below (see, e.g., [2]). In Sections 2.1 and 2.2 we treat the case when the metric on M has a warped product form. Perturbations are studied in Section 2.3. Finally, in Section 2.4, we derive Theorem 1.1 as an easy consequence of the results obtained for manifolds.

2.1. Preliminaries

We suppose that the metric on M is given by

$$ds_M^2 = dx^2 + f(x)^2 ds_N^2, \quad (2.1)$$

where f is a positive, $C^\infty[1, \infty)$ function, and that

$$\text{Vol}(M) = \text{Vol}(N) \int_1^\infty f(x)^d dx < \infty. \quad (2.2)$$

If $d = 1$, (2.2) is a consequence of (H1). H_N , the Laplace–Beltrami operator on M with Neumann boundary conditions, acts on a Hilbert space $L^2(M, dm_M)$ and is the unique self-adjoint operator whose quadratic form is given by the closure of

$$q(\phi, \phi) = \int_M |\nabla\phi|^2 dm_M \quad (2.3)$$

on $C_0^2(\bar{M})$. In (2.2), $dm_M = f^d dm_N dx$ and ∇ is the gradient on M . Of equal importance for us is the Laplace–Beltrami operator $H_{N,D}$ on M with the Dirichlet boundary condition along $\{1\} \times N$, and the Neumann one on the rest of the boundary. It is defined as a closure of the form (2.3) on a subspace of $C_0^2(\bar{M})$ consisting of functions which vanish along $\{1\} \times N$. The analog of (1.5) is the one-dimensional Schrödinger operator of the form

$$H_V = -\frac{d^2}{dx^2} + V(x), \quad V(x) = \frac{d^2}{4} \left(\frac{f'}{f} \right)^2 + \frac{d}{2} \left(\frac{f'}{f} \right)', \quad (2.4)$$

with the Dirichlet boundary condition at 1. Denote by

$$C_d = ((4\pi)^{(d+1)/2} \Gamma((d+3)/2))^{-1}. \quad (2.5)$$

The following lemma, which we prove in Section 2.2, is the main technical ingredient of our approach.

LEMMA 2.1. *Suppose that $V(x) \rightarrow \infty$, $f(x)^2 V(x) \rightarrow 0$ as $x \rightarrow \infty$. Then*

$$N_E(H_N) \sim N_E(H_{N,D}) \sim N_E(H_V) + E^{(d+1)/2} C_d \text{Vol}(M). \quad (2.6)$$

In the sequel we collect, for the reader's convenience, a few simple results which will be needed later. Let

$$D_N = \{ \phi : \phi \in C_0^2(\bar{M}), v\phi = 0 \},$$

where v is the outward unit normal vector field on ∂M . H_N acts on D_N as

$$H_N(\phi) = -\frac{1}{f(x)^d} \frac{\partial}{\partial x} f(x)^d \frac{\partial}{\partial x} \phi + \frac{1}{f(x)^2} H_N^N(\phi),$$

where H_N^N is a Laplace–Beltrami operator of N . H_N^N has a compact resolvent [2]; its spectrum consists of discrete eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_n \rightarrow \infty$, and we denote by ϕ_n^N the corresponding eigenfunctions. Introducing

$$L_n^2(M) = \left\{ g : g(x, t) = \psi(x) \phi_n^N(t), \int_1^\infty |\psi(x)|^2 f(x)^d dx < \infty \right\},$$

we obtain the decomposition

$$L^2(M) = \bigoplus_{n \geq 0} L_n^2(M) = \bigoplus_{n \geq 0} L_n^2([1, \infty), f(x)^d dx).$$

The operator H_N splits accordingly,

$$H_N = \bigoplus_{n \geq 0} H_{N,n},$$

where $H_{N,n}$ acts on $L_n^2([1, \infty), f(x)^d dx)$ as

$$H_{N,n} = -\frac{1}{f(x)^d} \frac{\partial}{\partial x} f(x)^d \frac{\partial}{\partial x} + \frac{\lambda_n}{f^2(x)}.$$

Under the unitary map

$$U : L^2([1, \infty), dx) \rightarrow L_n^2([1, \infty), f(x)^d dx), \quad U(\phi) = f^{-d/2} \phi$$

$H_{N,n}$ transforms as

$$H_n = U^{-1} H_{N,n} U = -\frac{d^2}{dx^2} + V(x) + \frac{\lambda_n}{f^2(x)}, \quad (2.7)$$

and (if V is bounded below) is essentially self-adjoint on

$$D_V = \{ \psi : \psi \in C_0^2[1, \infty), \psi'(1) = d/2(f'(1)/f(1)) \psi(1) \}. \quad (2.8)$$

H_N is unitarily equivalent to the operator $\bigoplus_{n \geq 0} H_n$ acting on $\bigoplus_{n \geq 0} L^2([1, \infty), dx)$. Similarly, $H_{N,D}$ is unitarily equivalent to $\bigoplus_{n \geq 0} H_n^D$, where H_n^D is the operator (2.7) with the Dirichlet boundary condition at 1. The spectral analysis of H_N and $H_{N,D}$ reduces to the spectral analysis of the one-dimensional Schrödinger operators H_n, H_n^D . We will need

LEMMA 2.2. *If $V(x) \rightarrow \infty$ as $x \rightarrow \infty$, we have*

$$N_E(H_n^D) \leq N_E(H_n) \leq 1 + N_E(H_n^D) \quad (2.9)$$

for all $n \geq 0$.

Proof. We just sketch the well-known argument. $C_0^\infty[1, \infty)$ is a form core for H_n^D , and thus $N_E(H_n^D) \leq N_E(H_n)$ follows from the min-max principle [15]. Let

$$D = \{\psi : \psi \in C_0^2[1, \infty), \psi(1) = 0\}.$$

If L stands for an arbitrary vector subspace of $L^2[1, \infty)$, the min-max principle yields

$$N_E(H_n) = \sup_{\substack{L \subset D_V \\ (H_n \psi, \psi) < E \\ \psi \in L, \|\psi\| = 1}} \dim L, \quad N_E(H_n^D) = \sup_{\substack{L \subset D \\ (H_n^D \psi, \psi) < E \\ \psi \in L, \|\psi\| = 1}} \dim L.$$

Fix $L \subset D_V$ and let $L_0 = \{\psi \in L : \psi(1) = 0\}$. Observing that $\dim L/L_0 \leq 1$ we derive (2.9). ■

We finish with the following

LEMMA 2.3. *If $d = 1$ and (H1) is satisfied, we have*

$$f(x) + |f'(x)| + f'(x)^2/f(x) + f(x)V(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (2.10)$$

Furthermore, for large x , f is convex and strictly decreasing.

Proof. The result follows from

$$f''(x) = 2f(x)V(x) + \frac{f'(x)^2}{2f(x)}. \quad \blacksquare$$

2.2. Proof of Lemma 2.1

Denote $A_N = \bigoplus_{n \geq 1} H_n$, $A_D = \bigoplus_{n \geq 1} H_n^D$. Relations (2.6) will follow if we prove that

$$\lim_{E \rightarrow \infty} \frac{N_E(A_N)}{E^{(d+1)/2}} = \frac{N_E(A_D)}{E^{(d+1)/2}} = C_d \text{Vol}(M). \quad (2.11)$$

Let

$$m = \max_{x>1} f(x)^2, \quad M = \max_{x>1} |V(x) f(x)^2|.$$

We have that $N_E(H_n^D) = 0$ if $\lambda_n > mE + M$, and thus

$$\begin{aligned} N_E(A_D) &\leq N_E(A) \leq N_E(A_D) + \#\{\lambda_n : \lambda_n \leq mE + M\} \\ &\leq N_E(A_D) + O(E^{d/2}), \end{aligned}$$

since Weyl's law applies for H_N^N . Consequently, it suffice to prove (2.11) for A_D . By the Karamata–Tauberian theorem [19], (2.11) will follow if we prove

$$\lim_{t \rightarrow 0} t^{(d+1)/2} \text{Tr}(\exp(-tA_D)) = (4\pi)^{-(d+1)/2} \text{Vol}(M). \quad (2.12)$$

First, note that

$$\lim_{t \rightarrow 0} t^{d/2} \sum_{k \geq 1} \exp(-t\lambda_k) = (4\pi)^{-d/2} \text{Vol}(N) \quad (2.13)$$

[2], and in addition

$$t^{d/2} \sum_{k \geq 1} \exp(-t\lambda_k) < L \quad (2.14)$$

for a uniform constant L and for all $t > 0$. The Golden–Thompson inequality [19] yields

$$\text{Tr}(\exp(-tH_k^D)) \leq \frac{1}{2\sqrt{\pi t}} \int_1^\infty \exp(-t \cdot (V(x) + \lambda_k/f(x)^2)) dx.$$

Fix $\varepsilon > 0$, $\lambda_1 > \varepsilon > 0$, and let $R > 0$ be big enough that $|f^2(x) V(x)| < \varepsilon$ if $x > R$. Let $c = \inf_{x \in [1, R]} V(x)$. We have

$$\begin{aligned} \text{Tr}(\exp(-tH_k)) &\leq \frac{e^{-ct}}{2\sqrt{\pi t}} \int_1^\infty \exp(-t \cdot (\lambda_k - \varepsilon)/f(x)^2) dx \\ t^{(d+1)/2} \text{Tr}(\exp(-tA_D)) &= \sum_{k \geq 1} t^{(d+1)/2} \text{Tr}(\exp(-tH_k)) \\ &\leq \frac{e^{-ct}}{2\sqrt{\pi}} \int_1^\infty f(x)^d \cdot \sum_k (t/f(x)^2)^{d/2} \\ &\quad \times \exp(-t(\lambda_k - \varepsilon)/f(x)^2) dx. \end{aligned}$$

Using (2.14) and the Lebesgue dominated convergence theorem we obtain

$$\limsup_{t \rightarrow 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \leq \frac{\operatorname{Vol}(N)}{(4\pi)^{(d+1)/2}} \int_1^\infty f(x)^d dx.$$

It remains to show

$$\liminf_{t \rightarrow 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \geq (4\pi)^{-(d+1)/2} \operatorname{Vol}(M). \quad (2.15)$$

Let $R > 1$ be a positive number; make a partition of $[1, R]$ into k intervals I_m of equal size. Denote by H_D^m the Dirichlet Laplacian on I_m , and let

$$d_m = \sup_{x \in I_m} f(x)^{-2}, \quad c = \sup_{x \in [1, R]} |V(x)|.$$

Dirichlet–Neumann bracketing yields

$$\operatorname{Tr}(\exp(-tH_n)) \geq \sum_{m=1}^k \exp(-t(c + \lambda_n d_m)) \operatorname{Tr}(\exp(-tH_D^m)).$$

Obviously, $\operatorname{Tr}(\exp(-tH_D^m)) = \operatorname{Tr}(\exp(-tH_D^1))$ for all m , and

$$\lim_{t \rightarrow 0} t^{1/2} \operatorname{Tr} \exp(-tH_D^1) = \frac{1}{2\sqrt{\pi}} \frac{R}{k}.$$

We have

$$\begin{aligned} & \liminf_{t \rightarrow 0} t^{(d+1)/2} \sum_{n>0} \operatorname{Tr}(\exp(-tH_n)) \\ & \geq \frac{1}{2\sqrt{\pi}} \sum_{m=1}^k \frac{R}{k} d_m^{-d/2} \liminf_{t \rightarrow 0} (td_m)^{d/2} \sum_{n>0} \exp(-td_m \lambda_n) \\ & = \operatorname{Vol}(N) (4\pi)^{-(d+1)/2} \sum_{m=1}^k \frac{R}{k} d_m^{-d/2}. \end{aligned}$$

Using that f is continuous and passing to the limit $k \rightarrow \infty$ we have

$$\liminf_{t \rightarrow 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \geq (4\pi)^{-(d+1)/2} \operatorname{Vol}(N) \int_1^R f(x)^d dx.$$

Letting $R \rightarrow \infty$ we obtain (2.15) and (2.12).

2.3. Metric Perturbations

In this section we suppose that a metric on M is given by

$$ds_M^2 = \alpha(t, x)^2 dx^2 + \beta(t, x)^2 ds_N^2, \quad (2.16)$$

where α, β are two positive, C^∞ functions on M . We also suppose that

$$\text{Vol}(M) = \int_M \alpha \cdot \beta^d dm_N dx < \infty.$$

After a suitable coordinate change, the region Ω , given by (1.3) (if (H1) is satisfied), transforms into $(-1, 1) \times (1, \infty)$ with a metric of the form (2.16). That is the reason why we choose to discuss (2.16), even if a much larger class of perturbations can be treated along the same lines (see [9, 14] for related discussions). $H_N, H_{N,D}$ are defined, as in the previous section, via the closure of the quadratic form (2.3) (with $dm_M = \alpha \cdot \beta^d dm_N dx$) on the appropriate subspace. If there exists a function f , satisfying the condition of Lemma 2.1, such that $\alpha \rightarrow 1, \beta \rightarrow f$ as $x \rightarrow \infty$, one expects that $N_E(H_N)$ should not be too far from $N_E(\hat{H}_N)$, where \hat{H}_N is the Laplace–Beltrami operator on M for the metric

$$ds_M^2 = dx^2 + f(x)^2 ds_N^2. \quad (2.17)$$

It is indeed the case. Denote $M_L = N \times [L, \infty)$,

$$\|g\|_L = \sup_{(t,x) \in M_L} |g(t,x)|,$$

and let

$$v(L) = \|\alpha - 1\|_L + \|f/\beta - 1\|_L + \|\hat{\nabla}\alpha\|_L + \|\hat{\nabla}(f/\beta)\|_L, \quad (2.18)$$

where $\hat{\nabla}$ is the gradient on M with the metric (2.17). For H_v given by (2.4) we have

LEMMA 2.4. *Suppose that $v(L) \rightarrow 0$ as $L \rightarrow \infty$, that f and V satisfy the conditions of Lemma 2.1, and that $N_E(H_v)$ satisfies (H2). Then*

$$N_E(H_N) \sim N_E(H_{N,D}) \sim N_E(H_v) + E^{(d+1)/2} C_d \text{Vol}(M).$$

We remark that, while $\text{Vol}(M)$ is calculated in the metric (2.16), the operator H_v arises from the metric (2.17).

Proof. We will consider only H_N . A virtually identical argument applies for $H_{N,D}$. For $L > 1$ denote by $H_{L,D}^-, H_{L,D}^+$ the Laplace–Beltrami operators acting on $N \times [1, L], M_L$, with metric (2.16) and the Dirichlet boundary condition along $N \times \{L\}$; on the rest of boundary we take the Neumann one. Denote by $\hat{H}_{L,D}^+$ the Laplace–Beltrami operator on M_L with the metric (2.17) and with same boundary condition as $H_{L,D}^+$. Let

$$U: L^2(M_L, \alpha\beta^d dm_N dx) \rightarrow L^2(M_L, f^d dm_N dx)$$

be a unitary mapping defined as

$$U(\phi) = (\alpha \cdot (\beta/f)^d)^{1/2} \phi = (1/g) \cdot \phi.$$

The operator $H_{L,D}^+$ is then unitarily equivalent to the operator acting on $L^2(M_L, f^d dm_N dx)$, which we again denote by $H_{L,D}^+$ and whose quadratic form is given by the closure of

$$\int_{M_L} |\nabla(g\phi)|^2 \alpha \beta^d dm_N dx \quad (2.19)$$

on the subspace

$$C_{0,L}^2(\bar{M}_L) = \{\phi : \phi \in C_0^2(\bar{M}_L), \phi(t, L) = 0\}.$$

Vector fields $\nabla\phi$, $\hat{\nabla}\phi$ are given as

$$\nabla\phi = \frac{1}{\alpha^2} \frac{\partial\phi}{\partial x} \frac{\partial}{\partial x} + \frac{1}{\beta^2} \nabla_N \phi,$$

$$\hat{\nabla}\phi = \frac{\partial\phi}{\partial x} \frac{\partial}{\partial x} + \frac{1}{f^2} \nabla_N \phi,$$

where ∇_N is the gradient on N . If $\phi \in C_{0,L}^2(\bar{M}_L)$ and has norm 1 as an element of the $L^2(M_L, f^d dm_N dx)$, we estimate

$$\begin{aligned} & |((H_{L,D}^+ - \hat{H}_{L,D}^+) \phi, \phi)| \\ & \leq \int_{M_L} ||\nabla(g\phi)|^2 \alpha \beta^d - |\hat{\nabla}(\phi)|^2 f^d| dm_N dx \\ & \leq \int_{M_L} A(L) |\hat{\nabla}(g\phi)|^2 + ||\hat{\nabla}(g\phi)|^2 - |\hat{\nabla}(\phi)|^2| f^d dm_N dx, \end{aligned}$$

where

$$A(L) = \|1/g\|_L^2 (\|(1/\alpha)^2 - 1\|_L + \|(f/\beta)^2 - 1\|_L) + \|(1/g)^2 - 1\|_L.$$

Furthermore, we have

$$\begin{aligned} |\hat{\nabla}(g\phi)|^2 & \leq |\hat{\nabla}(\phi)|^2 \cdot (\|g\|_L^2 + \|g\|_L \cdot \|\hat{\nabla}g\|_L) \\ & \quad + |\phi|^2 \cdot (\|\hat{\nabla}g\|_L^2 + \|g\|_L \cdot \|\hat{\nabla}g\|_L), \\ | |\hat{\nabla}(g \cdot \phi)|^2 - |\hat{\nabla}(\phi)|^2 | & \leq |\hat{\nabla}(\phi)|^2 \cdot (\|g^2 - 1\|_L + \|g\|_L \cdot \|\hat{\nabla}g\|_L) \\ & \quad + |\phi|^2 \cdot (\|\hat{\nabla}g\|_L^2 + \|g\|_L \cdot \|\hat{\nabla}g\|_L). \end{aligned}$$

Because $v(L) \rightarrow 0$, all the constants in the above estimates are $O(v(L))$, and we conclude that

$$|((H_{L,D}^+ - \hat{H}_{L,D}^+) \phi, \phi)| \leq Dv(L)((\hat{H}_{L,D}^+ \phi, \phi) + 1), \quad (2.20)$$

for an L -independent constant D . In the sequel we take L large enough that $Dv(L) < 1$, and then absorb D into $v(L)$. The inequality (2.20) and the min-max principle yield (recall that $f(E) \gg g(E)$ means $\liminf_{E \rightarrow \infty} f(E)/g(E) \geq 1$)

$$N_E(H_{L,D}^+) \gg N_{E-v(L)}((1+v(L)) \hat{H}_{L,D}^+) \sim N_E((1+v(L)) \hat{H}_{L,D}^+). \quad (2.21)$$

If $H_{V,L}$ is operator (2.4) acting on $L^2[L, \infty)$, we observe that the asymptotics of $N_E(H_{V,L})$ does not depend on the boundary condition at L , nor on L itself. Consequently, in the sequel we will deal only with H_V . Denote

$$C_1(L) = \liminf_{E \rightarrow \infty} \frac{N_E((1-v(L)) H_V)}{N_E(H_V)},$$

$$C_2(L) = \limsup_{E \rightarrow \infty} \frac{N_E((1+v(L)) H_V)}{N_E(H_V)}.$$

Hypotheses (H2) implies

$$\lim_{L \rightarrow \infty} C_1(L) = 1, \quad \lim_{L \rightarrow \infty} C_2(L) = 1. \quad (2.22)$$

The relations (2.20), (2.22) were the two essential ingredients in the argument. Denote by $\text{Vol}(\hat{M}_L)$ the volume of M_L in the metric (2.17). We have

$$\begin{aligned} N_E(H_N) &\gg N_E(H_{L,D}^-) + N_E(H_{L,D}^+) \\ &\gg E^{(d+1)/2} C_d \text{Vol}(N \times [1, L]) + N_E((1+v(L)) \hat{H}_{L,D}^+) \\ &\gg I(L) (E^{(d+1)/2} C_d (\text{Vol}(N \times [1, L]) + \text{Vol}(\hat{M}_L)) + N_E(H_V)), \end{aligned} \quad (2.23)$$

where

$$I(L) = \min\{(1+v(L))^{-((1+d)/2)}, C_2(L)\}.$$

Formula (2.23) follows from Dirichlet–Neumann bracketing, Lemma 2.1, and the fact that the eigenvalue distribution of a Laplace–Beltrami operator on a compact manifold with a piecewise smooth boundary and

mixed boundary conditions satisfies Weyl's law [2]. Replacing the boundary condition along $N \times \{L\}$ with the Neumann b.c., we obtain the operators $H_{L,N}^-$ and $H_{L,N}^+$, and a completely analogous argument gives

$$\begin{aligned} N_E(H_N) &\leq N_E(H_{L,N}^-) + N_E(H_{L,N}^+) \\ &\leq S(L)(E^{(d+1)/2} C_d(\text{Vol}(N \times [1, L]) + \text{Vol}(\hat{M}_L)) + N_E(H_\nu)), \end{aligned} \quad (2.24)$$

where

$$S(L) = \max\{(1 - \nu(L))^{-(1+d)/2}, C_1(L)\}.$$

As $L \rightarrow \infty$, $I(L) \rightarrow 1$, $S(L) \rightarrow 1$, $\text{Vol}(\hat{M}_L) \rightarrow 0$, $\text{Vol}(N \times [1, L]) \rightarrow \text{Vol}(M)$, and the lemma follows from (2.23), (2.24). ■

It is now obvious why our argument fails in the case when $N_E(H_\nu)$ grows exponentially fast ($C_1(L) = C_2(L) = \infty$). It is natural to conjecture that in such cases $N_E(H_N) \sim N_E(H_\nu)$, but it is unlikely that the above argument can be modified to prove it.

2.4. Proof of Theorem 1.1.

One consequence of hypothesis (H1) (see Lemma 2.3) is that f is strictly decreasing function for large x . The familiar Dirichlet–Neumann bracketing argument, which will be repeated in detail once again below, implies that without loss of generality we can assume $f'(x) < 0$ for $x > 1$. We construct a change of variable as follows: Let

$$\varepsilon(x, y) = \frac{y}{f(x)}, \quad -1 \leq \varepsilon \leq 1.$$

ε is the first integral of the equation

$$\frac{dy}{dx} = y \cdot \frac{f'}{f}.$$

The equation for the orthogonal lines is given by

$$y \cdot \frac{dy}{dx} = -\frac{f}{f'},$$

whose first integral is

$$\frac{y^2}{2} + \int_1^x \frac{f(t)}{f'(t)} dt = c.$$

Any C^1 function of this first integral is an orthogonal coordinate to ε . Let

$$F(x) = \int_1^x \frac{f(t)}{f'(t)} dt;$$

note that F is a decreasing function ($f' < 0$) and denote $a = \lim_{x \rightarrow \infty} F(x)$. The inverse function F^{-1} is well defined on $(a, 0]$, and for R large enough we have

$$\frac{y^2}{2} + F(x) \in (a, 0], \quad x > R, (x, y) \in \Omega.$$

Let

$$\eta(x, y) = F^{-1}(y^2/2 + F(x)), \quad (x, y) \in \Omega, x > R.$$

It is easy to check that (ε, η) is one-one, and that Jacobian $D(\varepsilon, \eta)/D(x, y) \sim 1/f(x) \neq 0$ for x large. Denoting (for $c > R$)

$$\Omega_1 = \{(x, y) : (x, y) \in \Omega, \eta(x, y) > c\}, \quad (2.25)$$

we conclude that, for a large c , (ε, η) is a C^∞ -bijection between Ω_1 and half-strip $M = (-1, 1) \times (c, \infty)$ with a C^∞ -inverse. The eigenvalue asymptotics of a Laplacian on a bounded region with piecewise C^∞ boundary and with mixed boundary conditions satisfies Weyl's law. Consequently, putting an additional Dirichlet or Neumann b.c. along $\eta(x, y) = c$ we observe that it is enough to prove the statement for $H_N, H_{N,D}$, the Laplacians on Ω_1 with respectively Neumann or Dirichlet b.c. along $\eta(x, y) = c$, and the Neumann b.c. on the rest of the boundary. The above change of variables transforms $H_N, H_{N,D}$ into Laplace-Beltrami operators on M , with the metric $ds_M^2 = dx(\varepsilon, \eta)^2 + dy(\varepsilon, \eta)^2$, and the Neumann or Dirichlet b.c. along $[-1, 1] \times \{c\}$ and the Neumann b.c. along $\{\pm 1\} \times [c, \infty)$. An easy calculation shows

$$dx^2 + dy^2 = \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1} \left(d\varepsilon^2 f^2(x) + d\eta^2 \left(\frac{f'(x)}{f(x)}\right)^2 \left(\frac{f(\eta)}{f'(\eta)}\right)^2\right). \quad (2.26)$$

In the notation of Section 2.3

$$\alpha(\varepsilon, \eta) = \left(\frac{f'(x)}{f(x)}\right) \left(\frac{f(\eta)}{f'(\eta)}\right) \cdot \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1/2}$$

$$\beta(\varepsilon, \eta) = f(x) \cdot \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1/2},$$

and it is a straightforward (but rather long) exercise in differentiation to show that

$$\begin{aligned}
|\alpha(\varepsilon, \eta) - 1| &= O(F(\eta)), & |f(\eta)/\beta(\varepsilon, \eta) - 1| &= O(F(\eta)), \\
|\nabla\alpha(\varepsilon, \eta)| &\leq \left| \frac{1}{f(\eta)} \frac{\partial\alpha(\varepsilon, \eta)}{\partial\varepsilon} \right| + \left| \frac{\partial\alpha(\varepsilon, \eta)}{\partial\eta} \right| = O(F(\eta)), & (2.27) \\
|\nabla(f(\eta)/\beta(\varepsilon, \eta))| &\leq \left| \frac{1}{f(\eta)} \frac{\partial(f/\beta)}{\partial\varepsilon} \right| + \left| \frac{\partial(f/\beta)}{\partial\eta} \right| = O(F(\eta)),
\end{aligned}$$

where

$$F(\eta) = |f(\eta)| + |f'(\eta)| + |f''(\eta)| + |f'(\eta)^2/f(\eta)|.$$

In obtaining (2.27) we have used the fact that the third derivative of f is bounded (recall (1.3)). Theorem 1.1. is now an immediate consequence of Lemmas 2.3, 2.4.

3. DIRICHLET LAPLACIANS ON REGIONS WITH CUSPS

3.1. Some Generalities

There have been quite a few results [1, 3, 4, 8, 11, 17, 18, 20] on the asymptotics of the eigenvalue distribution of H_D in regions Ω given by (1.3) when $f(x) \rightarrow 0$ and $\text{Vol}(\Omega) = \infty$. Here we give a new treatment which, besides being elementary, seems to cover most of the interesting examples. We refer to the papers of Rosenbljum [17] and Davies [5] for a detailed discussion of the spectral properties of H_D in limit-cylindrical domains.

We suppose that f is convex and that

$$f(x) + f''(x) + f'(x)^2/f(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty. \quad (3.1)$$

If $\text{Vol}(\Omega) = \infty$, $\lim_{E \rightarrow \infty} N_E(H_D)/E = 0$ and we restrict ourselves to studying the operators H_D , $H_{D,N}$ on Ω_1 given by (2.25), with respectively Dirichlet or Neumann boundary condition on $\eta(x, y) = c$. Performing the same change of variable as that in the previous section, we obtain the Laplace–Beltrami operators on $M = (-1, 1) \times (c, +\infty)$ with the metric (2.26) and with the Dirichlet boundary conditions on $\{\pm 1\} \times [c, \infty)$ and the Dirichlet or Neumann b.c. on $[-1, 1] \times \{c\}$. Let us first analyze $H_{D,N}$. If $\hat{H}_{D,N}$ is the Laplace–Beltrami operator on M with metric (1.9) with the same boundary condition as $H_{D,N}$ we obtain as in Sections 2.3, 2.4 that for any $\varepsilon > 0$ we can find c big enough that

$$N_E((1 - \varepsilon) \hat{H}_{N,D}) \geq N_E(H_{N,D}) \geq N_E((1 + \varepsilon) \hat{H}_{N,D}). \quad (3.2)$$

Separating the variables, we obtain that $\hat{H}_{D,N}$ is unitarily equivalent to $\bigoplus_{n \geq 1} H_n$, given by (2.7), acting on $\bigoplus_{n \geq 1} L^2[c, \infty)$, and with the bound-

ary conditions (2.8) at $x = c$. Formula (3.1) implies that $V(x)f(x)^2 \rightarrow 0$, and (eventually increasing ε in (3.2)) we can restrict ourselves to studying

$$A = \bigoplus_{n \geq 1} -\frac{d^2}{dx^2} + \left(\frac{n\pi}{2f(x)}\right)^2. \quad (3.3)$$

Starting with H_D on Ω_1 , we end up with operator (3.3) with the Dirichlet boundary condition at c , which we denote by A_D . Formula (3.2) implies that $\lim_{E \rightarrow \infty} N_E(A)/E = \infty$, and, as in Section 2.2, we observe that the asymptotics of $N_E(A)$ does not depend on the boundary condition at c , nor on c itself. Consequently, we can restrict ourselves to studying A_D with $c = 1$. The strategy is now clear: If we show that

$$\lim_{\varepsilon \rightarrow 0} \lim_{E \rightarrow \infty} \frac{N_E((1 \pm \varepsilon)A_D)}{N_E(A_D)} = 1, \quad (3.4)$$

we have $N_E(H_D) \sim N_E(A_D)$, and the asymptotics of the original problem follows.

To demonstrate the effectiveness of the above strategy, we prove Theorem 1.2.

3.2. Proof of Theorem 1.2

We can obviously restrict ourselves to studying only the horn $\Omega_1 = \{x : x > 1, |y| < x^{-\alpha}\}$, and multiplying the result by 2 if $0 < \alpha < 1$, or with 4 if $\alpha = 1$. Formula (3.1) is obviously valid. The operators A_D become

$$\bigoplus_{n \geq 1} -\frac{d^2}{dx^2} + \left(\frac{n\pi}{2}\right)^2 x^{2\alpha}$$

acting on $L^2[1, \infty)$. Suppose that we prove

$$\lim_{t \rightarrow 0} t^{1/2+1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) = \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + 1\right), \quad (3.5)$$

if $0 < \alpha < 1$

and

$$\lim_{t \rightarrow 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) = \frac{1}{4\pi}, \quad \text{if } \alpha = 1. \quad (3.6)$$

Then, by the Karamata-Tauberian theorem [18, 19]

$$N_E(A_D) \sim \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta(1/\alpha) \frac{\Gamma(1/(2\alpha) + 1)}{\Gamma(\frac{3}{2} + 1/(2\alpha))} E^{1/2+1/(2\alpha)}, \quad \text{if } 0 < \alpha < 1,$$

$$N_E(A_D) \sim \frac{1}{4\pi} E \ln E \quad \text{if } \alpha = 1,$$

formula (3.4) is immediate and the theorem follows. It remains to prove (3.5), (3.6). It should not come as a surprise that the argument closely follows that of Section 2.2.

Case $0 < \alpha < 1$. The Gordon–Thompson inequality yields

$$\begin{aligned} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) &\leq t^{1/2 + 1/2\alpha} \sum_{n>0} \frac{1}{2\sqrt{\pi t}} \int_1^\infty \exp(-t(n\pi/2)^2 x^{2\alpha}) dx \\ &\leq \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \int_0^\infty \exp(-x^{2\alpha}) dx \\ &\leq \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + 1\right), \end{aligned}$$

and it is immediate that

$$\limsup_{t \rightarrow 0} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) \leq \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + 1\right).$$

It remains to prove

$$\liminf_{t \rightarrow 0} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) \geq \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + 1\right). \quad (3.7)$$

Make a partition of $[1, \infty)$ into intervals I_k of equal size $1/m$. Denote by H_k^D the Dirichlet Laplacian on I_k , and by

$$d_k = \sup_{x \in I_k} x^{2\alpha}, \quad Q_m(t) = mt^{1/2} \operatorname{Tr}(\exp(-tH_k^D)), \quad V_m(x) = \sum_{k>0} d_k \cdot \chi_k(x), \quad (3.8)$$

where χ_k is the characteristic function of the interval I_k . Putting additional Dirichlet boundary conditions at the end points of intervals I_k we obtain

$$\begin{aligned} &t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) \\ &\geq t^{1/2 + 1/2\alpha} \sum_{k>0} \operatorname{Tr}(\exp(-tH_k^D)) \sum_{n>0} \exp(-t(n\pi/2)^2 d_k) \\ &\geq Q_m(t) t^{1/2\alpha} \sum_{n>0} \int_1^\infty \exp(-t(n\pi/2)^2 V_m(x)) dx \\ &= Q_m(t) \left(\frac{2}{\pi}\right)^{1/\alpha} \sum_{n>0} \left(\frac{1}{n}\right)^{1/\alpha} \int_{t^{1/2\alpha}(n\pi/2)^{1/\alpha}}^\infty \exp(-V_m(x)) dx \\ &\geq Q_m(t) \left(\frac{2}{\pi}\right)^{1/\alpha} \sum_1^N \left(\frac{1}{n}\right)^{1/\alpha} \int_{t^{1/2\alpha}(N\pi/2)^{1/\alpha}}^\infty \exp(-V_m(x)) dx. \end{aligned}$$

Using that

$$\lim_{t \rightarrow 0} Q_m(t) = \frac{1}{2\sqrt{\pi}}$$

we obtain

$$\begin{aligned} & \liminf_{t \rightarrow 0} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) \\ & \geq \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \sum_1^N \left(\frac{1}{n}\right)^{1/\alpha} \int_0^\infty \exp(-V_m(x)) dx. \end{aligned}$$

Letting $N \rightarrow \infty$ and $m \rightarrow \infty$ we obtain (3.7) and (3.5).

Case $\alpha = 1$. As before

$$\begin{aligned} \operatorname{Tr}(\exp(-tA_D)) & \leq \frac{1}{2\sqrt{\pi t}} \sum_{n>0} \int_t^\infty \exp(-t(n\pi/2)^2 x^2) dx \\ & = \frac{1}{t\pi\sqrt{\pi}} \sum_{n>0} \frac{1}{n} \int_{\sqrt{tn\pi/2}}^\infty \exp(-x^2) dx. \end{aligned} \quad (3.9)$$

Split the positive integers into two sets, $I_1 = \{n : n\sqrt{t} \leq 2/\pi\}$ and $I_2 = \{n : n\sqrt{t} \geq 2/\pi\}$. We have

$$\begin{aligned} \sum_{n \in I_2} \frac{1}{n} \int_{\sqrt{tn\pi/2}}^\infty \exp(-x^2) dx & \leq \frac{\pi\sqrt{t}}{2} \sum_{n \in I_2} \exp(-\sqrt{tn\pi/2}) \\ & = O(1) \quad \text{as } t \rightarrow 0, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \sum_{n \in I_1} \frac{1}{\pi\sqrt{\pi}} \int_{\sqrt{tn\pi/2}}^\infty \exp(-x^2) dx & \leq \frac{1}{2\pi} \sum_{n \in I_1} \frac{1}{n} \\ & \sim \frac{1}{4\pi} \ln t^{-1} \quad \text{as } t \rightarrow 0. \end{aligned} \quad (3.11)$$

In (3.11) we used that $1 + \frac{1}{2} + \dots + 1/n - \ln n \rightarrow \gamma$, as $n \rightarrow \infty$, where γ is the Euler constant. From (3.9), (3.10), (3.11) we obtain

$$\limsup_{t \rightarrow 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \leq \frac{1}{4\pi}.$$

To prove that

$$\liminf_{t \rightarrow 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \geq \frac{1}{4\pi} \quad (3.12)$$

we proceed as follows. Let $I = \{n : n \sqrt{t} < 2\varepsilon/\pi\}$, and with notation (3.8) we have

$$\begin{aligned} & t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \\ & \geq Q_m(t) t^{1/2} (\ln t^{-1})^{-1} \sum_{n>0} \int_1^\infty \exp(-t(n\pi/2)^2 V_m(x)) dx \\ & = Q_m(t) \frac{2}{\pi} \sum_{n>0} \frac{1}{n} (\ln t^{-1})^{-1} \int_{\sqrt{tn}\pi/2}^\infty \exp(-V_m(x)) dx \\ & \geq Q_m(t) \frac{2}{\pi} (\ln t^{-1})^{-1} \sum_{n \in I} \frac{1}{n} \int_\varepsilon^\infty \exp(-V_m(x)) dx. \end{aligned}$$

As $t \rightarrow 0$,

$$Q_m(t) \rightarrow \frac{1}{2\sqrt{\pi}}, \quad (\ln t^{-1})^{-1} \sum_{n \in I} \frac{1}{n} \rightarrow \frac{1}{2},$$

and consequently,

$$\liminf_{t \rightarrow 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \geq \frac{1}{2\pi\sqrt{\pi}} \int_\varepsilon^\infty \exp(-V_m(x)) dx.$$

Letting $\varepsilon \rightarrow 0$ and $m \rightarrow \infty$ we obtain (3.12) and (3.6).

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