Interval dimension is a comparability invariant

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Abstract

We allow orders (ordered sets) to be infinite. An interval order is an order that does not contain $2 + 2$ as an induced suborder. The interval dimension of an order is the minimum number of interval orders (on the same set) whose intersection is the given order. We show that orders with the same comparability graph have the same interval dimension, answering a question raised by Dagan, Golumbic and Pinter for finite orders. We also obtain the analogous result for some other notions of dimension.

1. Introduction

Recall that a family of order relations on the same set is said to realize the intersection of these relations. New notions of dimension for orders can be obtained by relaxing the usual condition that a realizer consists of linear orders. The first such example is the interval dimension of Trotter and Bogart [16–17] and Trotter and Moore [18]. Let $\mathcal{S}$ denote the class of orders from which realizers are to be chosen; this class should satisfy the following two ‘standard’ conditions: it contains all chains (linear orders) and it is hereditary (closed under isomorphism).

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ism and taking induced suborders). Mimicking the definition of dimension in Dushnik and Miller [5], we define the \( I \)-dimension of an order to be the minimum size of a realizer that is contained in \( I \). The two standard conditions ensure that the \( I \)-dimension is bounded above by the ordinary dimension, isomorphic orders have the same dimension, and \( I \)-dimension is an isotone function (with respect to inclusion). For further background on (ordinary) dimension, see Kelly and Trotter [14]. See Fishburn [6] for more background on interval orders and Habib [9] for more information on comparability invariants of finite orders.

Unless the contrary is stated explicitly, we allow an order to be infinite. For each class of finite orders that can be characterized by finitely many forbidden subposets (up to isomorphism), we shall use the same name for the class of arbitrary orders that exclude the same list. In particular, by an interval order, we mean an order that does not contain \( 2 + 2 \). We first study interval dimension (where \( I \) consists of all interval orders) and prove its comparability invariance. Later, we apply similar ideas to obtain the analogous result for other classes of orders.

For the purpose of this paper, it is reasonable to impose a third standard condition: membership in \( I \) is a comparability invariant. Given a class \( I \) of orders that satisfies the three standard conditions, consider the question: Is \( I \)-dimension a comparability invariant? For ordinary dimension, this question was answered affirmatively by Arditti and Jung [1]. The finite case was shown previously by Gysin [8] using ideas of Gallai [7] and Hiraguchi [11].

We shall prove the comparability invariance of interval dimension, answering Question 1 of Dagan, Golumbic and Pinter [4]. In our proof, we shall use an operation, called pushing up, that transforms orders. In Section 2, we define pushing up only for finite interval orders and prove that interval dimension is a comparability invariant for finite orders. The proof for arbitrary orders is given later in Theorem 1 of Section 3. In some respects, we follow the comparability invariance proof given by Kelly [12] for ordinary dimension. For example, we use patchwork orders (which are defined in Section 3) to construct our realizers. We shall also apply the graph decomposition theory from [12]. Although our separate proof for the finite case is logically unnecessary, it motivates the general argument. In the finite case, pushing up is simpler and induction allows a shorter, less complicated proof.

After proving Theorem 1, we use the methods of its proof to prove the comparability invariance of \( I \)-dimension for many other classes \( I \) of orders. We also discover some related comparability invariants.

We say that a class \( I \) of orders has finite character if it is closed under isomorphism and membership of any order \( P \) in \( I \) is equivalent to every finite subposet of \( P \) being in \( I \). Observe that any class having finite character is hereditary. We use classes of finite character to extend definitions from classes of finite orders to classes of arbitrary ones. A natural extension of a property
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originally defined for finite orders is any property that satisfies the following condition: any class of orders having finite character has the new property iff the subclass of its finite orders has the original property.

A class $\mathcal{I}$ of orders is closed under substitution if, whenever an order in $\mathcal{I}$ is substituted for a point in another order in $\mathcal{I}$, the result is also in $\mathcal{I}$. This property is satisfied by chains but fails for interval orders (even for finite ones). Section 4 deals with a natural extension of this property, being superclosed under substitution.

We present more comparability invariants in Section 5. Let $\mathcal{I}$ be an hereditary class of orders that contains all chains and is closed under duality. In Theorem 2, we show that $\mathcal{I}$-dimension is a comparability invariant whenever $\mathcal{I}$ is superclosed under substitution. For example, series-parallel dimension is a comparability invariant. Theorem 3 gives conditions on $\mathcal{I}$ that ensure that $\mathcal{I}$-dimension is a comparability invariant. Since the class of interval orders satisfies these conditions, Theorem 3 is a generalization of Theorem 1. Other classes that satisfy all the hypotheses of Theorem 3 are weak orders and, for any integer $n$, interval orders whose dimension does not exceed $n$.

It remains open whether semiorder dimension is a comparability invariant. (Semiorders exclude $2 + 2$ and $3 + 1$ as subposets.) All classes $\mathcal{I}$ for which we prove that $\mathcal{I}$-dimension is a comparability invariant are superclosed under substitution or are subclasses of interval orders. It would be interesting to find examples that fail both of these conditions.

Using the classes of orders for which we prove, in Theorems 2 and 3, that the corresponding dimension concept is a comparability invariant, we create many new comparability invariants in Theorem 4. For a fixed index set and a fixed assignment of one of these classes to each index, the corresponding invariant is having a realizer with this index set such that each member of the realizer belong to the class that was assigned to its index. (In the proof, the realizer is constructed by combining the different constructions used in proving Theorems 2 and 3.) For example, it is a comparability invariant for an order to have a realizer consisting of one series-parallel order and one interval order.

Although there are many properties of orders that are comparability invariants only in the finite case (see, for example, Kelly [13]), all of our positive results hold for arbitrary orders.

We conclude this introduction with a discussion of finite orders that have interval dimension at most two. Dagan, Golumbic and Pinter [4] used such orders for modelling the routing of nets across a channel in integrated circuits. They called the incomparability graphs of such orders trapezoid graphs. The number of layers needed to route the nets equals the chromatic number of the corresponding incomparability graph. They gave an $O(n + k)$ algorithm to compute the chromatic number of a trapezoid graph, where $n$ is the number of vertices and $k$ is the chromatic number. Cogis [2—3] has shown that the computation of interval dimension and ordinary dimension are polynomially equivalent. (The interval
dimension of an order equals the Ferrers dimension—the terminology used in [2–3]—of the corresponding strict order relation.) Therefore, the computational complexity of recognizing finite orders with interval dimension at most two is polynomial. In fact, Cogis gave such a recognition algorithm that has complexity $O(n^4)$ in [2]; a very different—and more efficient—recognition algorithm is given by Habib and Möhring [10]. Thus, by our comparability invariance result, the computational complexity of recognizing a trapezoid graph is also polynomial, answering another question of [4].

2. Interval dimension of finite orders

In this section, all orders are finite. The intervals that represent finite interval orders are understood to be closed; moreover, no point is the endpoint of more than one interval in the representation. (Zero-length intervals are allowed.)

For an element $u$ of a finite interval order $E$ (with a given representation by intervals), we obtain a new interval order, denoted by $E \uparrow u$, by replacing the interval representing $u$ by a single point that is infinitesimally below the least lower endpoint of elements above $u$. We say that $E \uparrow u$ is obtained from $E$ by pushing up $u$. (If $u$ was originally maximal, then it becomes the maximum element of $E \uparrow u$.) Observe that $E \uparrow u$ is an extension of $E$ and that the new comparabilities in $E \uparrow u$ are exactly those of the form $x < u$, where $x$ satisfies the implication $u < t \Rightarrow x < t$ in $E$. The important property of $E \uparrow u$ is that it remains an interval order whenever $u$ is substituted by any interval order. (First, slightly expand the zero-length interval that represents $u$ to provide room for the substitution.) Observe that $E \uparrow u$ can be defined independently of the interval representation of $E$ by saying that it is the extension of $E$ formed by putting $u$ above as many new elements as possible subject to the condition that the only new valid inequalities are of the form $x < u$.

A non-empty subset $S$ of an order is order autonomous if, for any $y \notin S$, whenever $x < y$, $x > y$ or $x \parallel y$ holds for some $x \in S$, then the same relation holds for every $x \in S$. If $(a, b)$ is an order autonomous subset of an order $Q$, where $a < b$, and the family $(E_i \mid i \in I)$ of interval orders realizes $Q$, then $(E_i \uparrow a \mid i \in I)$ also realizes $Q$. We give a proof here although it is a special case of Lemma 2 in Section 3. Let $x \parallel a$ in $Q$; thus, $x \parallel b$ in $Q$. Since the inequality $a < x$ fails in $E_i$ for some $i$, it also fails in $E_i \uparrow a$. In addition, $x < a$ fails in some $E_i \uparrow a$ because, otherwise, $x < b$ would hold in each $E_i \uparrow a$ and thus in each $E_i$. We have shown that $(E_i \uparrow a \mid i \in I)$ realizes $Q$.

Let $G$ be the comparability graph of a finite order $P$. We shall prove that the interval dimension of $P$ depends only on $G$. We can assume that $G$ is nontrivial (has more than one vertex). We shall write $G[S]$ for the subgraph of $G$ induced on a subset $S$ of vertices. Let $G = G^*(H_1, H_2, \ldots, H_m)$ be the Gallai decomposition of $G$; i.e., $H_1, H_2, \ldots, H_m$ are the successor nodes of $G$ (quasimaximal
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strongly autonomous subsets of $G$) in the decomposition tree of $G$ and $G^*$ is the quotient of $G$. (We are using the terminology and notation of Kelly [12] which follows that of Gallai [7]. We shall give further details on the decomposition theory in Section 3.) Let $G^* = G^*(H_1', H_2', \ldots, H_n')$, where $H_i'$ is $K_2$ (the complete graph on two vertices) if $H_i$ contains an edge, and a single vertex otherwise. We call $G^*$ the augmented quotient of the graph $G$. Since $G^*$ is prime, a complete graph or an empty graph, any two orderings of $G^*$ are isomorphic or dually isomorphic. Consequently, all the orderings of $G^*$ have the same interval dimension, say $d$. (We have just used the trivial observation that the dual of an interval order is also an interval order.) We claim that the interval dimension of $P$ equals the maximum of $d$ and the interval dimensions of the orderings induced on each $H_i$ by $P$. Clearly, the latter number is a lower bound for the interval dimension of $P$. We shall prove the other direction by constructing a suitable realizer for $P$.

Let $Q$ be the order induced on $G^*$ by $P$, and let $Q^*$ denote a corresponding ordering of $G^*$. We write $H_i'$ as $\{a_i, b_i\}$, where $a_i \leq b_i$ in $Q^*$. We also let $Q_i$ denote the order induced on $H_i$ by $P$. Let $(E_j \mid 1 \leq j \leq n)$ be a realizer for $Q^*$ consisting of interval orders, where $n$ is the maximum of $d$ and the interval dimensions of the $Q_i$'s. For each $i$, we fix a realizer $(F_j \mid 1 \leq j \leq n)$ for $Q_i$ consisting of interval orders, subject to the condition that, whenever $Q_i$ is an antichain, then so is each $F_j$. By stages, we shall transform $(E_j \mid 1 \leq j \leq n)$ into a realizer for $P$ consisting of interval orders. At each stage, $R$ denotes the order being realized; $R$ starts as $Q^*$ and finishes as $P$. Each stage will construct a realizer—still denoted by $(E_j \mid 1 \leq j \leq n)$—for the new $R$. For $i$ from 1 to $m$, we repeat the following steps:

Step 1. If $|H_i'| = 1$, then go to step 4.
Step 2. Delete $b_i$ in $R$.
Step 3. Push up $a_i$ and then delete $b_i$ in $E_j$ (1 \leq j \leq n).
Step 4. Substitute $Q_i$ for $a_i$ in $R$.
Step 5. Substitute $F_j$ for $a_i$ in $E_j$ (1 \leq j \leq n).

We have shown that we still have a realizer for $R$ consisting of interval orders after Step 3. If $|H_i'| = 1$, then we are replacing a single element by an antichain in Step 5. Otherwise, we are substituting an interval order for a pushed-up element (which is represented by a zero-length interval). Thus, in both cases, Step 5 produces a realizer for the new $R$ consisting of interval orders. Since $R$ finishes as $P$, we have proved our claim.

Using our claim, it follows easily by induction that interval dimension of $P$ depends only on $G$. In fact, it follows that the interval dimension of any ordering of $G$ equals the maximum of the interval dimension of any ordering of $(G[A])^*$ as $A$ runs over all the nontrivial strongly autonomous subsets of $G$. (Theorem 1 is the corresponding result for arbitrary orders.) This formula shows that the calculation of interval dimension for orders with bounded decomposition diameter has polynomial complexity. (The decomposition diameter of an order is
the maximum size of the quotient of a node in the decomposition tree of its comparability graph.) Notice that this expression for interval dimension in terms of decomposition (or substitution) is more complicated than for ordinary dimension. The analogous formula of Habib [9] for the jump number involves a similar substitution of graphs with two vertices for certain nodes. However, his two vertices are independent while ours are joined.

3. Pushing up and interval dimension for arbitrary orders

Henceforth, orders may be infinite. First, we generalize pushing up and define it for arbitrary orders. The new definition involves a linear extension of the induced order on the set to be pushed up. For a finite order, the effect of pushing up the set $U$ with respect to its linear extension $L$ is exactly the same as successively pushing up the individual elements of $U$, where the elements of $U$ are selected in the order opposite to $L$. The single element $u$ is pushed up in Fig. 1.

**Definition.** For a subset $U$ of an order $P = (X; <)$ and a linear extension $L$ of the subposet $U$, let $<_Q$ be the binary relation on $X$ defined as follows: For $x, y \in X$, $x <_Q y$ holds iff one of the following two conditions is satisfied (where $\supset$ denotes proper containment):

(i) $\{t \in X - U \mid x \leq t\} \supset \{t \in X - U \mid y \leq t\}$.

(ii) Both $x$ and $y$ are in $U$, $\{t \in X - U \mid x \leq t\} = \{t \in X - U \mid y \leq t\}$, and $x <_L y$.

In Lemma 1, it will be shown that $<_Q$ is a strict order relation on $X$. Observe that $x <_Q y$ implies $x < y$ whenever $y \notin U$. In other words, the upper element of each new valid inequality must be in $U$. We shall denote $(X; <_Q)$ by $P \uparrow_L U$ (often with the subscript $L$ omitted) and say that it is obtained from $P$ by **pushing up $U$**.

![Fig. 1. Before and after pushing up an element.](image-url)
Lemma 1. For a subset $U$ of an order $P$ and a linear extension $L$ of the subposet $U$, $P \uparrow_L U$ is an order that extends $P$.

Proof. Let $P = (X; <)$ and $P \uparrow_L U = (X; <_Q)$. Clearly, the relation $<_Q$ is irreflexive. Moreover, it is obvious that the relation $<_Q$ contains $<$. Finally, we must show that $<_Q$ is transitive. Assume that $x <_Q y$ and $y <_Q z$. Let $t \notin U$ satisfy $z \leq t$. Since $x \leq t$ obviously holds, we can assume that condition (ii) in the definition of pushing up applies to both $x <_Q y$ and $y <_Q z$. The transitivity of $<_L$ now implies that $x <_Q z$, completing the proof. □

Lemma 2. Let $A$ and $A' = \{a' \mid a \in A\}$ be disjoint subsets of a set $X$. For each $i \in I$, let the order $F_i$ be an extension of the order $E_i$ in which all new valid inequalities are of the form $x < a$ with $a \in A$. If $P$ is an order on $X$ such that $a < a'$ is an order autonomous chain for each $a \in A$ and the family $(E_i \mid i \in I)$ realizes $P$, then $(F_i \mid i \in I)$ also realizes $P$.

Proof. To show that the intersection (as relations) of $(F_i \mid i \in I)$ equals $P$, we only need to consider incomparabilities involving at least one element of $A$. For each $i \in I$ and $a \in A$, the inequality $a < a'$ holds in $F_i$ because it holds in $E_i$. For $x \notin A$, we define $x' = x$. Let $x \parallel a$ in $P$ with $a \in A$. Since the inequality $a < x'$ fails in $E_i$ for some $i$, it follows that $a < x$ fails in $F_i$. In addition, $x < a'$ fails in some $F_i$ because, otherwise, $x < a'$ would hold in each $F_i$ and thus in each $E_i$, contradicting that $x \parallel a'$ in $P$. We have shown that $(F_i \mid i \in I)$ realizes $P$. □

By Lemma 1, we can take $F_i$ to be $E_i \uparrow A$ in Lemma 2. The more general formulation of Lemma 2 will be applied in Section 5. Let us define an element $u$ of an order $P$ to be immovable if, for any element $x$ of $P$ that is incomparable with $u$, there is an element $y$ such that $u < y$ and $x \parallel y$. This name is justified by the observation that, for any finite set $U$ of immovable elements in an order $P$, $P \uparrow U$ equals $P$. (The previous statement can fail when $U$ is infinite.) A fact about pushing up that will be useful for us is that any element of $U$ is immovable in $P \uparrow U$, even when $U$ is infinite. (We shall not apply the related statement, that for any subset $U$ of an order $P$, $(P \uparrow U) \uparrow U$ equals $P \uparrow U$.) Observe that the previous two lemmas were for arbitrary orders but that the next three lemmas apply only to interval orders. Note that the initial linear extension on the set $U$ to be pushed up is used only as a ‘tie-breaker’ and may not agree with the final order on $U$ (even for the interval order $2 + 1$).

Lemma 3. If $Q - P \uparrow U$ for a subset $U$ of an interval order $P$, then $Q$ is also an interval order.

Proof. Suppose that $x <_Q y$ and $z <_Q t$ form $2 + 2$ in $Q$. Since $z \leq Q y$ in $Q$, there is $y' \notin U$ such that $y \leq y'$ and $z \leq y'$ hold in $P$. Therefore, by symmetry, there are $y', t' \notin U$ such that $Q$ induces $2 + 2$ on $S = \{x, y', z, t'\}$. Since both maximal
elements of $S$ are outside $U$, it follows (by Lemma 1 and our observation regarding new valid inequalities) that $P$ also induces $2 + 2$ on $S$, completing the proof. □

**Lemma 4.** The subposet of all immovable elements of an interval order is a chain. Moreover, whenever $2 + 1$ is a subposet of an interval order, the single element of the trivial chain is not immovable.

**Proof.** If $u$ and $v$ are incomparable immovable elements of an order, then there exists a $2 + 2$ in which $u$ and $v$ are the minimal elements, proving the first statement. The second statement follows in a similar manner. □

Lemma 4 is obvious for finite orders because the immovable elements can be represented by zero-length intervals (at different levels). We require the following substitution property that we saw in the finite case.

**Lemma 5.** If $U$ is a subset of an interval order $P$ such that each element of $U$ is immovable and $Q_u$ is an interval order for each $u \in U$, then the order $R$ obtained from $P$ by substituting $Q_u$ for each $u \in U$ is also an interval order.

**Proof.** If $R$ contained $2 + 2$, then by the second part of Lemma 4, there are $u$ and $v$ in $U$ such that the first chain is in $Q_u$ and the second one is in $Q_v$. Since the restriction of $P$ to $U$ is linear by the first part of Lemma 4, we conclude that $u = v$. Thus, $Q_u$ contains $2 + 2$, a contradiction. □

We shall apply the decomposition theory for arbitrary graphs presented in Kelly [12] that generalizes Gallai’s [7] theory for finite graphs. We review some of the details. If $xy$ is an edge of a graph, then we say that $x$ is joined to $y$ and write $x \sim y$. Both the binary relation $\sim$ and its irreflexive negation $\nabla$ are extended to pairs of sets in the usual way. A non-empty subset $A$ of vertices of a graph $G$ is called autonomous if every vertex outside of $A$ is joined either to all vertices of $A$ or to no vertex of $A$; in other words, for each $x \in V(G) \setminus A$, either $x \sim A$ or $x \nabla A$. A non-empty subset of vertices $B$ is strongly autonomous if it is autonomous and one of the following three conditions is satisfied for every autonomous subset $A : A \cap B = \emptyset$, $A \subseteq B$, or $A \supseteq B$. An autonomous subset of the comparability graph of an order does not have to be an order autonomous subset of the order; in fact, such a subset is order autonomous exactly when it is convex. However, a strongly autonomous subset of a comparability graph is order autonomous in each of its orderings. In the appendix to this paper, we correct an error in [12].

For a graph $G$ and $S \subseteq V(G)$, we write $G[S]$ for the subgraph induced on $S$ and $G(S)$ for the strongly autonomous closure of $S$ (the smallest strongly autonomous subset containing $S$). A strongly autonomous subset $A$ of a graph $G$ is
quasimaximal if \( A \) is a proper subset of \( V(G) \) and there is no strongly autonomous subset between \( A \) and \( V(G) \). A nontrivial graph \( G \) (or its set of vertices when \( G \) is understood) has nonlimit type if each vertex is contained in a quasimaximal strongly autonomous subset; equivalently, \( V(G) \) can be written as \( G(x, y) \) for two distinct vertices \( x \) and \( y \). The quotient \( G^\# \) of a graph \( G \) of nonlimit type is the graph whose vertices are the quasimaximal strongly autonomous subsets of \( G \), with two subsets joined in \( G^\# \) whenever they were joined in \( G \). The graph \( G \) can be obtained from its quotient by substituting the corresponding induced subgraph of \( G \) for each vertex. The concepts of nonlimit type and quotient are relativized to each subgraph induced on a nontrivial strongly autonomous subset of a graph. Observe that \( G(S) \) has nonlimit type for any finite non-empty set \( S \) of vertices.

We denote the collection of all strongly autonomous subsets of a graph \( G \) by \( \mathcal{F}(G) \) and call it the decomposition tree of \( G \) (although it is obviously not a real tree when \( G \) is infinite). To preserve the analogy with the finite case, we shall call the elements of \( \mathcal{F}(G) \) nodes. There are three kinds of nodes: trivial (a leaf), nonlimit, and limit. For a nonlimit node \( A \), the elements of \( A^\# \), the vertex set of the quotient of \( G[A] \), are the successors of \( A \). Observe that a decomposition tree of any graph satisfies the following definition, but that this definition does not mention any graph.

**Definition.** A decomposition system on a nonempty set \( X \) is a subset \( \mathcal{D} \) of the power set of \( X \) that satisfies the following four conditions:

(D1) If \( A, B \in \mathcal{D} \), then \( A \subseteq B \) or \( B \subseteq A \) or \( A \cap B = \emptyset \).

(D2) \( \mathcal{D} \) contains \( X \) and all singletons of \( X \), but it does not contain the empty set.

(D3) The intersection of any non-empty subchain of \( \mathcal{D} \) is in \( \mathcal{D} \).

(D4) The union of any non-empty subchain of \( \mathcal{D} \) is in \( \mathcal{D} \).

If \( X \) is finite, then conditions (D3) and (D4) are superfluous. By (D1) and (D2), the members of \( \mathcal{D} \) that contain any fixed nonempty set form a nonempty chain of sets. Thus, by (D3), each nonempty set has a \( \mathcal{D} \)-closure, the smallest member of \( \mathcal{D} \) that contains it. As when \( \mathcal{D} \) is the decomposition tree of a graph, we can define nonlimit sets and their quotients. A nonlimit set is the \( \mathcal{D} \)-closure of a 2-element set and a quotient is the collection of quasimaximal subsets of some nonlimit set. For any nonempty subset \( Y \) of a set \( X \), observe that a decomposition system \( \mathcal{D} \) on \( X \) induces the following decomposition system on \( Y \):

\[
\{ A \cap Y \mid A \in \mathcal{D}, A \cap Y \neq \emptyset \}.
\]

By Theorem 4.7 of Kelly [12], if each quotient of a decomposition system \( \mathcal{D} \) on \( X \) is endowed with an order relation, then there is a unique order \( Q \) on \( X \) that satisfies the following two conditions:

(i) Each set in \( \mathcal{D} \) is order autonomous in \( Q \).

(ii) The original order and the one induced by \( Q \) agree on each quotient of \( \mathcal{D} \).
We shall call $Q$ the \textit{patchwork order} associated with the orders given on all the quotients of $\mathcal{S}$. If $G$ is the comparability graph of an order $P$, the decomposition system is $\mathcal{S}(G)$, and the order on each quotient extends the order induced by $P$, then the patchwork order is clearly an extension of $P$.

We now generalize a construction that we gave for finite graphs in Section 2. The \textit{augmented quotient} $G^*$ of a graph $G$ of nonlimit type is constructed by starting with $G^*$ and substituting $K_2$ for each vertex of $G^*$ for which the corresponding subgraph of $G$ contains an edge. If $G^*$ is not a complete graph, then it is a prime graph or an empty graph, so that any two of its orderings are isomorphic or dually isomorphic. Observe that $G^*$ is the quotient of $G$ unless $G^*$ is a complete graph or no $K_2$ was substituted. The ordering of a graph of nonlimit type is uniquely determined by the order induced on its quotient and the order induced on each of its quasimaximal strongly autonomous subsets. Therefore, either every ordering of $G^*$ is a chain or any two orderings of $G^*$ are isomorphic or dually isomorphic. Consequently, all orderings of $G^*$ have the same interval dimension.

**Theorem 1.** Interval dimension is a comparability invariant. In fact, the interval dimension of any ordering of a nontrivial comparability graph $G$ is the supremum of the interval dimension of any ordering of $(G[A])^*$ as $A$ runs over the nonlimit nodes of $\mathcal{S}(G)$.

**Proof.** Let $P$ be an ordering of the comparability graph $G$. The letter $A$ will always denote a nonlimit node of $\mathcal{S}(G)$. The cardinal given in the statement, which we denote by $m$, is a lower bound for the interval dimension of $P$ because an ordering of $(G[A])^*$ is a subposet of $P$ for each $A$. We shall prove the other direction by constructing a realizer of size $m$ for $P$. We shall choose our notation so that $A^*$ is a subset of $V((G[A])^*)$. For each $H \in A^*$, let $\{H, H'\}$ be the set of one or two vertices that is substituted for $H$ in forming $(G[A])^*$. We define an ordering of $(G[A])^*$ by setting $H < H'$ for each $H$ and requiring that this ordering extend the ordering induced on $(G[A])^*$ by $P$. We fix an index set $I$ of cardinality $m$. Let $(F_{A,i} | i \in I)$, consisting of interval orders, be a realizer for the ordering of $(G[A])^*$, subject to the condition that $F_{A,i}$ is an antichain for all $i$ whenever $G[A]$ does not contain an edge.

For each $A$ and $i \in I$, let $E_{A,i}$ be the order on $A^*$ obtained from $F_{A,i}$ by pushing up the subset $\{H | H \in A^*, H \neq H'\}$ and then deleting the subset $\{H' | H \in A^*, H \neq H'\}$. By Lemma 2 and the subsequent observation, $(E_{A,i} | i \in I)$ realizes the ordering of $(G[A])^*$. For each $i \in I$, let $E_i$ be the patchwork order associated with the family $E_{A,i}$ indexed by $A$. By a previous observation, $E_i$ extends $P$.

We now want to show that $(E_i | i \in I)$ realizes $P$. We follow the proof of Lemma 4.11(ii) of [12], the version of this result in which $E_{A,i}$ is a linear order. Suppose $b \parallel c$ in $P$. Let $A = G(b, c)$ and let $B, C \in A^*$ be such that $b \in B$ and $c \in C$. Clearly, $B \neq C$. If $B < C$ holds in $E_{A,i}$ and $C < B$ holds in $E_{A,j}$, then $b < c$.
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holds in $E_i$ and $c < b$ holds in $E_i$. Otherwise, $B \parallel C$ holds in some $E_{A,i}$, and therefore, $b \parallel c$ holds in $E_i$. Thus, $(E_i \mid i \in I)$ is a realizer.

It only remains to show that each $E_i$ is an interval order. Suppose, to the contrary, that some $E_i$ contains $2 + 2$, which we assume is formed by $x < y$ and $z < t$. Let $A = G(x, y, z, t)$ and let $X, Y, Z, T \in A^x$ be such that $x \in X$, $y \in Y$, $z \in Z$, and $t \in T$. Clearly, the vertices $X, Y, Z$ and $T$ cannot all be equal. A consideration of the order autonomous subsets of $2 + 2$ shows that both $X$ and $Y$ are distinct from both $Z$ and $T$. Since $E_{A,i}$ is an interval order by Lemma 3, $X, Y, Z$ and $T$ cannot all be distinct. By symmetry, we can assume that $X = Y$. Let $B = G(x, y)$. Since $E_{B,i}$ is not an antichain, $G[B]$ contains an edge. Therefore, $G[X]$ also contains an edge and $X$ was pushed up (and is therefore immovable) in $F_{A,i}$. If $Z \neq T$, then we can create $2 + 2$ inside $E_{A,i}$ by substituting a 2-element chain for $X$, contradicting Lemma 5. Therefore, we can assume that $Z = T$. By our previous argument, both $X$ and $Z$ are immovable in $F_{A,i}$. Thus, $X$ and $Z$ are comparable in $E_{A,i}$ by the first part of Lemma 4. Since this implies that $x$ and $z$ are comparable in $E_i$, we have a contradiction that completes the proof. □

4. Superclosed under substitution

In Corollary 2 of Proposition 1, we shall show that the following definition defines a natural extension of being closed under substitution.

**Definition.** An hereditary class of orders is **superclosed under substitution** if it contains all chains (respectively, antichains) whenever it contains a nontrivial one, and it contains an order $P$ whenever it contains the order induced by $P$ on each quotient of $\mathcal{F}(G)$, where $G$ is the comparability graph of $P$.

To appreciate this definition, we consider the infinite order $P$ of Fig. 2 (an example from [12]). Although the comparability graph $G$ of $P$ has nonlimit type, each quotient of $\mathcal{F}(G)$ is a graph on two vertices. Thus, $\mathbb{N}$, the prime order on four elements, is not contained in $P$. Since finite orders that exclude $\mathbb{N}$ are called **series-parallel**, $P$ is series-parallel by our naming convention. Let $\mathcal{H}$ be the smallest hereditary class that contains all chains and antichains, and is closed under substitution. The class $\mathcal{H}$ contains all finite series-parallel orders but does not contain $P$ because each node of the decomposition tree for each order in $\mathcal{H}$ is nonlimit or a leaf. By Theorem 3.6 of [12] a series-parallel order induces a chain or antichain on any quotient of the corresponding decomposition tree. By Corollary 4 of the following proposition, the smallest hereditary class that contains both 2-element orders and is superclosed under substitution equals the class of series-parallel orders. In particular, $P$ is in this class.
We say that an order is minimally outside a class of orders if it is outside but every proper subposet is inside. Clearly, any order that is outside a class of finite character contains an order that is minimally outside (and is therefore finite). Except for the chains and antichains, observe that each order whose inclusion is required by the above definition is a patchwork order for which the decomposition system is the decomposition tree of its comparability graph. The next result allows much more freedom.

**Proposition 1.** If an hereditary class \( \mathcal{H} \) of orders is superclosed under substitution, then it contains any patchwork order for which the order on each quotient of the corresponding decomposition system is in \( \mathcal{H} \).

**Proof.** Let \( \mathcal{H} \) satisfy the stated hypotheses. If \( \mathcal{H} \) is nontrivial and consists only of chains (respectively, antichains), then it is easy to see that each patchwork order as in the statement is also a chain (antichain). Therefore, we can assume that \( \mathcal{H} \) contains all chains and antichains. Assume that the patchwork order \( P \) over the decomposition system \( \mathcal{D} \) is such that the induced order on each quotient of \( \mathcal{D} \) is in \( \mathcal{H} \). Let \( G \) be the comparability graph of \( P \), let \( A \) be a nonlimit node of \( \mathcal{I}(G) \) and let \( Q \) be the order induced on \( A^\# \) by \( P \). It suffices to show that \( Q \) is in \( \mathcal{H} \). By our previous argument, we can assume that \( Q \) is prime. Choose \( x \) and \( y \) so that
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\[ A = G(x, y) \], and let \( D \) be the \( \mathcal{D} \)-closure of \( \{x, y\} \). Since \( D \) is order autonomous in \( P \), it is autonomous in \( G \) and, therefore, either \( D \subseteq A \) or \( A \subseteq D \). If \( D \subseteq A \), then \( G(D) = A \) and \( D \notin \mathcal{F}(G) \), and so, by Theorem 4.6 of [12], \( A \) is a nonlimit node of series or parallel type, a contradiction. Therefore, we can assume that \( A \subseteq D \).

We shall show that each successor \( A' \) of \( A \) in \( \mathcal{F}(G) \) contains a \( \mathcal{D} \)-successor \( D' \) of \( D \); since \( Q \) will then be isomorphic to a subposet of the order on the \( \mathcal{D} \)-quotient of \( D \), the proof will be complete. Since \( A \subseteq D \) and \( D \) is the union of its \( \mathcal{D} \)-successors, there is a \( \mathcal{D} \)-successor \( D' \) of \( D \) such that \( A' \cap D' \) is non-empty. If \( A \subseteq D' \), then \( D' \) would be a proper subset of \( D \) containing both \( x \) and \( y \), a contradiction. Therefore, \( D' \subset A \). Since \( G(D') = A \), \( A \) is a prime node in \( \mathcal{F}(G) \), and \( D' \) is autonomous in \( G \), \( A' \subset D' \) is impossible by the same theorem from [12] that we used above. Therefore, \( D' \subseteq A' \), completing the proof. \( \square \)

**Corollary 1.** If an hereditary class \( \mathcal{H} \) of orders is superclosed under substitution then it is closed under substitution.

**Proof.** Let \( Y \) be the underlying set of an order that is substituted into a second one to yield the order \( P \) on the set \( X \). The order \( P \) is patchwork order over the decomposition system consisting of \( Y, X \), and all singletons of \( X \). If the original orders are in \( \mathcal{H} \), then so is \( P \) by the proposition. \( \square \)

**Corollary 2.** A class of orders having finite character is superclosed under substitution iff its subclass of finite members is closed under substitution. In other words, being superclosed under substitution is a natural extension of being closed under substitution.

**Proof.** Let \( \mathcal{H} \) be a class of orders that has finite character. The forward implication is immediate by Corollary 1. We now assume that \( \mathcal{H} \) is not superclosed under substitution. Let the order \( P \) with comparability graph \( G \) be outside \( \mathcal{H} \) although its induced order on each quotient of \( \mathcal{F}(G) \) is in \( \mathcal{H} \). Let \( Q \) be a subposet of \( P \) that is minimally outside \( \mathcal{H} \). The underlying set \( Y \) of \( Q \) is finite. Let \( \mathcal{D} \) be the substitution system induced on \( Y \) by \( \mathcal{F}(G) \). If \( \mathcal{D} \) consisted only of \( Y \) and its singletons, then each successor of \( A = G(Y) \) would contain at most one element of \( Y \), implying that \( Q \) is a subposet of the induced order on \( \mathcal{A}^m \), a contradiction. Therefore, \( \mathcal{D} \) contains a set in addition to \( Y \) and the singletons. Since every proper subposet of \( Q \) is in \( \mathcal{H} \), \( Q \) can be obtained by substituting one finite member of \( \mathcal{H} \) into another. (For the underlying set of the first order, choose any set in \( \mathcal{D} \) that contains no other nontrivial set in \( \mathcal{D} \).) Thus, the subclass of finite orders in \( \mathcal{H} \) is not closed under substitution. \( \square \)

**Corollary 3.** A class \( \mathcal{H} \) of orders having finite character is superclosed under substitution iff each order that is minimally outside \( \mathcal{H} \) is prime or a two-element order.
**Proof.** Apply Corollary 2 to reduce to substitution in the finite case. We omit the simple argument that is then required. □

**Corollary 4.** The smallest hereditary class \( \mathcal{H} \) that is superclosed under substitution and contains a given non-empty class \( \mathcal{A} \) of orders consists of all the following orders:

(i) all trivial (one-element) orders;
(ii) all chains if \( \mathcal{A} \) contains an order that is not an antichain;
(iii) all antichains if \( \mathcal{A} \) contains an order that is not a chain;
(iv) each nontrivial order \( P \) with comparability graph \( G \) such that the order induced by \( P \) on each quotient of \( \mathcal{I}(G) \) is either prime and isomorphic to a subposet of an order in \( \mathcal{A} \) or is a chain (respectively, antichain) and \( \mathcal{A} \) contains a nontrivial one.

**Proof.** Let \( \mathcal{H}' \) be the class of orders satisfying all of the four stated conditions. Clearly, \( \mathcal{A} \subseteq \mathcal{H}' \subseteq \mathcal{H} \). It only remains to show that \( \mathcal{H}' \) is hereditary (because being superclosed under substitution then follows because of the way we have defined \( \mathcal{H}' \)). For notational simplicity, we assume that \( \mathcal{H} \) is hereditary and contains all the orders in (i), (ii), and (iii). Let \( Q = \langle Y; \ll \rangle \) be a nontrivial subposet of an order \( P \) in \( \mathcal{H}' \) and let \( G \) be the comparability graph of \( P \). Observe that each set in the decomposition system \( \mathcal{D} \) induced on \( Y \) by \( \mathcal{I}(G) \) is order autonomous in \( Q \). Moreover, each nonlimit set in \( \mathcal{D} \) is the restriction of a nonlimit node of \( \mathcal{I}(G) \). Certainly, \( Q \) is a patchwork order over \( \mathcal{D} \). Moreover, because \( \mathcal{A} \) contains the order induced by \( P \) on each quotient of \( \mathcal{I}(G) \), the order on each \( \mathcal{D} \)-quotient is the restriction of an order in \( \mathcal{A} \). By the proof of the proposition, the order induced by \( Q \) on any prime quotient of its comparability graph is a subposet of the order on some \( \mathcal{D} \)-quotient, and is therefore a subposet of an order in \( \mathcal{A} \). Thus, \( Q \) is in \( \mathcal{H}' \), completing the proof. □

5. More comparability invariants

Let \( \mathcal{I} \) be an hereditary class of orders that contains all chains, is closed under duality, and is superclosed under substitution. With only minor modifications, we can use the proof of Theorem 1 to prove the comparability invariance of \( \mathcal{I} \)-dimension for such a class \( \mathcal{I} \). The argument is similar to the one given for ordinary dimension in Theorem 4.12 of [12]. Pushing up is no longer required and we do not need the augmented quotient, but only the quotient. Thus, the expression for the \( \mathcal{I} \)-dimension (given in Theorem 2 below) involves the quotient rather than the augmented quotient. To see that the proposed value of the \( \mathcal{I} \)-dimension depends only on \( G \), recall that \( \mathcal{I} \) contains all chains and is closed under duality. The constructed patchwork orders will be in \( \mathcal{I} \) by Proposition 1 and they will form a realizer by our extension of Lemma 4.11(ii) of [12]. This analysis shows the validity of the next theorem.
Theorem 2. If an hereditary class \( \mathcal{F} \) of orders contains all chains, is closed under duality, and is supereclosed under substitution, then \( \mathcal{F} \)-dimension is a comparability invariant. In fact, for such a class \( \mathcal{F} \), the \( \mathcal{F} \)-dimension of any ordering of a nontrivial comparability graph \( G \) is the supremum of the \( \mathcal{F} \)-dimension of any ordering of \( (G[A])^* \) as \( A \) runs over the nonlimit nodes of \( \mathcal{F}(G) \).

Corollary 1. Let \( \mathcal{G} \) be a family of finite comparability graphs, where each graph is prime or is the empty graph on two vertices. If \( \mathcal{F} \) is the class of orders that exclude both the orderings of each graph in \( \mathcal{G} \) as a subposet, then \( \mathcal{F} \)-dimension is a comparability invariant. Moreover, \( \mathcal{F} \)-dimension is given by the formula of Theorem 2.

Proof. Combine Theorem 2 and Corollary 3 of Proposition 1.

Corollary 2. Ordinary dimension and series-parallel dimension are both comparability invariants.

Proof. This result follows by Corollary 1 because the respective excluded comparability graphs are the empty graph on two vertices and the path on four vertices.

We now generalize Theorem 1.

Theorem 3. Let \( \mathcal{F} \) be an hereditary class of orders that contains all chains, is closed under duality, and is also closed under each of the following three operations:

(i) replacing points by antichains;
(ii) pushing up;
(iii) formation of a patchwork order when the order on each quotient is already in \( \mathcal{F} \) and any nontrivial set in a quotient is immovable in the order on that quotient.

Then, \( \mathcal{F} \)-dimension is a comparability invariant. In fact, for such a class \( \mathcal{F} \), the \( \mathcal{F} \)-dimension of any ordering of a nontrivial comparability graph \( G \) is the supremum of the \( \mathcal{F} \)-dimension of any ordering of \( (G[A])^* \) (the augmented quotient of \( G[A] \)) as \( A \) runs over the nonlimit nodes of \( \mathcal{F}(G) \).

Proof. Although this proof is similar to that of Theorem 1, many details are different. Suppose that the class \( \mathcal{F} \) is as above. We shall prove the final statement for an arbitrary ordering \( P \) of a comparability graph \( G \). The cardinal of that statement, which we denote by \( m \), is a comparability invariant, because \( \mathcal{F} \) is closed under duality and contains all chains. As in the proof of Theorem 1, \( m \) is a lower bound for the \( \mathcal{F} \)-dimension of \( P \).
Let $H$ be the comparability graph that is obtained from $G$ by collapsing (to a point) each strongly autonomous set that does not contain an edge of $G$. Condition (i) allows us to assume that $G = H$. (After applying the result for $H$, replace each collapsed point by a suitable antichain in each realizer.) If $G$ is trivial, then $P$ is an antichain. Henceforth, we assume that $G$ is nontrivial. Observe that every nontrivial set in $\mathcal{T}(G)$ contains an edge of $G$. Let $X = V(G)$ and let $\mathcal{N}$ consist of each nontrivial set that is in $A^*$ for some nonlimit $A$ in $\mathcal{T}(G)$. Let $Y$ be a set that is disjoint from $X$ and equipotent to $\mathcal{N}$, and let $\varphi$ be the corresponding bijection from $\mathcal{N}$ to $Y$.

We define an order $Q$ on $Z = X \cup Y$ by requiring that:

(a) $P$ is a subposet of $Q$.
(b) If $x \in A \in \mathcal{N}$, then $x < \varphi(A)$ in $Q$.
(c) If $A \in \mathcal{N}$ and $x \in X - A$, then $x$ and $\varphi(A)$ satisfy exactly the same comparability in $Q$ as do $x$ and the set $A$ with respect to $P$.
(d) For distinct $A, B \in \mathcal{N}$, $\varphi(A) < \varphi(B)$ holds in $Q$ iff $A \subset B$, or $A \cap B = \emptyset$ and $A < B$ in $P$.

We omit the straightforward verification that $Q$ is an order. We define

$$\hat{A} = A \cup \{\varphi(B) \mid B \in \mathcal{N} \text{ and } B \subset A\}$$

whenever $A \in \mathcal{T}(G)$. Let $\mathcal{D}$ consist of all sets of the form $\hat{A}$ with $A \in \mathcal{T}(G)$, together with all singletons of $Y$. Clearly, $\mathcal{D}$ is a decomposition system on $Z$ and every set in $\mathcal{D}$ is order autonomous in $Q$. It is notationally convenient to extend the definition of $\varphi$ so that, for any singleton $\{x\}$ in $\mathcal{T}(G)$, $\varphi(\{x\}) = x$.

It is not difficult to show that the nonlimit sets of $\mathcal{D}$ are exactly those of the form $\hat{A}$ with $A$ nonlimit in $\mathcal{T}(G)$. In fact, the $\mathcal{D}$-quotient of such a set $\hat{A}$ consists of all $B$ and $\{\varphi(B)\}$ with $B \in A^*$. Observe that the order that $Q$ induces on the $\mathcal{D}$-quotient of $\hat{A}$ is isomorphic, in an obvious way, to an ordering of $(G[A])^*$.

As in the proof of Theorem 1, it suffices to construct a realizer of size $m$ for $P$ that consists of orders in $\mathcal{F}$. In fact, we shall construct such a realizer for $Q$ (which suffices because $\mathcal{F}$ is hereditary). Whenever $A$ is nonlimit in $\mathcal{T}(G)$, let $(F_{A,i} \mid i \in I)$, consisting of orders in $\mathcal{F}$, be a realizer for the ordering of the $\mathcal{D}$-quotient $S$ of $\hat{A}$ that is induced by $Q$. For $i \in I$, let $E_{A,i}$ be the order on $S$ that obtained from $F_{A,i}$ by pushing up all elements of the form $B$ where $B$ is nontrivial. By (ii), $E_{A,i}$ is in $\mathcal{F}$. By Lemma 2, $(E_{A,i} \mid i \in I)$ realizes the ordering of $S$. For each $i \in I$, let $E_i$ be the patchwork order over $\mathcal{D}$ associated to the family $E_{A,i}$ indexed by $\hat{A}$. Since each nontrivial $\hat{B}$ was pushed up in forming $F_{A,i}$, (iii) is satisfied by the patchwork order $E_i$. Therefore, $E_i$ is in $\mathcal{F}$. Since $(E_i \mid i \in I)$ realizes $Q$ by the reasoning used in Theorems 1 and 2, the proof is complete.

**Corollary.** Let $\mathcal{F}$ be a class of orders having finite character whose subclass $\mathcal{F}$ of finite orders contains all finite chains, is closed under duality, and is also closed
Interval dimension is a comparability invariant under each of the following three operations:

(i) replacing a point by a finite antichain;
(ii) pushing up a point;
(iii) replacing any immovable point by an order in \( \mathcal{F} \).

Then, \( \mathcal{F} \)-dimension is a comparability invariant. Moreover, \( \mathcal{F} \)-dimension is given by the formula of Theorem 3.

**Proof.** We claim that each condition of the Theorem 3 is a natural extension of the corresponding condition of the corollary. If there is an order formed according to one of the conditions of Theorem 3 that is outside an hereditary class of finite character, then the same statement holds for a suitably chosen finite subposet. (The arguments, which we omit, are similar to the proof of Corollary 2 of Proposition 1. Sufficiently many elements must be kept in (ii) to prevent new inclusions between certain sets, and also in (iii) to preserve immovability.) Thus, our claim holds for (i) and (iii). For (ii), use our description of pushing up any finite set in terms of pushing up single elements (given at the beginning of Section 3). Let \( \mathcal{I} \) be as in the corollary. Our claim shows that \( \mathcal{I} \) satisfies conditions (i), (ii) and (iii) of Theorem 3. Since \( \mathcal{I} \) has finite character, it contains all chains and is closed under duality. Therefore, \( \mathcal{I} \) satisfies all the hypotheses of Theorem 3, completing the proof. □

It is easily seen that condition (i) in the above corollary means that each order that is minimally outside \( \mathcal{I} \) does not have a nontrivial antichain as an order autonomous subset. Let us derive Theorem 1 from the above corollary without considering any infinite orders. In Section 2, we obtained an interval order when we pushed up a single point in a finite interval order. Moreover, an immovable element in a finite interval order can be represented by a zero-length interval. Therefore, the class of finite interval orders satisfies all the conditions of the corollary. Applying the corollary yields Theorem 1. Recall that a weak order is defined by the exclusion of \( 2 + 1 \) as a subposet (see, for example, [15]). Obviously, weak orders are interval orders and an immovable element in a weak order is comparable with all other elements. We can now apply the above corollary to conclude that weak-order dimension is a comparability invariant.

We have already mentioned that the class of interval orders satisfies the hypotheses of Theorem 3, but fails those of Theorem 2. We can use Fig. 1 to provide an example with the opposite behavior. If \( \mathcal{I} \) is the class that excludes the right-hand order of Fig. 1 and its dual as subposets, then \( \mathcal{I} \) satisfies Theorem 2 because the excluded orders are prime. However, Fig. 1 shows that \( \mathcal{I} \) is not closed under pushing up. As well as giving new comparability invariants, the following theorem implies both Theorem 2 and Theorem 3 (except for the \( \mathcal{F} \)-dimension formulas).

**Theorem 4.** For each \( i \) in a set \( I \), let \( \mathcal{I}_i \) be a class of orders that satisfies all the
hypotheses for $\mathcal{I}$ given in Theorem 2 or in Theorem 3. Then, it is a comparability invariant to require an order to have a realizer of the form $(E_i \mid i \in I)$, where $E_i \in \mathcal{I}_i$ whenever $i \in I$.

**Proof.** Throughout this proof, we shall call a realizer good if it is of the form $(E_i \mid i \in I)$ with $E_i \in \mathcal{I}_i$ whenever $i \in I$. Let $G$ be a comparability graph that has an ordering with a good realizer and let $P$ be an arbitrary ordering of $G$. For each nonlimit $A \in \mathcal{F}(G)$, an ordering of $(G[A])^*$ is a subposet of $P$. Since each $\mathcal{I}_i$ contains all chains and is closed under duality, the assumption on $G$ implies that every ordering of $(G[A])^*$ has a good realizer for each nonlimit node $A$ of $\mathcal{F}(G)$.

At first, we follow the proof of Theorem 3. As in that proof, we can assume that any nontrivial strongly autonomous subsets of $G$ contains an edge. Starting with $P$, we construct the order $Q$ and the decomposition system $\mathcal{D}$ as in that proof. Since each $\mathcal{I}_i$ is hereditary, it suffices to construct a good realizer for $Q$. For each nonlimit $A \in \mathcal{F}(G)$, let $(F_{A,i} \mid i \in I)$ be a good realizer for the ordering of the $\mathcal{D}$-quotient $S$ of $A$ that is induced by $Q$. Let $i \in I$. If $\mathcal{I}_i$ satisfies the hypotheses of Theorem 2, then we set $E_{A,i} = F_{A,i}$. Otherwise, $E_{A,i}$ is the order on $S$ that obtained from $F_{A,i}$ by pushing up all elements of the form $a$ where $B$ is nontrivial. Clearly, $E_{A,i}$ is in $\mathcal{I}_i$ (by (ii) of Theorem 3 if $\mathcal{I}_i$ satisfies its hypotheses). By Lemma 2, $(E_{A,i} \mid i \in I)$ realizes the ordering of $S$. For each $i \in I$, let $F_i$ be the patchwork order over $\mathcal{D}$ associated to the family $E_{A,i}$ indexed by $\hat{A}$. By the argument of Theorem 2 or Theorem 3, it follows that $E_i$ is in $\mathcal{I}_i$ for all $i$. Since $(E_i \mid i \in I)$ realizes $Q$ for the same reason as in Theorems 1, 2 and 3, the proof is complete. □

Our final result provides some more examples for Theorem 3.

**Proposition 2.** Let $n$ be a positive integer. The class of orders whose dimension is at most $n$ has finite character and is closed under pushing up.

**Proof.** It is well known that this class has finite character. Let $P$ be a finite order whose dimension at most $n$, let $a \in P$, let $Q = P \uparrow a$, and let $(L_i \mid 1 \leq i \leq n)$, consisting of linear orders, be a realizer for $P$. For $1 \leq i \leq n$, we define the linear order $M_i$ on the underlying set of $P$ as follows: $M_i - \{a\}$ is a subposet of $L_i$ and the upper cover of $a$ in $M_i$ is the first element $x$ in $L_i - \{a\}$ that satisfies $a < x$ in $P$ ($a$ is the maximum of $M_i$ when there is no such $x$). Since $(M_i \mid 1 \leq i \leq n)$ realizes $Q$, the proof is complete. □

Since the class orders of dimension at most $n$ is superclosed under substitution, it follows by Proposition 2 that this class satisfies all the hypotheses of Theorem 3. Therefore, the class of interval orders of dimension at most $n$ also satisfies all the hypotheses of Theorem 3. Thus, this second class defines a dimension concept that is a comparability invariant, and can also be used to give further comparability invariants in Theorem 4.
Appendix—proof of a lemma

As observed by G. H. Wenzel, the proof given in [12] for the following lemma (Lemma 3.3) is incorrect.

**Lemma.** If $A$ is an independent set of vertices that is a nontrivial autonomous subset of a graph $G$, then there is a strongly autonomous subset $B$ of $G$ containing $A$ such that $B \setminus A \neq A$.

**Proof.** We define $B$ to be the union of all the autonomous subsets $S$ that contain $A$ and satisfy $S \setminus A \neq A$. Clearly, $B$ is autonomous, $A \subseteq B$, and $B \setminus A \neq A$. Also, $B$ contains each autonomous subset that intersects $B$ but does not contain $A$. Let $T$ be an autonomous subset that contains $A$ and suppose that $B \setminus T$ is non-empty. Starting from $A \setminus B \setminus A$, we successively conclude that $A \setminus B \setminus T$, $T \setminus B \setminus T$, $T \setminus B \setminus B$, and $T \setminus B \setminus A$. From the last conclusion and $B \setminus A \neq A$, we infer that $T \setminus A \neq A$. Therefore, $T \subseteq B$, completing the proof. □

**References**