Equivalence theorems of the convergence between Ishikawa and Mann iterations with errors for generalized strongly successively $\Phi$-pseudocontractive mappings without Lipschitzian assumptions

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Abstract


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1. Introduction

Let $E$ be a real Banach space. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$J(x) = \{ f^* \in E^*: \|x\|^2 = \langle x, f^* \rangle = \|f^*\| = \|x\| \},$$

where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known (see [1, 2]) that if $E$ is uniformly smooth, then $J$ is single-valued and is uniformly continuous on any bounded subsets of $E$. We shall denote the single-valued duality map by $j$.

Definition 1.1. A mapping $T: E \to E$ is called

(i) strongly successively pseudo-contractive if for all $x, y \in E$, there exist $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq (1 - k) \|x - y\|^2;$$

(ii) strongly pseudo-contractive if for all $x, y \in E$, there exist $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq (1 - k) \|x - y\|^2.$$

Definition 1.2. A mapping $T$ is called

(i) strongly successively $\phi$-pseudo-contractive if for all $x, y \in E$, there exist $j(x - p) \in J(x - p)$ and a strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|;$$

(ii) strongly $\phi$-pseudo-contractive if for all $x, y \in E$, there exist $j(x - p) \in J(x - p)$ and a strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|.$$

Definition 1.3. A mapping $T$ is called

(i) generalized strongly successively $\Phi$-pseudo-contractive if for all $x, y \in E$, there exist $j(x - p) \in J(x - p)$ and a strictly increasing function $\Phi: [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle T^n x - T^n y, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|);$$

(ii) generalized strongly $\Phi$-pseudo-contractive if for all $x, y \in E$, there exist $j(x - p) \in J(x - p)$ and a strictly increasing function $\Phi: [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \Phi(\|x - y\|).$$

Obviously if we replace $T^n$ by $T$ in (1), we will obtain (2).
We define the modified Mann iteration with errors by
\begin{equation}
    u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T^n u_n + \xi_n,
\end{equation}
and the modified Ishikawa iteration with errors by
\begin{equation}
    y_n = (1 - \beta_n)x_n + \beta_n T^n x_n + \sigma_n,
\end{equation}
\begin{equation}
    x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n + \nu_n,
\end{equation}
where the sequences \( \{\alpha_n\}, \{\beta_n\} \subseteq [0, 1] \) satisfy
\begin{equation}
    \lim_{n \to \infty} \alpha_n = 0, \quad \lim_{n \to \infty} \beta_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,
\end{equation}
and the sequences \( \{\xi_n\}, \{\nu_n\}, \{\sigma_n\} \) satisfy
\begin{equation}
    \|\xi_n\| = o(\alpha_n), \quad \|\nu_n\| = o(\alpha_n), \quad \lim_{n \to \infty} \|\sigma_n\| = 0.
\end{equation}

If \( \xi_n = 0, n \in \mathbb{N}, \nu_n = \sigma_n = 0, n \in \mathbb{N} \), in (3) and (4), then the corresponding iterations (3) and (4) are called the modified Mann and Ishikawa iterations respectively, which have been discussed by Rhoades and Soltuz in [21,22] for the equivalence of convergence between these two iterations.

We need the following lemmas to prove our main results.

**Lemma 1.1.** (See [3, Lemma 2.1, p. 97].) Let \( E \) be a real Banach space and \( J \) be a normality duality mapping. Then for any given \( x, y \in E \), the following inequality holds:
\begin{equation}
    \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \text{for all } j(x + y) \in J(x + y).
\end{equation}

**Lemma 1.2.** Let \( \Phi : [0, \infty) \to [0, \infty) \) be a strictly increasing function with \( \Phi(0) = 0 \) and let \( \{\theta_n\}, \{\sigma_n\}, \{\lambda_n\} \) and \( \{e_n\} \) be nonnegative real sequences such that
\begin{equation}
    \lim_{n \to \infty} \lambda_n = 0, \quad \sigma_n = o(\lambda_n), \quad \sum_{n=1}^{\infty} \lambda_n = \infty, \quad \lim_{n \to \infty} e_n = 0.
\end{equation}
Suppose that there exists an integer \( N^* > 0 \) such that for all \( n \geq N^* \),
\begin{equation}
    \theta_{n+1}^2 \leq \theta_n^2 - 2\lambda_n \Phi(\theta_{n+1} - e_n) + \sigma_n.
\end{equation}
Then \( \lim_{n \to \infty} \theta_n = 0. \)

**Proof.** The proof follows from the following two claims.

**Claim 1.** \( \lim \inf_{n \to \infty} \theta_n = 0. \)

Otherwise, suppose the contrary and assume that
\begin{equation}
    \lim \inf_{n \to \infty} \theta_n = \delta > 0
\end{equation}
for some real constant \( \delta > 0 \). Then, there exists an integer \( N_0 > 0 \) such that for all \( n \geq N_0, \theta_n \geq \delta > 0 \). Since \( \lim_{n \to \infty} e_n = 0 \), then there exists an integer \( N_1 > 0 \) such that for all \( n \geq N_1, 0 \leq e_n \leq \delta/2 \). Since \( \sigma_n = o(\lambda_n) \), then there exists an integer \( N_2 > 0 \) such that for all \( n \geq N_2, \)}
0 \leq \sigma_n \leq \lambda_n \Phi\left(\frac{\delta}{2}\right). \text{ Set } N = \max\{N^*, N_0, N_1, N_2\}. \text{ Then for all } n \geq N, \text{ we have } \sigma_n \leq \lambda_n \Phi\left(\frac{\delta}{2}\right) \text{ and } -\Phi(\theta_{n+1} - e_n) \leq -\Phi\left(\frac{\delta}{2}\right), \text{ and hence }

\theta_{n+1}^2 \leq \theta_n^2 - \lambda_n \Phi(\theta_{n+1} - e_n) + \sigma_n \leq \theta_n^2 - 2\lambda_n \Phi\left(\frac{\delta}{2}\right) + \lambda_n \Phi\left(\frac{\delta}{2}\right) = \theta_n^2 - \lambda_n \Phi\left(\frac{\delta}{2}\right).

Therefore, we have

\Phi\left(\frac{\delta}{2}\right) \cdot \sum_{n=N}^{\infty} \lambda_n \leq \sum_{n=N}^{\infty} \left[\theta_n^2 - \theta_{n+1}^2\right] \leq \theta_N^2,

which contradicts the condition \(\sum_{n=1}^{\infty} \lambda_n = \infty\). Thus, Claim 1 is true and \(\liminf_{n \to \infty} \theta_n = 0\).

(The fact \(\inf_{n=1}^{\infty} \theta_n = 0\) can be proved similarly as well.)

Hence from the well-known Weierstrass–Bolzano theorem, there exists a subsequence \(\{\theta_{n_j}\} \subseteq \{\theta_n\}\) such that \(\lim_{j \to \infty} \theta_{n_j} = 0\).

Let \(\varepsilon > 0\) be arbitrary. Then there exists an integer \(j_0 > 0\) such that for all \(j \geq j_0\), \(0 \leq \theta_{n_j} < \varepsilon\). Since \(\sigma_n = o(\lambda_n)\), then there exists an integer \(N_3 > 0\) such that for all \(n \geq N_3\), \(\sigma_n < \lambda_n \Phi\left(\frac{\delta}{2}\right)\). Since \(\lim_{n \to \infty} e_n = 0\), then there exists an integer \(N_4 > 0\) such that for all \(n \geq N_4\), \(0 \leq e_n < \frac{\varepsilon}{2}\).

Let \(n^*\) denote \(n_{j^*}\), where \(j^*\) is the minimum integer such that \(n_{j^*} \geq N^*, n_{j_0}, N_3, N_4\).

Claim 2. \(\theta_{n^*+k} < \varepsilon\) for all \(k \geq 0\).

The claim is clearly true for \(k = 0\).

Suppose that Claim 2 is true for some \(k \geq 0\), but not for \(k + 1\). That means \(\theta_{n^*+k} < \varepsilon\), but \(\theta_{n^*+k+1} \geq \varepsilon\). From the proof of Claim 1, for all \(n \geq n^*, \sigma_n < \lambda_n \Phi\left(\frac{\delta}{2}\right)\), and \(0 \leq e_n < \frac{\varepsilon}{2}\). Then, \((\theta_{n^*+k+1} - e_{n^*+k}) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}\) and hence \((-\Phi(\theta_{n^*+k+1} - e_{n^*+k})) < (-\Phi\left(\frac{\delta}{2}\right))\). Then from (8), we get

\[
\varepsilon^2 \leq \theta_{n^*+k+1}^2
\leq \theta_{n^*+k}^2 - 2\lambda_{n^*+k} \Phi(\theta_{n^*+k+1} - e_{n^*+k}) + \sigma_{n^*+k}
< \varepsilon_{n^*+k}^2 - 2\lambda_{n^*+k} \Phi\left(\frac{\varepsilon}{2}\right) + \lambda_{n^*+k} \Phi\left(\frac{\varepsilon}{2}\right)
< \varepsilon^2,
\]

which is impossible. This completes the proof. \(\square\)

In 2001, Moore and Nnoli [17] proved the following lemma.

Lemma MN. (See [17, Lemma 2.1, p. 134].) Let \(\Phi : [0, \infty) \to [0, \infty)\) be a strictly increasing function with \(\Phi(0) = 0\) and let \(\theta_n, \{\sigma_n\}, \{\lambda_n\}\) be nonnegative real sequences such that

\[
\lim_{n \to \infty} \lambda_n = 0, \quad \sigma_n = o(\lambda_n), \quad \sum_{n=1}^{\infty} \lambda_n = \infty.
\]

Suppose that for all \(n \geq 1\),

\[
\theta_{n+1}^2 \leq \theta_n^2 - 2\lambda_n \Phi(\theta_{n+1}) + \sigma_n.
\]

Then \(\lim_{n \to \infty} \theta_n = 0\).
Remark 1.1. It should be remarked here that our Lemma 1.2 extends Lemma MN under the more generalized case in (8) with error terms $e_n \geq 0$. Our Lemma 1.2 will reduces to Lemma MN if $e_n = 0$ for all $n \geq 1$ as a special case of our Lemma 1.2. Since $e_n \geq 0$, $	heta_{n+1} - e_n \leq \theta_{n+1}$, then $-\Phi(\theta_{n+1}) \leq -\Phi(\theta_{n+1} - e_n)$. Hence, if $\theta_1$ satisfies the inequality in Lemma MN, then it must satisfy inequality (8) in our Lemma 1.2, but the converse is not true.

After 1990, several researchers around the world proved that the Mann and Ishikawa iterative sequences with errors converge to the fixed points of strongly $\phi$-pseudocontractive mappings under suitable conditions. In 1996, Chidume [4], Ding [6], Osilike [18] began the work on $\phi$-pseudocontractive mappings with strict conditions in $q$-uniformly smooth Banach spaces. In 1998, the author [9] extended [4,6,18] to the Ishikawa iterative sequences with errors for $\phi$-pseudocontractive mappings in uniformly smooth Banach spaces.

In 2001, Chidume and Mutangadura [5] constructed a counter example showing that every nontrivial Mann iteration fails to converge while the Ishikawa iteration converges.

Therefore an open question arises:

Are there any differences between these two kinds of sequences? Can we prove the equivalence of the convergence between these two kinds of sequences?

Rhoades and Soltuz in [21,22] partially answered the question by showing the equivalence of the convergence between the original Ishikawa iterative sequence [12] and the original Mann iterative sequence [16] under some strict conditions for strongly successively pseudocontractive mappings. In 2006, Huang and Bu in [11] continued the study on the equivalence of the convergence between Mann and Ishikawa iterations with errors under milder conditions. We would like to emphasize that the study on equivalency has not been accomplished yet.

Many scientists have been working on the strong convergence of Mann and Ishikawa iterations with errors for various mappings, not only the strongly (successively) pseudo-contractive mappings, but also the more generalized case as the strongly (successively) $\phi$-pseudocontractive mappings, and the most general case up to date the generalized strongly (successively) $\phi$-pseudocontractive mappings as well. For the strong convergence results on these various mappings, the readers may consult [3–10,12,14–20,23].

Among these various mappings, the class of generalized strongly successively $\Phi$-pseudocontractive mappings is the most general up to date. By setting $n = 1$, then the generalized strongly successively $\Phi$-pseudocontractive mappings reduce to the generalized strongly $\Phi$-pseudocontractive mappings. Moreover, the class of generalized strongly (successively) $\Phi$-pseudocontractive mappings includes the class of strongly (successively) $\phi$-pseudocontractive mappings by setting $\Phi(s) = s\phi(s)$ for all $s \in [0, \infty)$, while the class of strongly successively $\phi$-pseudocontractive mappings includes the class of strongly successively pseudo-contractive mappings by setting $\phi(s) = ks$ for all $s \in [0, \infty)$. However, the converse is not true. An example by Hirano and Huang (see [10, Example 1, p. 1462]) showed that a strongly pseudo-contractive operator $T$ is not always a strongly $\phi$-pseudocontractive operator. Another example was provided by Moore and Nnoli (see [17, Example 1.4, p. 133]) to show that if $T$ is generalized strongly $\Phi$-pseudocontractive, then $T$ is strongly $\phi$-pseudocontractive, but the converse is not always true.

Hence it is of interest to know whether the equivalency of the convergence between iterations still holds for the most general class of the generalized strongly (successively) $\Phi$-pseudocontractive mappings. We will prove the equivalence of convergence between the modified sequences with errors defined by Liu [14] and Xu [23] for the most general class of the
generalized strongly (successively) $\Phi$-pseudocontractive mappings. Then as a conclusion of our results, in uniformly smooth Banach space and without Lipschitzian (and even without continuous) assumption and without any geometric restriction on the iteration parameters whatever, for any initial point $u_1, x_1 \in E$ (even not necessarily $u_1 = x_1$), these modified sequences with errors converge equivalently. Consequently, our theorems will include the results presented recently in [11,21,22] as special cases and hence generalize all of the recent results in [1–23].

2. Main results

**Theorem 2.1.** Let $E$ be a real uniformly smooth Banach space and let $T : E \to E$ be a generalized strongly successively $\Phi$-pseudocontractive mapping with bounded range. The sequences $\{u_n\}$ and $\{x_n\}$ are defined by (3) and (4) respectively, with $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$ satisfying (5), and $\{\xi_n\}, \{\sigma_n\}, \{\nu_n\}$ satisfying (6). Then for any initial point $u_1, x_1 \in E$, the following two assertions are equivalent:

(i) modified Mann iteration with errors (3) converges to the unique fixed point $x^* \in F(T)$;
(ii) modified Ishikawa iteration with errors (4) converges to the unique fixed point $x^* \in F(T)$.

**Proof.** The uniqueness of the fixed point $x^*$ comes from the definition of generalized strongly successively $\Phi$-pseudocontractive mapping.

If the modified Ishikawa iteration with errors (4) converges to $x^* \in F(T)$, then setting $\beta_n = 0$, $\sigma_n = 0$, $\forall n \in \mathbb{N}$, in (4), we can get the convergence of modified Mann iteration with errors. Conversely, we shall prove (i) $\Rightarrow$ (ii).

Since the range of $T$ is bounded and $T(TE) \subseteq TE$, then $T^n E$ is bounded. By induction we can conclude that $\{x_n\}, \{y_n\}$ are also bounded. Since $\lim_{n \to \infty} u_n = x^*$, then $\{u_n\}$ is bounded. Set

$$M := \sup_n \left\{ \| T^n y_n - T^n u_n \|, \| x_n - u_n \|, \| x_n - T^n y_n \|, \| u_n - T^n u_n \|, \| x_n - T^n x_n \| \right\}.$$ 

Then $M < \infty$.

Then from the iteration schemes with errors (3), (4), Lemma 1.1 and the definition of generalized strongly successively $\Phi$-pseudocontractive mapping $T$, we have

$$\| x_{n+1} - u_{n+1} \|^2 = \| \left[ (1 - \alpha_n) x_n + \alpha_n T^n y_n + \nu_n - \left( (1 - \alpha_n) u_n + \alpha_n T^n u_n + \xi_n \right) \right] \|^2$$

$$= \| (1 - \alpha_n) (x_n - u_n) + \alpha_n (T^n y_n - T^n u_n) + (\nu_n - \xi_n) \|^2$$

$$\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \langle T^n y_n - T^n u_n, j(x_{n+1} - u_{n+1}) \rangle$$

$$= (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \| T^n y_n - T^n u_n, j(y_n - u_n) \|^2$$

$$+ 2 \alpha_n \| T^n y_n - T^n u_n, j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \|^2 + 2 \| \nu_n - \xi_n, j(x_{n+1} - u_{n+1}) \|^2$$

$$\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \| y_n - u_n \|^2 - \Phi \left( \| y_n - u_n \| \right)$$

$$+ 2 \alpha_n \| y_n - T^n u_n \| \cdot \| j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \|$$

$$+ 2 \| \nu_n - \xi_n \| \cdot \| x_{n+1} - u_{n+1} \|$$

$$\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2 \alpha_n \| y_n - u_n \|^2 - 2 \alpha_n \Phi \left( \| y_n - u_n \| \right).$$
+ 2\alpha_n M \cdot \| j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \| + 2M \cdot (\| v_n \| + \| \xi_n \|). \tag{9}

Set

$$\delta_n = \| j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \|.$$ \tag{10}

Next we will prove that \( \lim_{n \to \infty} \delta_n = 0. \)

We know that if \( E \) is a uniformly smooth Banach space, then \( J \) is a single-valued mapping and uniformly continuous on bounded sets (see [13]). From conditions (5) and (6), we have

$$\| (x_{n+1} - u_{n+1}) - (y_n - u_n) \|
\leq \| x_{n+1} - y_n \| + \| u_{n+1} - u_n \|
= \| (1 - \alpha_n)x_n + \alpha_n T^n y_n + v_n \| - \| (1 - \beta_n)x_n + \beta_n T^n x_n + \sigma_n \|
+ \| (1 - \alpha_n)u_n + \alpha_n T^n u_n + \xi_n - u_n \|
\leq \alpha_n \| x_n - T^n y_n \| + \| v_n \| + \beta_n \| x_n - T^n x_n \| + \| \sigma_n \| + \alpha_n \| u_n - T^n u_n \| + \| \xi_n \|
\leq \alpha_n M + \| v_n \| + \beta_n M + \| \sigma_n \| + \alpha_n M + \| \xi_n \|
= M \cdot [2\alpha_n + \beta_n] + \| v_n \| + \| \sigma_n \| + \| \xi_n \| \to 0 \quad \text{as} \quad n \to \infty.$$ \hspace{1cm}

Then \( \delta_n \to 0 \) as \( n \to \infty. \) Furthermore, from (4) and Lemma 1.1,

$$\| y_n - u_n \|^2 = \| (1 - \beta_n)x_n + \beta_n T^n x_n + \sigma_n \| - u_n \|^2
= \| (1 - \beta_n)(x_n - u_n) + \beta_n (T^n x_n - u_n) + \sigma_n \|^2
\leq (1 - \beta_n)^2 \| x_n - u_n \|^2 + 2\beta_n (T^n x_n - u_n) + \sigma_n, j(y_n - u_n)
\leq (1 - \beta_n)^2 \| x_n - u_n \|^2 + 2M^2 \beta_n + 2M \| \sigma_n \|
\leq \| x_n - u_n \|^2 + 2M^2 \beta_n + 2M \| \sigma_n \|. \tag{11}
$$

Taking (10) and (11) into (9), then

$$\| x_{n+1} - u_{n+1} \|^2 \leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2\alpha_n \| y_n - u_n \|^2 - 2\alpha_n \Phi(\| y_n - u_n \|)
+ 2M \alpha_n \delta_n + 2M \cdot (\| v_n \| + \| \xi_n \|)
\leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2\alpha_n \cdot \{ \| x_n - u_n \|^2 + 2M^2 \beta_n + 2M \| \sigma_n \| \}
- 2\alpha_n \Phi(\| y_n - u_n \|) + 2M \alpha_n \delta_n + 2M \cdot (\| v_n \| + \| \xi_n \|)
= (1 + \alpha_n^2) \| x_n - u_n \|^2 - 2\alpha_n \Phi(\| y_n - u_n \|)
+ \{ 4M^2 \alpha_n \beta_n + 4M^2 \alpha_n \| \sigma_n \| + 2M \alpha_n \delta_n + 2M \| v_n \| + 2M \| \xi_n \| \}
\leq \| x_n - u_n \|^2 - 2\alpha_n \Phi(\| y_n - u_n \|) + \{ M^2 \alpha_n^2 + 4M^2 \alpha_n \beta_n
+ 4M \alpha_n \| \sigma_n \| + 2M \alpha_n \delta_n + 2M \| v_n \| + 2M \| \xi_n \| \}. \tag{12}
$$

Furthermore, from (3) and (4),

$$\| x_{n+1} - u_{n+1} \| \equiv \| (1 - \alpha_n)(x_n - u_n) + \alpha_n (T^n y_n - T^n u_n) + v_n - \xi_n \|
\leq (1 - \alpha_n) \| x_n - u_n \| + \alpha_n \| T^n y_n - T^n u_n \| + \| v_n \| + \| \xi_n \|
\leq \| x_n - u_n \| + \alpha_n M + \| v_n \| + \| \xi_n \|
\leq \| y_n - u_n \| + \| y_n - x_n \| + \alpha_n M + \| v_n \| + \| \xi_n \|. \tag{13}$$
Then we can rewrite (15) into
\[
\|y_n - x_n\| = \|(1 - \beta_n)x_n + \beta_n T^n x_n + \omega_n - x_n\|
\]
\[
= \|\beta_n (x_n - T^n x_n) - \sigma_n\| \leq \beta_n M + \|\sigma_n\|.
\]
Then from (13) we have
\[
\|x_{n+1} - u_{n+1}\| \leq \|y_n - u_n\| + \beta_n M + \|\omega_n\| + \alpha_n M + \|\nu_n\| + \|\xi_n\|.
\] (14)

Set \(e_n = \beta_n M + \|\omega_n\| + \alpha_n M + \|\nu_n\| + \|\xi_n\|\). Then obviously \(e_n \geq 0\) and \(\lim_{n \to \infty} e_n = 0\) from conditions (5) and (6).

Therefore from (14), \(\|y_n - u_n\| \geq (\|x_{n+1} - u_{n+1}\| - e_n)\) and then, \(\Phi (\|y_n - u_n\|) \geq \Phi (\|x_{n+1} - u_{n+1}\| - e_n)\) for any strict increasing function \(\Phi (s)\). Hence, from (12), we get
\[
\|x_{n+1} - u_{n+1}\|^2 \leq \|x_n - u_n\|^2 - 2\alpha_n \Phi (\|x_{n+1} - u_{n+1}\| - e_n)
\]
\[
+ \left\{ M^2 \alpha_n^2 + 4M^2 \alpha_n \beta_n + 4M \alpha_n \|\omega_n\| + 2M \alpha_n \delta_n + 2M \|\nu_n\| + 2M \|\xi_n\| \right\}.
\] (15)

Set
\[
\sigma_n = \alpha_n \left[ M^2 \alpha_n + 4M^2 \beta_n + 4M \|\omega_n\| + 2M \delta_n + 2M \frac{\|\nu_n\|}{\alpha_n} + 2M \frac{\|\xi_n\|}{\alpha_n} \right],
\]
\[
\theta_n = \|x_n - u_n\|, \quad \lambda_n = \alpha_n.
\]

Then we can rewrite (15) into
\[
\theta_{n+1}^2 \leq \theta_n^2 - 2\lambda_n \Phi (\theta_{n+1} - e_n) + \sigma_n,
\]
where \(\{\theta_n\}, \{\sigma_n\}, \{\lambda_n\}\) and \(\{e_n\}\) are nonnegative real sequences satisfying (7) from the conditions of (5) and (6) on \(\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]\), and \(\{\xi_n\}, \{\omega_n\}, \{\nu_n\}\).

Hence from Lemma 1.2, \(\lim_{n \to \infty} \|u_n - x_n\| = \lim_{n \to \infty} \theta_n = 0\).

Since the modified Mann iteration with errors (3) converges to \(x^*\) and \(\lim_{n \to \infty} \|u_n - x^*\| = 0\), then from the inequality \(0 \leq \|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\|\), we have \(\lim_{n \to \infty} \|x_n - x^*\| = 0\). This completes the proof. \(\square\)

Now we consider another form of Mann and Ishikawa iterations with errors. We define the modified Mann iteration with errors by
\[
u_{n+1} = (1 - \alpha_n - \gamma_n)u_n + \alpha_n T^n u_n + \gamma_n \xi_n,
\] (16)
and the modified Ishikawa iteration with errors by
\[
\begin{align*}
x_{n+1} &= (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n x_n + \gamma_n \omega_n, \\
y_n &= (1 - \alpha_n' - \gamma_n')x_n + \alpha_n' T^n x_n + \gamma_n' \omega_n,
\end{align*}
\] (17)
where the sequences \(\{\alpha_n\}, \{\alpha_n'\}, \{\gamma_n\}, \{\gamma_n'\} \subseteq [0, 1]\) satisfy
\[
\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \alpha_n' = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \gamma_n = o(\alpha_n), \quad \lim_{n \to \infty} \gamma_n' = 0,
\] (18)
and the sequences \(\{\xi_n\}, \{\nu_n\}, \{\omega_n\}\) are bounded.
Theorem 2.2. Let $E$ be a real uniformly smooth space and let $T : E \to E$ be a generalized strongly successively $\Phi$-pseudocontractive mapping with bounded range. The sequences $\{u_n\}$ and $\{x_n\}$ are defined by (16) and (17) respectively, with $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\} \subseteq [0, 1]$ satisfying (18), and $\{\xi_n\}$, $\{\nu_n\}$, $\{\sigma_n\}$ being bounded. Then for $u_1, x_1 \in E$, the following two assertions are equivalent:

(i) modified Mann iteration with errors (16) converges to the fixed point $x^* \in F(T)$;

(ii) modified Ishikawa iteration with errors (17) converges to the fixed point $x^* \in F(T)$.

Proof. If the modified Ishikawa iteration with errors converges to $x^* \in F(T)$, then setting $\alpha'_n = \gamma'_n = 0$, $\forall n \in \mathbb{N}$, we can get the convergence of modified Mann iteration with errors. Next we will prove the result (i) $\Rightarrow$ (ii). Since $T^n E$, $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ are bounded, we set

$$M := \sup_n \left\{ \left\| T^n y_n - T^n u_n \right\|, \left\| v_n - \xi_n \right\|, \left\| x_n - T^n y_n \right\|, \left\| u_n - T^n u_n \right\|, \left\| u_n - x_n \right\|, \left\| x_n - T^n x_n \right\|, \left\| x_n - v_n \right\|, \left\| x_n - \sigma_n \right\|, \left\| \xi_n - u_n \right\| \right\},$$

(19)

we obtain $M < \infty$.

From (16), (17) and Lemma 1.1 with

$$x := (1 - \alpha_n - \gamma_n)(x_n - u_n), \quad y := \alpha_n(T^n y_n - T^n u_n) + \gamma_n(v_n - \xi_n),$$

then

$$\left\| x_{n+1} - u_{n+1} \right\|^2 = \left\| (1 - \alpha_n - \gamma_n)(x_n - u_n) + \alpha_n(T^n y_n - T^n u_n) + \gamma_n(v_n - \xi_n) \right\|^2$$

$$\leq (1 - \alpha_n - \gamma_n)^2 \left\| x_n - u_n \right\|^2 + 2\alpha_n\left\| T^n y_n - T^n u_n \right\| \left\| j(x_{n+1} - u_{n+1}) \right\|$$

$$+ 2\gamma_n \left\| v_n - \xi_n \right\| \left\| j(x_{n+1} - u_{n+1}) \right\|$$

$$\leq (1 - \alpha_n)^2 \left\| x_n - u_n \right\|^2 + 2\alpha_n\left\| T^n y_n - T^n u_n \right\| \left\| j(x_{n+1} - u_{n+1}) \right\|$$

$$+ 2\gamma_n \left\| v_n - \xi_n \right\| \left\| j(x_{n+1} - u_{n+1}) \right\|$$

$$\leq (1 - \alpha_n)^2 \left\| x_n - u_n \right\|^2 + 2\alpha_n \left[ \left\| y_n - u_n \right\|^2 - \Phi(\left\| y_n - u_n \right\|) \right]$$

$$+ 2\alpha_n M \left\| j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \right\| + 2\gamma_n M^2.$$  

(20)

Because $E$ is a uniformly smooth space, $J$ is a single-valued mapping and uniformly continuous on every bounded set. Set

$$\delta_n := \left\| j(x_{n+1} - u_{n+1}) - j(y_n - u_n) \right\|,$$

(21)

then

$$\left\| (x_{n+1} - u_{n+1}) - (y_n - u_n) \right\|$$

$$\leq \left\| (x_{n+1} - y_n) - (u_{n+1} - u_n) \right\|$$

$$\leq -\left( \alpha_n + \gamma_n \right) x_n + \left( \alpha'_n + \gamma'_n \right) x_n + \alpha_n T^n y_n - \alpha'_n T^n x_n + \gamma_n v_n - \gamma'_n \sigma_n$$

$$+ \left\| \alpha_n(T^n u_n - u_n) + \gamma_n(\xi_n - u_n) \right\|$$

$$\leq \alpha_n \left( \left\| x_n - T^n y_n \right\| + \left\| u_n - T^n u_n \right\| \right) + \alpha'_n \left\| x_n - T^n x_n \right\|$$

$$+ \left\| \alpha_n(T^n u_n - u_n) + \gamma_n(\xi_n - u_n) \right\|$$

$$\leq \alpha_n \left( \left\| x_n - T^n y_n \right\| + \left\| u_n - T^n u_n \right\| \right) + \alpha'_n \left\| x_n - T^n x_n \right\|$$

$$+ \left\| \alpha_n(T^n u_n - u_n) + \gamma_n(\xi_n - u_n) \right\|.$$
\[ + \gamma_n \left( \| x_n - v_n \| + \| \xi_n - u_n \| \right) + \gamma'_n \| x_n - \omega_n \| \]
\[ \leq (\alpha_n + \gamma_n)2M + (\alpha'_n + \gamma'_n)M \to 0 \quad \text{as } n \to \infty \]  
(22)

implies that \( \delta_n \to 0 \) as \( n \to \infty \). Moreover, from (17),
\[ \| y_n - u_n \|^2 \leq \left[ \| x_n - u_n \| + \| x_n - y_n \| \right]^2 \]
\[ = \left[ \| x_n - u_n \| + \| \alpha'_n (T^n x_n - x_n) + \gamma'_n (\sigma_n - x_n) \| \right]^2 \]
\[ \leq \left[ \| x_n - u_n \| + \alpha'_n \| T^n x_n - x_n \| + \gamma'_n \| \sigma_n - x_n \| \right]^2 \]
\[ \leq \left[ \| x_n - u_n \| + M (\alpha'_n + \gamma'_n) \right]^2 \]
\[ = \| y_n - u_n \|^2 + M (\alpha'_n + \gamma'_n) [2 \| u_n - x_n \| + M (\alpha'_n + \gamma'_n)] \]
\[ \leq \| y_n - u_n \|^2 + (\alpha'_n + \gamma'_n) \cdot 4M^2. \]  
(23)

Taking (21) and (23) into (20), then
\[ \| x_{n+1} - u_{n+1} \|^2 \leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2\alpha_n \| y_n - u_n \|^2 \]
\[ - 2\alpha_n \Phi (\| y_n - u_n \|) + 2\alpha_n \delta_n M + 2\gamma_n M^2 \]
\[ \leq (1 - \alpha_n)^2 \| x_n - u_n \|^2 + 2\alpha_n \left[ \| x_n - u_n \|^2 + (\alpha'_n + \gamma'_n) \cdot 4M^2 \right] \]
\[ - 2\alpha_n \Phi (\| y_n - u_n \|) + 2\alpha_n \delta_n M + 2\gamma_n M^2 \]
\[ = (1 + \alpha_n^2) \| x_n - u_n \|^2 - 2\alpha_n \Phi (\| y_n - u_n \|) \]
\[ + 2\alpha_n (\alpha'_n + \gamma'_n) \cdot 4M^2 + 2\alpha_n \delta_n M + 2\gamma_n M^2 \]
\[ \leq \| x_n - u_n \|^2 - 2\alpha_n \Phi (\| y_n - u_n \|) \]
\[ + \alpha_n \cdot \left[ \alpha_n M^2 + 8M^2 (\alpha'_n + \gamma'_n) + 2\delta_n M + 2 \frac{\gamma_n}{\alpha_n} M^2 \right]. \]  
(24)

Furthermore, from (16), (17) and Lemma 1.1, we get
\[ \| x_{n+1} - u_{n+1} \| = \left\| (1 - \alpha_n - \gamma_n) (x_n - u_n) + \alpha_n (T^n y_n - T^n u_n) + \gamma_n (v_n - \xi_n) \right\| \]
\[ \leq \| x_n - u_n \| + \alpha_n M + \gamma_n \left( \| v_n \| + \| \xi_n \| \right) \]
\[ \leq \| y_n - u_n \| + \| y_n - x_n \| + \alpha_n M + \gamma_n \left( \| v_n \| + \| \xi_n \| \right) \]
\[ \leq \| y_n - u_n \| + \left[ \alpha'_n M + \gamma'_n M + \alpha_n M + \gamma_n \left( \| v_n \| + \| \xi_n \| \right) \right]. \]  
(25)

Set
\[ e_n = \alpha'_n M + \gamma'_n M + \alpha_n M + \gamma_n \left( \| v_n \| + \| \xi_n \| \right), \quad \lambda_n = \alpha_n, \]
\[ \theta_n \equiv \| x_n - u_n \|, \quad \sigma_n \equiv \alpha_n \left[ \alpha_n M + 8M^2 (\alpha'_n + \gamma'_n) + 2\delta_n M + 2 \frac{\gamma_n}{\alpha_n} M^2 \right]. \]

Therefore from (25), \( \| y_n - u_n \| \geq \| x_{n+1} - u_{n+1} \| - e_n \) and hence \( \Phi (\| y_n - u_n \|) \geq \Phi (\| x_{n+1} - u_{n+1} \| - e_n) \) since \( \Phi (s) \) is a strict increasing function. Hence from (24) we obtain that
\[ \theta_{n+1}^2 \leq \theta_n^2 - 2\lambda_n \Phi (\theta_n + e_n) + \sigma_n, \]
where \( \{\theta_n\}, \{\sigma_n\}, \{\lambda_n\} \) and \( \{e_n\} \) are nonnegative real sequences satisfying (7) from the conditions of (18).
Then from Lemma 1.2, \(\lim_{n \to \infty} \|u_n - x_n\| = \lim_{n \to \infty} \theta_n = 0\). Since the modified Mann iteration with errors (16) converges to \(x^*\), \(\lim_{n \to \infty} \|u_n - x^*\| = 0\), then from the inequality \(0 \leq \|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\|\), we have \(\lim_{n \to \infty} \|x_n - x^*\| = 0\). This completes the proof. \(\square\)

It is well known that the generalized strongly \(\Phi\)-pseudocontractive mapping is a particular form of the generalized strongly successively \(\Phi\)-pseudocontractive mapping. Obviously, replacing \(T^n\) by \(T\) in (1), one obtains the definition of generalized strongly \(\Phi\)-pseudocontractive mapping (2). Then in Theorems 2.1 and 2.2, if \(T\) is a generalized strongly \(\Phi\)-pseudocontractive mapping, the conclusion will still hold.

Replacing \(T^n\) by \(T\) in (3) and (4), we obtain the following ordinary Mann and Ishikawa iterations with errors, respectively:

\[
\begin{align*}
  u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n Tu_n + \xi_n, \\
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n + v_n, \\
  y_n &= (1 - \beta_n)x_n + \beta_n Tx_n + \sigma_n.
\end{align*}
\] (26)

Let \(T, S : E \to E\), \(f \in E\) be given. It is well known that \(T\) is a generalized strongly \(\Phi\)-pseudocontractive mapping if and only if \((I - T)\) is generalized strongly \(\Phi\)-accretive (see [17]). Moreover, \(x^*\) is the fixed point for the mapping \(Tx = f + (I - S)x\) if and only if \(x^*\) is the solution for \(Sx = f\). Then we have the following results.

**Theorem 2.3.** Let \(E\) be a real uniformly smooth Banach space and let \(T : E \to E\) be a generalized strongly \(\Phi\)-pseudocontractive mapping with bounded range. The sequences \(\{u_n\}\) and \(\{x_n\}\) are defined by (26) and (27) respectively, with \(\{\alpha_n\} \subseteq [0, 1]\) satisfying (5), and \(\{v_n\}\), \(\{\sigma_n\}\), \(\{\xi_n\}\) satisfying (6). Then for \(u_1, x_1 \in E\), we have the following equivalences:

(i) Mann iteration with errors (26) converges to the fixed point \(x^* \in F(T)\);
(ii) Ishikawa iteration with errors (27) converges to the fixed point \(x^* \in F(T)\).

Considering iterations (26), (27) with \(Tx = f + (I - S)x\), we have the following results.

**Theorem 2.4.** Let \(E\) be a real uniformly smooth Banach space and let \(S : E \to E\) be a generalized strongly \(\Phi\)-accretive mapping with bounded range. The sequences \(\{u_n\}\) and \(\{x_n\}\) are defined by (26) and (27) respectively, with \(\{\alpha_n\} \subseteq [0, 1]\) satisfying (5), and \(\{v_n\}\), \(\{\sigma_n\}\), \(\{\xi_n\}\) satisfying (6). Then for \(u_1, x_1 \in E\), we have the following equivalences:

(i) Mann iteration with errors (26) converges to the solution \(x^*\) of \(Sx = f\);
(ii) Ishikawa iteration with errors (27) converges to the solution \(x^*\) of \(Sx = f\).

Replacing \(T^n\) by \(T\) in (16), (17) one obtains another form of ordinary Mann and Ishikawa iterations with errors respectively defined by Xu [23]:

\[
\begin{align*}
  u_{n+1} &= (1 - \alpha_n - \gamma_n)u_n + \alpha_n Tu_n + \gamma_n \xi_n, \\
  x_{n+1} &= (1 - \alpha_n - \gamma_n)x_n + \alpha_n Ty_n + \gamma_n v_n, \\
  y_n &= (1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n Tx_n + \gamma'_n \sigma_n.
\end{align*}
\] (28)
Theorem 2.5. Let $E$ be a real uniformly smooth Banach space and let $T : E \to E$ be a general-ized strongly $\Phi$-pseudocontractive mapping with bounded range. The sequences $\{u_n\}$, $\{x_n\}$ are defined by (28) and (29) respectively, with $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$, $\{\xi_n\}$, $\{v_n\}$, $\{\sigma_n\}$ being bounded. Then for $u_1, x_1 \in E$, we have the following equivalences:

(i) Mann iteration with errors (28) converges to the fixed point $x^* \in F(T)$;
(ii) Ishikawa iteration with errors (29) converges to the fixed point $x^* \in F(T)$.

Analogously we consider iterations (28), (29) with $Tx = f + (I - S)x$.

Theorem 2.6. Let $E$ be a real uniformly smooth Banach space and $S : E \to E$ be a generalized strongly $\Phi$-accretive mapping with bounded range. The sequences $\{u_n\}$ and $\{x_n\}$ are defined by (28) and (29), respectively, with $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$ satisfying (18), and $\{\xi_n\}$, $\{v_n\}$, $\{\sigma_n\}$ being bounded. Then for $u_1, x_1 \in E$, we have the following equivalences:

(i) Mann iteration with errors (28) converges to the solution $x^*$ of $Sx = f$;
(ii) Ishikawa iteration with errors (29) converges to the solution $x^*$ of $Sx = f$.

Remark 2.1. Our results cover all the results in [11] as a special case. As $\Phi(s) = ks^2$, $k \in (0, 1)$, our Theorems 2.1–2.6 will reduce to Theorems 1–6 of [11], respectively. Our better results remain true under the weaker conditions such that the range of $T$ is bounded with the wider selections of $\xi_n$, $v_n$, $\gamma_n$ under the conditions of (6) and (18) such that

$$
\|\xi_n\| = o(\alpha_n), \quad \|v_n\| = o(\alpha_n), \quad \gamma_n = o(\alpha_n),
$$

than those of the pre-requirements in [11] as

$$
\sum_{n=1}^{\infty} \|\xi_n\| < \infty, \quad \sum_{n=1}^{\infty} \|v_n\| < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n < \infty.
$$

Moreover, the unnecessary conditions such as the closeness and convexity in Refs. [21,22] can be dropped.

Remark 2.2. If $\xi_n = 0$, $v_n = 0$, $\sigma_n = 0$, or if $\gamma_n = 0$, $\gamma'_n = 0$, for all $n \in \mathbb{N}$, then our Theorems 1 and 2, Theorems 3 and 5, and Theorems 4 and 6 will reduce to Theorem 8 of [21], Theorem 2.1 of [22], and Corollary 3.3 of [22] respectively under a special case of $\Phi(s) = ks^2$, $k \in (0, 1)$. Hence, the results in [21] and [22] are the special cases of our paper. Then as a conclusion of our results, in uniformly smooth Banach space and without Lipschitzian assumption (even not necessarily continuous), for any initial point $u_1, x_1 \in E$ (even without any pre-requirement $u_1 = x_1$ in [11,21,22]), these modified sequences with errors converge equivalently. Consequently, our theorems include the results in [11,21,22] as special cases and hence generalize all of the recent results in [1–23].

References