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Note

An Inequality on Paths in a Grid

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Let $(c_{i,j})$ be a positive $n \times m$ log supermodular matrix, i.e., $c_{i,j} \cdot c_{i+1,j+1} \geqslant c_{i+1,j} \cdot c_{i,j+1}$. Then there exists a path from $c_{1,1}$ to $c_{n,m}$ whose average dominates that of all the entries of the matrix $(c_{i,j})$.

INTRODUCTION

Let $(P, \leqslant)$ denote a finite poset (partially ordered set), and let f be a function from P to the positive reals. For a subset S of P, let $f(S) = \prod \{f(x) \mid x \in S\}$, and let $\mathrm{Av}(P, \leqslant, f)$ denote the average of the family $\{f(I) \mid I$ is an (upper) ideal in $P\}$. Information theoretic considerations suggest the following question:

Does there exist a linear extension $\preccurlyeq$ of $\leqslant$ such that

$$\mathrm{Av}(P, \leqslant, f) \leqslant \mathrm{Av}(P, \preccurlyeq, f). \tag{A}$$

Even though the answer here is not *yes* in general, it is so in some interesting cases. If, e.g., P is a totally unordered set (an antichain) of cardinality n, then the truth of (A) is equivalent to the inequality

$$\frac{(1+x_1)\cdots(1+x_n)}{2^n} \leqslant \frac{1+x_1+x_1x_2+\cdots+x_1\cdots x_n}{n+1}, \tag{B}$$

whenever $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n > 0$.

A direct proof of this inequality is given by H.-J. Seiffert in [2]. In [1] certain generalizations of (B) are studied, and (B) follows from Corollary 1

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in that paper. In the present paper another generalization of (B) is considered, and our starting point is that (B) follows if one mangages to prove (A) for the case that P is a sum of two chains. We define $[u_1, ..., u_k] = 1 + u_1 + \cdots + u_1 \cdots u_k$, and the corresponding inequality says that

$$\frac{[x_1, ..., x_n] \cdot [y_1, ..., y_m]}{(n+1)(m+1)} \leqslant \frac{[z_1, ..., z_{n+m}]}{n+m+1} \tag{C}$$

for *some* sequence $\{z_k \mid 1 \leqslant k \leqslant n+m\}$ that is a *merging* of the positive sequences $\{x_i\}$ and $\{y_j\}$.

A *merging* has the obvious meaning, in that, e.g., x_1, y_1, x_2, x_3, y_2 is a merging of x_1, x_2, x_3 and y_1, y_2. Inequality (C) is illustrated in Fig. 1. In this figure there is a one-to-one correspondence between paths from 1 (lower left) to $x_1x_2x_3y_1y_2$ (upper right) and mergings of the sequences $\{x_i\}$ and $\{y_j\}$. The path indicated in Fig. 1 corresponds to the merging: x_1, x_2, y_1, y_2, x_3.

$$\begin{array}{c|cccc}
y_1y_2 & y_1y_2 & x_1y_1y_2 & x_1x_2y_1y_2 \;\text{—}\; & x_1x_2x_3y_1y_2 \\
 & & & | & \\
y_1 & y_1 & x_1y_1 & x_1x_2y_1 & x_1x_2x_3y_1 \\
 & & & | & \\
1 & 1 \;\text{—}\; & x_1 \;\text{—}\; & x_1x_2 & x_1x_2x_3 \\ \hline
 & 1 & x_1 & x_1x_2 & x_1x_2x_3
\end{array}$$

FIG. 1. $n = 3$ and $m = 2$.

With this in mind, inequality (C) is equivalent to:

Let $\{a_i \mid 1 \leqslant i \leqslant n\}$ and $\{b_j \mid 1 \leqslant j \leqslant m\}$ be sequences of positive real numbers, and put $c_{i,j} = a_i b_j$. Then there exists a path (see Fig. 2) from $c_{1,1}$ to $c_{n,m}$ whose average is not less than the average of the family $\{c_{i,j} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$. We note that the matrix $(c_{i,j})$ enjoys the property that $c_{i,j} \cdot c_{i+1,j+1} = c_{i+1,j} \cdot c_{i,j+1}$. The main theorem in this note proves this reformulation of inequality (C) for any positive log supermodular matrix, i.e., $c_{i,j} \cdot c_{i+1,j+1} \geqslant c_{i+1,j} \cdot c_{i,j+1}$. Log supermodular matrices often occur in combinatorial structures (see, for instance, [3]).

$$\begin{array}{c|cccc}
b_3 & c_{13} & c_{23} & c_{33} \;\text{—}\; & c_{43} \\
 & & & | & \\
b_2 & c_{12} & c_{22} & c_{32} & c_{42} \\
 & & & | & \\
b_1 & c_{11} \;\text{—}\; & c_{21} \;\text{—}\; & c_{31} & c_{41} \\ \hline
 & a_1 & a_2 & a_3 & a_4
\end{array}$$

FIG. 2. $c_{i,j} = a_i b_j$.

The Theorem

As indicated in the Introduction, a path from $(1, 1)$ to (n, m) is a sequence of $n+m-1$ pairs starting at $(1, 1)$ and ending at (n, m), such that in each step, either the first or the second component is increased by 1.

Theorem 1. *Let $\{c_{i,j} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ be positive real numbers. Assume that*

$$c_{i,j} \cdot c_{i+1,j+1} \geqslant c_{i,j} \cdot c_{i+1,j} \qquad \text{for all} \quad 1 \leqslant i < n \quad \textit{and} \quad j < m.$$

Then

$$\frac{c_{i_1,j_1} + \cdots + c_{i_k,j_k}}{n+m-1} \geqslant \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} c_{i,j}}{nm} \tag{1}$$

for some path $\{(i_t, j_t) \mid 1 \leqslant t \leqslant k = n+m-1\}$ from $(1, 1)$ to (n, m).

Remark. Inequality (1) says that the (arithmetic) average over the path is at least as big as the average over all $c_{i,j}$. By similar techniques as in the proof below, one can also prove Theorem 1 (with identical conditions) for the *geometric* average, but this will not be further discussed in this note.

Proof. The proof is by induction on $n+m$. The claim is certainly true if $n=1$ or $m=1$. Thus, assume both $n, m > 1$. The main objects in the proof are illustrated in Fig. 3.

$$\begin{matrix} \begin{bmatrix} c_{1,m} \\ \vdots \\ c_{1,2} \end{bmatrix} & \begin{bmatrix} c_{2,m} & \cdots & c_{n,m} \\ \vdots & & \vdots \\ c_{2,2} & \cdots & c_{n,2} \end{bmatrix} \\ c_{1,1} & \begin{bmatrix} c_{2,1} & \cdots & c_{n,1} \end{bmatrix} \end{matrix}$$

Figure 3

Let $X = c_{1,1}$, $A' = c_{2,1} + \cdots + c_{n,1}$, $B' = c_{1,2} + \cdots + c_{1,m}$, and $C' = \sum_{i=2}^{n} \sum_{j=2}^{m} c_{i,j}$. A path from $c_{1,1}$ to $c_{n,m}$ must pass through $c_{1,2}$ or $c_{2,1}$. By the induction hypothesis there exists a path $S_A = c_{2,1} + \cdots + c_{n,m}$ from $c_{2,1}$ to $c_{n,m}$ such that

$$\frac{S_A}{n+m-2} \geqslant \frac{A' + C'}{(n-1)m} \tag{2}$$

and a path $S_B = c_{1,2} + \cdots + c_{n,m}$ from $c_{1,2}$ to $c_{n,m}$ such that

$$\frac{S_B}{n+m-2} \geqslant \frac{B' + C'}{n(m-1)}. \tag{3}$$

We wish to show that

$$\frac{X+S_A}{n+m-1} \geqslant \frac{A'+B'+C'+X}{nm} \tag{4}$$

or

$$\frac{X+S_B}{n+m-1} \geqslant \frac{A'+B'+C'+X}{nm}. \tag{5}$$

Put $r=n-1$ and $s=m-1$. By inequalities (2) and (3) it is sufficient to show:

$$(C'+A')\frac{s}{r} - B'(r+s+1) + Xrs \geqslant 0 \tag{6}$$

or

$$(C'+B')\frac{r}{s} - A'(r+s+1) + Xrs \geqslant 0. \tag{7}$$

Put $A=A'/rX$, $B=B'/sX$, $C=C'/rsX$. Then (6) and (7) reduce to

$$sC + A - B(r+s+1) + r \geqslant 0 \tag{8}$$

$$rC + B - A(r+s+1) + s \geqslant 0. \tag{9}$$

By the log supermodularity we know $XC' \geqslant A'B'$, and hence $C \geqslant AB$. Assuming that both (8) and (9) are false implies

$$A < \frac{B(r+s+1)-r}{1+sB} \tag{10}$$

and

$$A(r+s+1-rB) > B+s. \tag{11}$$

Since $B+s>0$ and $A>0$, $r+s+1-rB$ must be positive. Thus (11) implies

$$A > \frac{B+s}{r+s+1-rB}. \tag{12}$$

Now (10) and (12) imply

$$(r+s)(r+1)(B^2-2B+1) < 0.$$

This contradiction shows that (8) or (9) has to be true, which proves Theorem 1.

Remarks. We can always find a path (i_t, j_t) that gives a strict inequality in (1), except when $n=1$ or $m=1$ or when all $c_{i,j}$ are equal. The discussion is omitted in this note.

It is interesting to observe that there are cases in which all paths $\{c_{i_t,j_t}\}$ from $c_{1,1}$ to $c_{n,m}$ in Theorem 1 give a strict inequality in (1). Two very simple examples are

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 8 & 16 & 32 \\ 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 \end{bmatrix}.$$

The second one (and, in general, $c_{i,j}=x^{i+j-2}$ for some positive $x \neq 1$) has the additional property that $c_{i,j} \cdot c_{i+1,j+1} = c_{i+1,j} \cdot c_{i,j+1}$.

Corollaries

In the Introduction, we defined $[x_1, ..., x_n] = 1 + x_1 + x_1 x_2 + \cdots + x_1 \cdots x_n$, and we defined a *merging* of two sequences in the obvious way.

Corollary 1. *Let $\{x_i \mid 1 \leqslant i \leqslant n\}$ and $\{y_j \mid 1 \leqslant j \leqslant m\}$ be the positive sequences. Then there exists a merging $\{z_k \mid 1 \leqslant k \leqslant n+m\}$ of these sequences such that*

$$\frac{[x_1, ..., x_n]}{n+1} \cdot \frac{[y_1, ..., y_m]}{m+1} \leqslant \frac{[z_1, ..., z_{n+m}]}{n+m+1}.$$

Proof. Let $a_1 = b_1 = 1$, $a_i = \prod_{k=1}^{i-1} x_k$ $(2 \leqslant i \leqslant n+1)$, and $b_j = \prod_{k=1}^{j-1} y_k$ $(2 \leqslant j \leqslant m+1)$. Let $c_{i,j} = a_i b_j$. Theorem 1 applied to the matrix $(c_{i,j})$ proves Corollary 1.

Corollary 2. *Assume $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_m > 0$. Let $z_1 \geqslant z_2 \geqslant \cdots \geqslant z_{n+m}$ be the uniquely defined sorted merging of $\{x_i\}$ and $\{y_j\}$, then*

$$\frac{[x_1, ..., x_n]}{n+1} \cdot \frac{[y_1, ..., y_m]}{m+1} \leqslant \frac{[z_1, ..., z_{n+m}]}{n+m+1}.$$

Proof. The sorted merging is the one that maximizes the value of $[z_1, ..., z_{n+m}]$. Thus Corollary 1 implies Corollary 2.

Remark. Corollary 2 was proven in [1] for the case that all the x's are greater than all the y's.

Corollaries 1 and 2 can easily be generalized to more than two sequences.

COROLLARY 3. *Let $\{a_i \mid 1 \leqslant i \leqslant n\}$ be a positive log convex sequence (i.e., $a_i^2 \leqslant a_{i+1} a_{i-1}$). Then*

$$\frac{\sum_{i=1}^n (s_i a_i)}{\sum_{i=1}^n s_i} \leqslant \frac{\sum_{i=1}^n a_i}{n}$$

for any positive sequence $\{s_i\}$ satisfying

(i) $s_i = s_{n-i+1}$ *for* $1 \leqslant i \leqslant n$

(ii) $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{\lceil n/2 \rceil}$

Proof. Let $c_{i,j} = a_{i+j-1}$ for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant n-1$. Then $c_{i,j}$ satisfies the condition of Theorem 1. It is also easily seen that any any path from $c_{1,1}$ to $c_{2,n-1}$ is the sequence $\{a_1, a_2, ..., a_n\}$:

$$(c_{i,j}) = \begin{bmatrix} a_{n-1} & a_n \\ \vdots & \vdots \\ a_2 & a_3 \\ a_1 & a_2 \end{bmatrix}.$$

Hence Theorem 1 implies

$$\frac{a_1 + \cdots + a_n}{n} \geqslant \frac{a_1 + 2(a_2 + \cdots + a_{n-1}) + a_n}{2(n-1)}$$

which implies

$$\frac{a_1 + \cdots + a_n}{n} \geqslant \frac{a_2 + \cdots + a_{n-1}}{n-2}.$$

The sequence $a_2, ..., a_{n-1}$ is also log convex, and hence

$$\frac{a_1 + \cdots + a_n}{n} \geqslant \frac{a_2 + \cdots + a_{n-1}}{n-2} \geqslant \frac{a_3 + \cdots + a_{n-2}}{n-4} \geqslant \cdots.$$

By multiplying these fractions by s_1, $s_2 - s_1$, $s_3 - s_2$, etc. and by repeatedly using the fact that $s/t \geqslant u/v$ implies

$$\frac{s}{t} \geqslant \frac{s+u}{t+v} \geqslant \frac{u}{v}$$

for positive s, t, u, v; the corollary easily follows.

References

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