

## Note

### An Inequality on Paths in a Grid

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Let  $(c_{i,j})$  be a positive  $n \times m$  log supermodular matrix, i.e.,  $c_{i,j} \cdot c_{i+1,j+1} \geq c_{i+1,j} \cdot c_{i,j+1}$ . Then there exists a path from  $c_{1,1}$  to  $c_{n,m}$  whose average dominates that of all the entries of the matrix  $(c_{i,j})$ . © 1992 Academic Press, Inc.

#### INTRODUCTION

Let  $(P, \leq)$  denote a finite poset (partially ordered set), and let  $f$  be a function from  $P$  to the positive reals. For a subset  $S$  of  $P$ , let  $f(S) = \prod \{f(x) \mid x \in S\}$ , and let  $\text{Av}(P, \leq, f)$  denote the average of the family  $\{f(I) \mid I \text{ is an (upper) ideal in } P\}$ . Information theoretic considerations suggest the following question:

Does there exist a linear extension  $\leq'$  of  $\leq$  such that

$$\text{Av}(P, \leq', f) \leq \text{Av}(P, \leq, f). \tag{A}$$

Even though the answer here is not *yes* in general, it is so in some interesting cases. If, e.g.,  $P$  is a totally unordered set (an antichain) of cardinality  $n$ , then the truth of (A) is equivalent to the inequality

$$\frac{(1+x_1) \cdots (1+x_n)}{2^n} \leq \frac{1+x_1+x_1x_2+\cdots+x_1 \cdots x_n}{n+1}, \tag{B}$$

whenever  $x_1 \geq x_2 \geq \cdots \geq x_n > 0$ .

A direct proof of this inequality is given by H.-J. Seiffert in [2]. In [1] certain generalizations of (B) are studied, and (B) follows from Corollary 1

in that paper. In the present paper another generalization of (B) is considered, and our starting point is that (B) follows if one manages to prove (A) for the case that  $P$  is a sum of two chains. We define  $[u_1, \dots, u_k] = 1 + u_1 + \dots + u_1 \cdots u_k$ , and the corresponding inequality says that

$$\frac{[x_1, \dots, x_n] \cdot [y_1, \dots, y_m]}{(n+1)(m+1)} \leq \frac{[z_1, \dots, z_{n+m}]}{n+m+1} \tag{C}$$

for some sequence  $\{z_k | 1 \leq k \leq n+m\}$  that is a *merging* of the positive sequences  $\{x_i\}$  and  $\{y_j\}$ .

A *merging* has the obvious meaning, in that, e.g.,  $x_1, y_1, x_2, x_3, y_2$  is a merging of  $x_1, x_2, x_3$  and  $y_1, y_2$ . Inequality (C) is illustrated in Fig. 1. In this figure there is a one-to-one correspondence between paths from 1 (lower left) to  $x_1 x_2 x_3 y_1 y_2$  (upper right) and mergings of the sequences  $\{x_i\}$  and  $\{y_j\}$ . The path indicated in Fig. 1 corresponds to the merging:  $x_1, x_2, y_1, y_2, x_3$ .

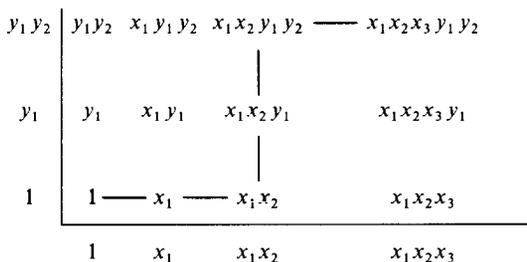


FIG. 1.  $n=3$  and  $m=2$ .

With this in mind, inequality (C) is equivalent to:

Let  $\{a_i | 1 \leq i \leq n\}$  and  $\{b_j | 1 \leq j \leq m\}$  be sequences of positive real numbers, and put  $c_{i,j} = a_i b_j$ . Then there exists a path (see Fig. 2) from  $c_{1,1}$  to  $c_{n,m}$  whose average is not less than the average of the family  $\{c_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m\}$ . We note that the matrix  $(c_{i,j})$  enjoys the property that  $c_{i,j} \cdot c_{i+1,j+1} = c_{i+1,j} \cdot c_{i,j+1}$ . The main theorem in this note proves this reformulation of inequality (C) for any positive log supermodular matrix, i.e.,  $c_{i,j} \cdot c_{i+1,j+1} \geq c_{i+1,j} \cdot c_{i,j+1}$ . Log supermodular matrices often occur in combinatorial structures (see, for instance, [3]).

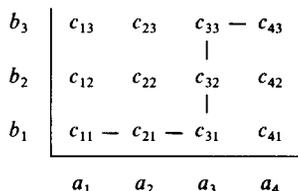


FIG. 2.  $c_{i,j} = a_i b_j$ .

THE THEOREM

As indicated in the Introduction, a path from  $(1, 1)$  to  $(n, m)$  is a sequence of  $n + m - 1$  pairs starting at  $(1, 1)$  and ending at  $(n, m)$ , such that in each step, either the first or the second component is increased by 1.

**THEOREM 1.** *Let  $\{c_{i,j} | 1 \leq i \leq n, 1 \leq j \leq m\}$  be positive real numbers. Assume that*

$$c_{i,j} \cdot c_{i+1,j+1} \geq c_{i,j} \cdot c_{i+1,j} \quad \text{for all } 1 \leq i < n \text{ and } j < m.$$

Then

$$\frac{c_{i_1, j_1} + \dots + c_{i_k, j_k}}{n + m - 1} \geq \frac{\sum_{i=1}^n \sum_{j=1}^m c_{i,j}}{nm} \tag{1}$$

for some path  $\{(i_t, j_t) | 1 \leq t \leq k = n + m - 1\}$  from  $(1, 1)$  to  $(n, m)$ .

*Remark.* Inequality (1) says that the (arithmetic) average over the path is at least as big as the average over all  $c_{i,j}$ . By similar techniques as in the proof below, one can also prove Theorem 1 (with identical conditions) for the *geometric* average, but this will not be further discussed in this note.

*Proof.* The proof is by induction on  $n + m$ . The claim is certainly true if  $n = 1$  or  $m = 1$ . Thus, assume both  $n, m > 1$ . The main objects in the proof are illustrated in Fig. 3.

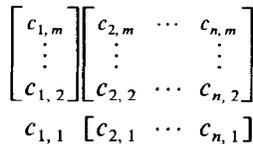


FIGURE 3

Let  $X = c_{1,1}$ ,  $A' = c_{2,1} + \dots + c_{n,1}$ ,  $B' = c_{1,2} + \dots + c_{1,m}$ , and  $C' = \sum_{i=2}^n \sum_{j=2}^m c_{i,j}$ . A path from  $c_{1,1}$  to  $c_{n,m}$  must pass through  $c_{1,2}$  or  $c_{2,1}$ . By the induction hypothesis there exists a path  $S_A = c_{2,1} + \dots + c_{n,m}$  from  $c_{2,1}$  to  $c_{n,m}$  such that

$$\frac{S_A}{n + m - 2} \geq \frac{A' + C'}{(n - 1)m} \tag{2}$$

and a path  $S_B = c_{1,2} + \dots + c_{n,m}$  from  $c_{1,2}$  to  $c_{n,m}$  such that

$$\frac{S_B}{n + m - 2} \geq \frac{B' + C'}{n(m - 1)}. \tag{3}$$

We wish to show that

$$\frac{X + S_A}{n + m - 1} \geq \frac{A' + B' + C' + X}{nm} \quad (4)$$

or

$$\frac{X + S_B}{n + m - 1} \geq \frac{A' + B' + C' + X}{nm}. \quad (5)$$

Put  $r = n - 1$  and  $s = m - 1$ . By inequalities (2) and (3) it is sufficient to show:

$$(C' + A') \frac{s}{r} - B'(r + s + 1) + Xrs \geq 0 \quad (6)$$

or

$$(C' + B') \frac{r}{s} - A'(r + s + 1) + Xrs \geq 0. \quad (7)$$

Put  $A = A'/rX$ ,  $B = B'/sX$ ,  $C = C'/rsX$ . Then (6) and (7) reduce to

$$sC + A - B(r + s + 1) + r \geq 0 \quad (8)$$

$$rC + B - A(r + s + 1) + s \geq 0. \quad (9)$$

By the log supermodularity we know  $XC' \geq A'B'$ , and hence  $C \geq AB$ . Assuming that both (8) and (9) are false implies

$$A < \frac{B(r + s + 1) - r}{1 + sB} \quad (10)$$

and

$$A(r + s + 1 - rB) > B + s. \quad (11)$$

Since  $B + s > 0$  and  $A > 0$ ,  $r + s + 1 - rB$  must be positive. Thus (11) implies

$$A > \frac{B + s}{r + s + 1 - rB}. \quad (12)$$

Now (10) and (12) imply

$$(r + s)(r + 1)(B^2 - 2B + 1) < 0.$$

This contradiction shows that (8) or (9) has to be true, which proves Theorem 1.

*Remarks.* We can always find a path  $(i, j)$  that gives a strict inequality in (1), except when  $n = 1$  or  $m = 1$  or when all  $c_{i,j}$  are equal. The discussion is omitted in this note.

It is interesting to observe that there are cases in which all paths  $\{c_{i,j}\}$  from  $c_{1,1}$  to  $c_{n,m}$  in Theorem 1 give a strict inequality in (1). Two very simple examples are

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 8 & 16 & 32 \\ 2 & 4 & 8 & 16 \\ 1 & 2 & 4 & 8 \end{bmatrix}.$$

The second one (and, in general,  $c_{i,j} = x^{i+j-2}$  for some positive  $x \neq 1$ ) has the additional property that  $c_{i,j} \cdot c_{i+1,j+1} = c_{i+1,j} \cdot c_{i,j+1}$ .

#### COROLLARIES

In the Introduction, we defined  $[x_1, \dots, x_n] = 1 + x_1 + x_1 x_2 + \dots + x_1 \cdots x_n$ , and we defined a *merging* of two sequences in the obvious way.

**COROLLARY 1.** *Let  $\{x_i | 1 \leq i \leq n\}$  and  $\{y_j | 1 \leq j \leq m\}$  be the positive sequences. Then there exists a merging  $\{z_k | 1 \leq k \leq n+m\}$  of these sequences such that*

$$\frac{[x_1, \dots, x_n]}{n+1} \cdot \frac{[y_1, \dots, y_m]}{m+1} \leq \frac{[z_1, \dots, z_{n+m}]}{n+m+1}.$$

*Proof.* Let  $a_1 = b_1 = 1$ ,  $a_i = \prod_{k=1}^{i-1} x_k$  ( $2 \leq i \leq n+1$ ), and  $b_j = \prod_{k=1}^{j-1} y_k$  ( $2 \leq j \leq m+1$ ). Let  $c_{i,j} = a_i b_j$ . Theorem 1 applied to the matrix  $(c_{i,j})$  proves Corollary 1.

**COROLLARY 2.** *Assume  $x_1 \geq x_2 \geq \dots \geq x_n > 0$  and  $y_1 \geq y_2 \geq \dots \geq y_m > 0$ . Let  $z_1 \geq z_2 \geq \dots \geq z_{n+m}$  be the uniquely defined sorted merging of  $\{x_i\}$  and  $\{y_j\}$ , then*

$$\frac{[x_1, \dots, x_n]}{n+1} \cdot \frac{[y_1, \dots, y_m]}{m+1} \leq \frac{[z_1, \dots, z_{n+m}]}{n+m+1}.$$

*Proof.* The sorted merging is the one that maximizes the value of  $[z_1, \dots, z_{n+m}]$ . Thus Corollary 1 implies Corollary 2.

*Remark.* Corollary 2 was proven in [1] for the case that all the  $x$ 's are greater than all the  $y$ 's.

Corollaries 1 and 2 can easily be generalized to more than two sequences.

**COROLLARY 3.** Let  $\{a_i | 1 \leq i \leq n\}$  be a positive log convex sequence (i.e.,  $a_i^2 \leq a_{i+1}a_{i-1}$ ). Then

$$\frac{\sum_{i=1}^n (s_i a_i)}{\sum_{i=1}^n s_i} \leq \frac{\sum_{i=1}^n a_i}{n}$$

for any positive sequence  $\{s_i\}$  satisfying

- (i)  $s_i = s_{n-i+1}$  for  $1 \leq i \leq n$
- (ii)  $s_1 \leq s_2 \leq \dots \leq s_{\lceil n/2 \rceil}$

*Proof.* Let  $c_{i,j} = a_{i+j-1}$  for  $1 \leq i \leq 2$  and  $1 \leq j \leq n-1$ . Then  $c_{i,j}$  satisfies the condition of Theorem 1. It is also easily seen that any any path from  $c_{1,1}$  to  $c_{2,n-1}$  is the sequence  $\{a_1, a_2, \dots, a_n\}$ :

$$(c_{i,j}) = \begin{bmatrix} a_{n-1} & a_n \\ \vdots & \vdots \\ a_2 & a_3 \\ a_1 & a_2 \end{bmatrix}$$

Hence Theorem 1 implies

$$\frac{a_1 + \dots + a_n}{n} \geq \frac{a_1 + 2(a_2 + \dots + a_{n-1}) + a_n}{2(n-1)}$$

which implies

$$\frac{a_1 + \dots + a_n}{n} \geq \frac{a_2 + \dots + a_{n-1}}{n-2}$$

The sequence  $a_2, \dots, a_{n-1}$  is also log convex, and hence

$$\frac{a_1 + \dots + a_n}{n} \geq \frac{a_2 + \dots + a_{n-1}}{n-2} \geq \frac{a_3 + \dots + a_{n-2}}{n-4} \geq \dots$$

By multiplying these fractions by  $s_1, s_2 - s_1, s_3 - s_2$ , etc. and by repeatedly using the fact that  $s/t \geq u/v$  implies

$$\frac{s}{t} \geq \frac{s+u}{t+v} \geq \frac{u}{v}$$

for positive  $s, t, u, v$ ; the corollary easily follows.

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