F-Divisible modules and tilting modules over Prüfer domains

Luigi Salce

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy

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Abstract

Tilting modules over Prüfer domains are investigated. Tilting torsion classes over these domains correspond bijectively to finitely generated localizing systems of ideals. For each such system \( \mathcal{F} \), a generalized Fuchs divisible module \( \hat{\mathcal{F}} \) is constructed which generates the corresponding tilting torsion class.

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0. Introduction

Three recent papers investigated 1-tilting modules over commutative integral domains. The first one, by Trlifaj and Wallutis \[12\], considered 1-tilting modules over Dedekind domains; the second one, by the author \[10\], considered 1-tilting modules over valuation domains. Both papers made use of Gödel’s axiom of constructibility and additional conditions on the domains when dealing with tilting modules of uncountable torsionfree rank. In the third paper, Bazzoni et al. \[2\] made a substantial progress in the investigation of 1-tilting modules over arbitrary associative rings, proving general results on them; in particular, they proved that every 1-tilting module over a Prüfer domain is of finite type, and gave a new...
version of the main results on 1-tilting modules over Dedekind domains, eliminating extra set-theoretic and ring-theoretic hypotheses.

One further result by Bazzoni [1] clarifies the situation for Prüfer domains; in fact, she proves that general tilting modules over these domains have projective dimension 1, that is, they are 1-tilting. Therefore, when dealing with Prüfer domains, we will simply refer to tilting modules and tilting torsion classes.

In the papers [12,10] quoted above it is shown that there exists a strong connection between the tilting torsion classes and relative divisibility. To be more precise, the 1-tilting torsion classes over a Dedekind domain \( R \) coincide with the classes of \( \Sigma \)-divisible modules, where \( \Sigma \) denotes an arbitrary set of maximal ideals of \( R \); in this setting a module \( M \) is said to be \( \Sigma \)-divisible if \( M = PM \) for every \( P \in \Sigma \).

On the other side, the tilting torsion classes over a valuation domain coincide with the classes of \( S \)-divisible modules, where \( S \) denotes an arbitrary multiplicative set of non-zero elements of the domain.

These results cannot be interchanged in the two situations, in the sense that \( \Sigma \)-divisibility does not work for general valuation domains and \( S \)-divisibility does not work for general Dedekind domains (see [10, Example 4.14]).

In this paper we consider a common generalization of the two situations described above, investigating tilting modules over Prüfer domains. The notion that generalizes \( \Sigma \)-divisibility and \( S \)-divisibility in this setting is that of \( \mathcal{F} \)-divisibility (see [11, VI.9]), where \( \mathcal{F} \) denotes a localizing system of ideals of the domain; \( \mathcal{F} \) is also called a Gabriel topology (see [5,11, p. 146]). With any module one can associate a localizing system of divisibility. It turns out that, over Prüfer domains, the localizing system of divisibility of a tilting module is finitely generated; furthermore, there exists a bijection between finitely generated localizing systems of ideals \( \mathcal{F} \) of the Prüfer domain and tilting torsion classes, which coincide with the classes of \( \mathcal{F} \)-divisible modules. So, by a result by Fontana and Popescu [6], tilting torsion classes over a Prüfer domain \( R \) correspond bijectively to the overrings of \( R \).

The above result is achieved by showing that, given any finitely generated localizing system of ideals \( \mathcal{F} \) of a Prüfer domain \( R \), there is a canonical tilting module associated with it. This module, denoted by \( \hat{\mathcal{F}} \), generalizes both Fuchs’ divisible module \( \hat{\mathcal{F}} \) (see [8, VII.5.1]) and its generalization \( \hat{\mathcal{F}}_S \), investigated in [7] and used in [10].

1. \( \mathcal{F} \)-divisibility and localizing systems of divisibility of tilting modules

In this section \( R \) will always denote a commutative integral domain with 1 and \( R^\times \) the multiplicative monoid of its non-zero elements. Recall that a set \( \mathcal{F} \) of non-zero ideals of a domain \( R \) is a localizing system (or a Gabriel filter) if it satisfies the following two conditions:

(LS1) if \( I \in \mathcal{F} \) and \( I \subseteq J \subseteq R \), then \( J \in \mathcal{F} \);
(LS2) if \( I \in \mathcal{F} \) and \( J \subseteq R \) satisfies \( a^{-1}J \cap R \in \mathcal{F} \) for all \( 0 \neq a \in I \), then \( J \in \mathcal{F} \).

Localizing systems are in bijective correspondence with hereditary torsion theories. According to [5, p. 126], a localizing system \( \mathcal{F} \) is said to be finitely generated (resp., principal) if every ideal \( I \in \mathcal{F} \) contains a finitely generated (resp., principal) ideal \( L \in \mathcal{F} \).
It is well known (see [5, 5.1]) that a localizing system is closed under products, hence also under intersections. Actually, the finitely generated localizing systems of ideals are exactly the finitely generated filters of ideals closed under products. Furthermore, with any localizing system \( \mathcal{F} \) it is associated an overring \( R_\mathcal{F} \) of \( R \), called the ring of fractions with respect to \( \mathcal{F} \), defined by

\[
R_\mathcal{F} = \bigcup_{I \in \mathcal{F}} (R : I).
\]

If the localizing system \( \mathcal{F} \) is finitely generated, then obviously \( R_\mathcal{F} = \bigcup \{(R : I) \mid I \in \mathcal{F}, I \text{ finitely generated} \} \).

Given an \( R \)-module \( M \), denote by \( \mathcal{D}(M) \) the multiplicative set of ideals \( I \) of \( R \) such that \( M = IM \). Denote by \( \mathcal{D}_0(M) \) (respectively, \( \mathcal{D}_p(M) \)) the submonoid of \( \mathcal{D}(M) \) consisting of those ideals \( I \) which contain a finitely generated (respectively, principal) ideal \( J \) such that \( JM = M. \mathcal{D}(M) \) (respectively, \( \mathcal{D}_0(M), \mathcal{D}_p(M) \)) is called the localizing system of divisibility of \( M \) (respectively, finitely generated, principal localizing system of divisibility of \( M \)). The localizing system of divisibility \( \mathcal{D}(M) \) of the \( R \)-module \( M \) is finitely generated (resp., principal) exactly if \( \mathcal{D}(M) = \mathcal{D}_0(M) \) (resp., \( \mathcal{D}(M) = \mathcal{D}_p(M) \)).

The above terminology is justified by the following.

**Lemma 1.1.** Given a domain \( R \) and an \( R \)-module \( M \), \( \mathcal{D}(M), \mathcal{D}_0(M) \) and \( \mathcal{D}_p(M) \) are localizing systems.

**Proof.** Condition (LS1) is trivially satisfied for the three systems of ideals. Assume that \( I \in \mathcal{D}(M) \) and \( a^{-1}J \cap R \in \mathcal{D}(M) \) for all \( 0 \neq a \in I \); then \( (a^{-1}J \cap R)M = M \), so \( (J \cap aR)M = aM \) for all \( a \in I \). There follows that \( JM \subseteq JM \), hence \( JM = M \) and consequently \( J \in \mathcal{D}(M) \). Assume now that \( I \in \mathcal{D}_0(M) \) and \( a^{-1}J \cap R \in \mathcal{D}_0(M) \) for all \( 0 \neq a \in I \); then \( I \) contains a finitely generated ideal \( L \) such that \( LM = M \). Let \( a_1, \ldots, a_k \) be non-zero generators of \( L \); then \( a_i^{-1}J \cap R \) contains a finitely generated ideal \( J_i \) such that \( J_iM = M \) for all \( i \). Thus \( J \geq a_iJ_i \) for all \( i \), hence \( J \geq LJ_1 \cdots J_k \), which is finitely generated and satisfies \( LJ_1 \cdots J_k M = M \). Therefore \( J \in \mathcal{D}_0(M) \). The proof for \( \mathcal{D}_p(M) \) is similar. \( \square \)

If \( R \) is a valuation domain, \( \mathcal{D}_0(M) = \mathcal{D}_p(M) \) for every module \( M \), and it is easy to find examples for the strict inclusion \( \mathcal{D}_0(M) \subset \mathcal{D}(M) \) (take \( M = P \), where \( P = P^2 \) is the idempotent maximal ideal of \( R \)). If \( R \) is a Dedekind (or Noetherian) domain, then \( \mathcal{D}_0(M) = \mathcal{D}(M) \); an example of \( R \)-module \( T \) such that \( \mathcal{D}_p(T) = \{R\} \) and \( \mathcal{D}_0(T) = \{P^n \mid n \geq 0\} \), where \( P \) is a maximal ideal with \( P^n \) not principal for every \( n \geq 1 \), is given in [10, Example 4.14].

We introduce now two notions that are of main importance in this paper.

**Definition.** Given a localizing system \( \mathcal{F} \) of a domain \( R \), an \( R \)-module \( M \) is said to be \( \mathcal{F} \)-divisible if \( M = IM \) for every ideal \( I \in \mathcal{F} \). The module \( M \) is called \( h_\mathcal{F} \)-divisible if it is an epimorphic image of a direct sum of copies of \( R_\mathcal{F} \).
A warning for the reader is in order. Our definition of $F$-divisible modules coincides with that given in the more general context of modules over non-commutative rings in [11, p. 155]. It is different and independent from the definition of $\tau$-divisible modules given in [9, p. 117], where $\tau$ is the hereditary torsion theory associated with $F$.

If the localizing system $F$ is principal, then a module is $F$-divisible exactly if it is $S$-divisible, where $S$ is the submonoid of $R^\times$ consisting of the elements $r \in R$ such that $rR \in F$. In this case clearly $R_F = RS$.

The class of $F$-divisible modules is closed under epimorphic images, direct sums and extensions, thus it is a torsion class. It is not closed, in general, under direct products and submodules. The class of $h_F$-divisible modules is closed under epimorphic images and direct sums; it is not closed, in general, under extensions (look at the failure of this property for $h$-divisible modules).

Lemma 1.2. Let $F$ be a localizing system of a domain $R$.

(1) If $R_F$ is a flat $R$-module, then $h_F$-divisible modules are also $F$-divisible.
(2) If $M$ is a torsionfree $R$-module and $F \subseteq D(M)$, then $M$ is in a natural way a torsionfree $R_F$-module.
(3) If $0 \to A \to B \to C \to 0$ is an exact sequence of torsionfree $R$-modules such that $F \subseteq D(A) \cap D(C)$, then $B$ is a torsionfree $R_F$-module and the sequence is an exact sequence of $R_F$-modules.

Proof. (1) By [5, 5.1.10], $R_F$ is $F$-divisible, so the claim is obvious.
(2) Given $m \in M$ and $x = a/b \in R_F$, there exists an ideal $I$ of $R$ such that $xI \leq R$ and $M = IM$. So $m = a_1m_1 + \ldots + a_km_k$ $(a_j \in I, m_j \in M)$, hence we set $xm = (xa_1)m_1 + \ldots + (xa_k)m_k$, where $xa_j \in R$ for all $j$. Torsionfreeness ensures that this is a good definition and that $M$ is a torsionfree $R_F$-module.
(3) It is easy to check that $F \subseteq D(B)$, hence $B$ is a torsionfree $R_F$-module; the proof of the last claim is straightforward. □

Invertible ideals belonging to the localizing system of divisibility $D(M)$ of an $R$-module $M$ are related to the vanishing of certain modules of extensions.

Lemma 1.3. Let $I$ be an invertible ideal of the domain $R$ and $M$ an $R$-module. The following conditions are equivalent:

(1) $I \in D(M)$ (that is, $IM = M$);
(2) $\text{Ext}_R^1(R/I, M) = 0$;
(3) $\text{Ext}_R^1(I^{-1}/R, M) = 0$.

Proof. (1) $\Rightarrow$ (2) We will show that every homomorphism $\phi: I \to M$ extends to a homomorphism $\psi: R \to M$. We imitate the proof of [8, 1.7.2] with a slight modification. Let $1 = a_1x_1 + \ldots + a_kx_k$, where $a_i \in I$ and $x_i \in I^{-1}$ for all $i$, as above. From $M = IM$ we deduce that, for every $i$, $\phi(a_i) = b_{i1}m_{i1} + \ldots + b_{ir}m_{ir}$ for suitable elements $b_{ij} \in I$ and...
$m_{ij} \in M$. Set $m = \sum x_i \phi(a_i)$ and let $\psi(1) = m$. Then for all $a \in I$ we get:
\[
\phi(a) = \phi(aa_1 x_1 + \ldots + aa_k x_k) = ax_1 \phi(a_1) + \ldots + ax_k \phi(a_k) = \sum_{i} ax_i \phi(a_i) = am = \psi(a)
\]
hence $\psi$ extends $\phi$.

$(2) \Rightarrow (1)$ We must show that, fixed any $m \in M, m \in IM$. Let $l = a_1 x_1 + \ldots + a_k x_k$, where $a_i \in I$ and $x_i \in I^{-1}$ for all $i$. For each $i$ define a map $\phi_i: I \rightarrow M$ as follows: $\phi_i(a) = ax_i m (a \in I)$. Clearly $\phi_i$ is a homomorphism, so it can be extended to a homomorphism $\psi_i: R \rightarrow M$. Now we have
\[
m = a_1 x_1 m + \ldots + a_k x_k m = \phi_1(a_1) + \ldots + \phi_k(a_k)
\]
\[
= a_1 \psi_1(1) + \ldots + a_k \psi_k(1)
\]
hence $m \in IM$.

$(1) \Leftrightarrow (3)$ is obvious, since $\operatorname{Ext}^1_R(I^{-1}/R, M) \cong M/IM$, by [8, I. Exercise 5.5 (c)].

In [12], given an ideal $I$ of a domain $R$, an $R$-module $M$ is said to be $I$-divisible if $\operatorname{Ext}^1_R(R/I, M) = 0$. If this happens for every ideal $I \in \mathcal{F}$, where $\mathcal{F}$ is a localizing system, then usually $M$ is called $\mathcal{F}$-injective (see [11, p. 198]).

In general, given a localizing system $\mathcal{F}$ of ideals of a domain $R$, the two notions of $\mathcal{F}$-divisible and $\mathcal{F}$-injective modules are not related to each other. For instance, if $R$ is valuation domain with idempotent maximal ideal $P$, let $\mathcal{F} = \{R, P\}$. Then $\mathcal{F} = \mathcal{D}(P)$ and the module $M = P$ is $\mathcal{F}$-divisible but not $\mathcal{F}$-injective, since $\operatorname{Ext}^1_R(R/P, M) \neq 0$. Conversely, the module $N = R$ is $\mathcal{F}$-injective, since $\operatorname{Ext}^1_R(R/P, N) \cong (Q/R)[P] = 0$, but $N$ is clearly not $\mathcal{F}$-divisible. However, as an immediate consequence of the preceding lemma, and recalling that finitely generated ideals in Prüfer domains are invertible, we have the following

**Corollary 1.4.** Let $\mathcal{F}$ be a finitely generated localizing system of ideals of a Prüfer domain $R$, $\mathcal{F}_0$ its submonoid of invertible ideals. Then an $R$-module $M$ is $\mathcal{F}$-divisible if and only if $\operatorname{Ext}^1_R(R/I, M) = 0$ for all $I \in \mathcal{F}_0$. Hence an $\mathcal{F}$-injective $R$-module is $\mathcal{F}$-divisible.

The converse of Corollary 1.4 is not generally true; in fact, if $\mathcal{F}$ consists of all the non-zero ideals of $R$, $\mathcal{F}$-divisibility and $\mathcal{F}$-injectivity coincide with the usual notions of divisibility and injectivity, which coincide only over Dedekind domains.

We already recalled that a localizing system of ideals $\mathcal{F}$ of a domain $R$ gives rise to a hereditary torsion class in $\text{Mod}(R)$, consisting of those $R$-modules $M$ such that $\text{Ann}_R(x) \in \mathcal{F}$ for every element $x \in M$. The modules in this torsion class are called $\mathcal{F}$-torsion modules, according to [11, p. 146].

In the following, given a module $M$, we shall denote by $t(M)$ its usual torsion submodule and by $M$ the quotient module $M/t(M)$.

We recall also the definition of 1-tilting modules as formulated in [3]: a 1-tilting module (over an arbitrary ring $R$) is a module $T$ satisfying the following three conditions:

(T1) $\operatorname{p.d.} T \leq 1$;

(T2) $\operatorname{Ext}^1_R(T, T^{(\kappa)}) = 0$ for all cardinals $\kappa$;
(T3) there exists an exact sequence \( 0 \to R \to T_1 \to T_2 \to 0 \), where \( T_1, T_2 \in \text{Add}(T) \), the class of direct summands of direct sums of copies of \( T \).

We will use the characterization in [3] which says that \( T \) is 1-tilting exactly if \( \text{Gen}(T) = T^\perp \), where \( \text{Gen}(T) \) is the class of modules which are quotients of direct sums of copies of \( T \), and \( T^\perp = \{ M \mid \text{Ext}^1_R(T, M) = 0 \} \). Another result proved in [3] and used later on states that, given two 1-tilting modules \( T_1 \) and \( T_2 \), \( T_1 \in \text{Add}(T_2) \) if and only if \( T_2 \in \text{Add}(T_1) \), if and only if \( \text{Gen}(T_1) = \text{Gen}(T_2) \).

From now on we will consider modules over Prüfer domains. As mentioned in the introduction, all tilting modules over Prüfer domains are 1-tilting (see [1]); thus there is no danger of confusion in using the term “tilting” in our setting.

**Proposition 1.5.** Let \( R \) be a Prüfer domain and \( T \) a tilting \( R \)-module. Then the torsion part \( t(T) \) of \( T \) is a \( \mathcal{D}_0(T) \)-torsion module and a \( \mathcal{D}_0(T) \)-divisible module.

**Proof.** Let us assume that \( 0 \neq a \in t(T) \). Then \( \text{p.d.}(T/aR) \leq 1 \) and \( \text{p.d.}aR \leq 1 \), by [8, VI.6.4], hence \( aR \) is finitely presented, by [8, VI.6.2], thus \( aR \) is isomorphic to \( R/I \) for some invertible ideal \( I = \text{Ann}_R(a) \) of \( R \). From the exact sequence \( 0 \to aR \to T \to T/aR \to 0 \) we get the exact sequence

\[
0 = \text{Ext}^1_R(R/I, T) \to \text{Ext}^1_R(aR, T) \to \text{Ext}^2_R(T/aR, T) = 0,
\]

hence \( \text{Ext}^1_R(R/I, T) = 0 \). By Lemma 1.3, \( T = IT \), hence \( I = \text{Ann}_R(x) \in \mathcal{D}_0(T) \).

Let now \( I \in \mathcal{D}_0(T) \). There exists a finitely generated ideal \( J \leq I \) such that \( T = JT \). By Lemma 1.3 we have \( \text{Ext}^1_R(R/J, T) = 0 \). Since \( \text{Hom}_R(R/J, T/t(T)) = 0 \), we deduce that \( \text{Ext}^1_R(R/J, t(T)) = 0 \). Therefore \( t(T) = JT(T) = IT(T) \). \( \square \)

The next two lemmas deal with the tensor product of \( \mathcal{F} \)-torsion modules by \( R_{\mathcal{F}} \), for \( \mathcal{F} \) a finitely generated localizing system, and an application to 1-tilting modules.

**Lemma 1.6.** Let \( R \) be a Prüfer domain, \( \mathcal{F} \) a finitely generated localizing system of ideals of \( R \), and \( M \) an \( \mathcal{F} \)-torsion \( R \)-module. Then \( M \otimes_R R_{\mathcal{F}} = 0 \).

**Proof.** It follows by [11, VI.9.1], since \( R_{\mathcal{F}} \) is \( \mathcal{F} \)-divisible. \( \square \)

**Lemma 1.7.** Let \( T \) be a tilting module over the Prüfer domain \( R \), and \( A \in \text{Add}(T) \). Then \( t(A) \otimes_R R_{\mathcal{D}_0(T)} = 0 \), \( A \otimes_R R_{\mathcal{D}_0(T)} \cong \tilde{A} \), and \( \text{p.d.}_{R_{\mathcal{D}_0(T)}} \tilde{A} \leq 1 \).

**Proof.** The first equality immediately follows from Propositions 1.5 and Lemma 1.6; the latter isomorphism follows by tensoring the exact sequence \( 0 \to t(A) \to A \to \tilde{A} \to 0 \) by \( R_{\mathcal{D}_0(T)} \) and recalling that \( \tilde{A} \) is an \( R_{\mathcal{D}_0(T)} \)-module, by Lemma 1.2. The inequality \( \text{p.d.}_{R_{\mathcal{D}_0(T)}} \tilde{A} \leq 1 \) is obtained by tensoring by \( R_{\mathcal{D}_0(T)} \) a projective resolution of the \( R \)-module \( A \). \( \square \)

We can now prove the main result of this section.
Theorem 1.8. Let \( R \) be a Prüfer domain and \( T \) a tilting \( R \)-module. Then the localizing system of divisibility \( \mathcal{D}(T) \) of \( T \) is finitely generated.

Proof. Let \( I \) be an ideal of \( R \) such that \( T = IT \). Tensoring the exact sequence \( 0 \rightarrow R \rightarrow T_1 \rightarrow T_2 \rightarrow 0 \) (\( T_i \in \text{Add}(T) \)) with \( R_{\mathcal{D}(T)} \), by Lemma 1.7 we get the exact sequence \( 0 \rightarrow R_{\mathcal{D}(T)} \rightarrow \tilde{T}_1 \rightarrow \tilde{T}_2 \rightarrow 0 \). Clearly \( \tilde{T}_1 = IT \), hence, by [8, VI.9.5], \( I R_{\mathcal{D}(T)} = IT \cap R_{\mathcal{D}(T)} = R_{\mathcal{D}(T)} \). Obviously, there exists a finitely generated ideal \( J \) contained in \( I \) such that \( JR_{\mathcal{D}(T)} = R_{\mathcal{D}(T)} \). Then \( 1 = j_1 x_1 + \ldots + j_h x_h \) (\( j_i \in J, x_i \in R_{\mathcal{D}(T)} \)). We will prove that \( JT = T \).

If \( t \in T \), then \( t = 1 t = j_1 x_1 t + \ldots + j_h x_h t \); setting \( x_i = a_i/b_i \) for every index \( i (a_i, b_i \in R) \), we must show that \( j_i(a_i/b_i)t \in JT \) for all \( i \). Since \( a_i/b_i \in R_{\mathcal{D}(T)} \), there exists an ideal \( I_i \) such that \( I_i T = T \) and \( a_i I_i \leq b_i R \). Thus \( t = r_1 t_1 + \ldots + r_n t_n \) for suitable elements \( r_1, \ldots, r_n \in I_i \) and \( t_1, \ldots, t_n \in T \). There follows that \( j_i(a_i/b_i)t = j_i(a_i/b_i)r_1 t_1 + \ldots + j_i(a_i/b_i)r_n t_n \), where \( (a_i/b_i)r_h = s_h \in R \) for all \( h \leq n \). Consequently \( j_i(a_i/b_i)t = j_i s_1 t_1 + \ldots + j_i s_n t_n \in JT \), as desired. \( \square \)

2. The structure of tilting modules over Prüfer domains

Let \( R \) be an arbitrary ring. If \( T \) is a 1-tilting \( R \)-module, then \( T^\perp = \text{Gen}(T) \) is a torsion class, and a torsion class \( \mathcal{T} \) of \( R \)-modules is called a 1-tilting torsion class if there exists a 1-tilting \( R \)-module \( T \) such that \( \mathcal{T} = T^\perp \). The 1-tilting module \( T \), and the 1-tilting torsion class \( T^\perp \), are called of finite type if there exists a set \( \mathcal{S} \) of finitely presented \( R \)-modules such that \( T^\perp = \mathcal{S}^\perp \), where

\[
\mathcal{S}^\perp = \{ M \in \text{Mod}(R) \mid \text{Ext}_R^1(S, M) = 0 \text{ for all } S \in \mathcal{S} \}.
\]

It is well known that a 1-tilting torsion class of finite type is closed under taking pure submodules (equivalently, it is “definable”). One of the main results in [2], which is of interest in our setting, is the following theorem, that can be obtained by looking at Theorem 3.4 and its proofs in [2] (recall that tilting modules over Prüfer domains are 1-tilting).

Theorem 2.1 (Bazzoni et al. [2]). Every tilting torsion class \( \mathcal{T} \) over a Prüfer domain is of finite type: \( \mathcal{T} = \mathcal{S}^\perp \), where \( \mathcal{S} \) is the set of the cyclic finitely presented \( R \)-modules belonging to \( \mathcal{D}(T)^\perp \).

Remark that from the results quoted above it follows that every tilting torsion class over a Prüfer domain is closed under taking pure submodules. This result implies that Proposition 4.6, Theorems 4.11 and 4.13 in [10], proved for tilting modules \( T \) over valuation domains \( R \), remain true without assuming \( T \) of countable rank, or \( V = L \) and \( |\mathcal{R}| \leq 2^{\aleph_0} \). Furthermore, we can say a little more in the next Corollary, where \( \tau \) denotes the hereditary torsion theory associated with the localizing system \( \mathcal{D}(T) \).

Corollary 2.2. Let \( \mathcal{T} = T^\perp \) be a tilting torsion class over a Prüfer domain \( R \). Then \( \mathcal{T} \) coincides with the class of the \( \mathcal{D}(T) \)-divisible \( R \)-modules and it is closed under taking \( \tau \)-pure submodules.
Proof. Since every $R$-module generated by $T$ is $\mathcal{D}(T)$-divisible, it is enough to show that, conversely, a module $M$ which is $\mathcal{D}(T)$-divisible belongs to $T^\perp$. By Lemma 1.3, $\text{Ext}^1_R(R/I, M) = 0$ for every finitely generated ideal $I$ such that $T = IT$. By Theorem 2.1, $\mathcal{D}$ is the set of the cyclic modules of the form $R/I$, for $I$ an invertible ideal of $R$ such that $T = IT$. Consequently $M \in S^\perp = \mathcal{D}$.

Let now $N$ be a $\tau$-pure submodule of a $\mathcal{D}(T)$-divisible module $M$. This means that $\text{Hom}_R(R/I, M/N) = 0$ for all $I \in \mathcal{D}(T)$. If $I$ is a finitely generated ideal in $\mathcal{D}(T)$, then from the exact sequence

$$0 \to \text{Hom}_R(R/I, M/N) \to \text{Ext}^1_R(R/I, N) \to \text{Ext}^1_R(R/I, M) = 0$$

we deduce that $\text{Ext}^1_R(R/I, N) = 0$, so $N$ is $\mathcal{D}(T)$-divisible by Corollary 1.4. □

The converse of Corollary 2.2, namely, the fact that for every finitely generated localizing system of ideals $\mathcal{F}$ of a Prüfer domain, the class of the $\mathcal{F}$-divisible modules is a tilting torsion class, will be proved at the end of this section.

The next lemma can be obtained as a consequence of Corollary 2.2, since the ring $R\mathcal{D}(T)$ is $\mathcal{D}(T)$-divisible (see [5, 5.1.11]). We furnish an alternative easy proof, applying the closure property of tilting torsion classes under taking pure submodules.

**Lemma 2.3.** Let $R$ be a Prüfer domain and $T$ a tilting $R$-module. Then $R\mathcal{D}(T) \in T^\perp$.

**Proof.** As in the proof of Theorem 1.8, we have the exact sequence $0 \to R\mathcal{D}(T) \to \tilde{T}_1 \to \tilde{T}_2 \to 0$ with $T_i \in \text{Add}(T)$. The torsionfreeness of $\tilde{T}_2$ implies that $R\mathcal{D}(T)$ is pure in $\tilde{T}_1$, and since $\tilde{T}_1 \in \text{Gen}(T) = T^\perp$, the conclusion follows. □

The next lemma deals with modules over arbitrary rings.

**Lemma 2.4.** Let $R$ be any ring and $M$ an $R$-module such that $p.d. M \leq 1$, $\text{Ext}^1_R(M, M^{(\kappa)}) = 0$ for all cardinals $\kappa$, and with a direct summand isomorphic to $R$. Then $M$ is projective.

**Proof.** By the Eilenberg’s trick, we have an exact sequence

$$0 \to \bigoplus_{j \in J} R_j \to \bigoplus_{i \in I} R_i \to M \to 0,$$

where $R_j, R_i = R$ for every $j, i$. So we have the induced exact sequence

$$\text{Hom}_R\left( \bigoplus_{i \in I} R_i, \bigoplus_{j \in J} M_j \right) \to \text{Hom}_R\left( \bigoplus_{j \in J} R_j, \bigoplus_{j \in J} M_j \right) \to \text{Ext}^1_R\left( M, \bigoplus_{j \in J} M_j \right) = 0,$$

where $M_j = M$ for all $j$. Hence every map $\bigoplus_{j \in J} R_j \to \bigoplus_{j \in J} M_j$ extends to a map $\bigoplus_{i \in I} R_i \to \bigoplus_{j \in J} M_j$. In particular, the map $\phi$ sending each $R_j$ isomorphically onto the summand isomorphic to $R$ of the $j$th copy of $M$ extends to a map $\psi: \bigoplus_{i \in I} R_i \to \bigoplus_{j \in J} M_j$. But clearly there exists a map $\alpha: \bigoplus_{j \in J} M_j \to \bigoplus_{j \in J} R_j$ such that $\alpha \cdot \phi$ equals the identity.
map of $\bigoplus_{j \in J} R_j$, and $\alpha \cdot \psi$ is a splitting map for the exact sequence (1), whence $M$ is projective. \(\square\)

We can now prove easily the next result, which gives a necessary condition for the torsionfree quotient of a tilting module.

**Theorem 2.5.** Let $R$ be a Prüfer domain and $T$ a tilting $R$-module. Then $\bar{T}$ is a projective $R_{\mathcal{D}(T)}$-module.

**Proof.** Obviously $\bar{T}$ is a (torsionfree) $R_{\mathcal{D}(T)}$-module, since $\mathcal{D}(T) \subseteq \mathcal{D}(\bar{T})$ and in view of Lemma 1.2. By Lemma 2.3 there exists an epimorphism $\bigoplus T \to R_{\mathcal{D}(T)}$ which induces an epimorphism $\bigoplus \bar{T} \to R_{\mathcal{D}(T)}$. Since $\bigoplus \bar{T}$ is a torsionfree $R_{\mathcal{D}(T)}$-module, we deduce that $R_{\mathcal{D}(T)}$ is isomorphic to a summand of $\bigoplus \bar{T}$. In order to apply Lemma 2.4 to the ring $R_{\mathcal{D}(T)}$ and the $R_{\mathcal{D}(T)}$-module $\bigoplus \bar{T}$, it is enough to prove that $p.d. R_{\mathcal{D}(T)} \bar{T} \leq 1$ and that $\text{Ext}^1_{R_{\mathcal{D}(T)}}(\bar{T}, \bar{T}^{(k)}) = 0$ for any cardinal $k$. The first inequality follows by Lemma 1.6 and Theorem 1.8; furthermore, we have the exact sequence

$$0 = \text{Ext}^1_R(T, T^{(k)}) \to \text{Ext}^1_R(T, \bar{T}^{(k)}) \to \text{Ext}^2_R(T, t(T)^{(k)}) = 0,$$

hence $\text{Ext}^1_R(T, \bar{T}^{(k)}) = 0$. We also have the exact sequence

$$0 = \text{Hom}_R(t(T), \bar{T}^{(k)}) \to \text{Ext}^1_R(\bar{T}, \bar{T}^{(k)}) \to \text{Ext}^1_R(T, \bar{T}^{(k)}) = 0,$$

whence the middle term vanishes. Since $\text{Ext}^1_R(\bar{T}, \bar{T}^{(k)}) = \text{Ext}^1_{R_{\mathcal{D}(T)}}(\bar{T}, \bar{T}^{(k)})$ by Lemma 1.2, we are done. \(\square\)

An immediate consequence of Theorem 2.5, that derives also from Theorem 2.1 and condition (T3), is the following

**Corollary 2.6.** A torsionfree tilting module $T$ over a Prüfer domain is projective.

**Proof.** By Theorem 2.5, it is enough to prove that $R_{\mathcal{D}(T)} = R$. From the exact sequence $0 \to R \to T_1 \to T_2 \to 0$ ($T_i \in \text{Add}(T)$) we deduce that $R$ is pure in $T_1$, hence $R \in T^\perp$, by the remark after Theorem 2.1. This obviously implies that the only ideal in the localizing system $\mathcal{D}(T)$ is $R$, so the claim follows. \(\square\)

In the following proposition it is used the fact that every overring $S$ of a Prüfer domain $R$ is of the form $R_{\mathcal{F}}$, for a suitable finitely generated localizing system $\mathcal{F}$; actually, $\mathcal{F} = \mathcal{D}(S)$ (see [5, 5.1.10]).

**Proposition 2.7.** Let $R$ be a Prüfer domain and $S$ an overring of $R$ such that $p.d. R_S \leq 1$. Then

1. $\text{Ext}^1_R(S, D) = 0$ for every $h_{\mathcal{F}}$-divisible module $D$;
2. The $R$-module $S \oplus (S/R)$ is tilting.
Proof. (1) The proof is similar to that of [7, 3.1], by making use of Lemma 1.2, so it is left to the reader. (2) the conditions (T1) and (T3) are trivially satisfied. In order to verify (T2), we must check that

\[ \text{Ext}^1_R(S, S^{(k)}) = 0 = \text{Ext}^1_R(S/R, S^{(k)}). \]

The first equalities follow from point (1). We prove now the third equality. Let \( S = R_F \), which is the union of the \( R \)-submodules \( R : I \), ranging \( I \) in the set \( \mathcal{F}_0 \) of the finitely generated ideals of \( \mathcal{F} \). We have an exact sequence

\[ 0 \to H \to \bigoplus_{I \in \mathcal{F}_0} (R : I)/R \to S/R \to 0. \]

From this sequence we get the exact sequence

\[ \text{Hom}_R(H, S^{(k)}) \to \text{Ext}^1_R(S/R, S^{(k)}) \to \prod_I \text{Ext}^1_R((R : I)/R, S^{(k)}). \]

The first \( \text{Hom} \) is 0, since \( H \) is a torsion module, and the last \( \text{Ext} \) is also 0, by Lemma 1.3, and since \( S = IS \), by [5, 5.1.10], so the third equality holds. Finally, the last equality follows from the third one and the exact sequence \( \text{Ext}^1_R(S/R, S^{(k)}) \to \text{Ext}^1_R(S/R, (S/R)^{(k)}) \to \text{Ext}^2_R(S/R, R^{(k)}) = 0. \)

Our next goal is to prove the announced result which classifies the tilting torsion classes over a Prüfer domain \( R \) by the overrings of \( R \). By a well-known result by Fontana and Popescu [6] (see also [5, Theorem 5.1.15]), this amounts to classify the tilting torsion classes by means of the finitely generated localizing systems of ideals of \( R \). This is achieved by associating with every finitely generated localizing system of ideals \( \mathcal{F} \) a canonical tilting module. In case p.d. \( R \) \( \leq 1 \), a tilting module is already available by Proposition 2.7, namely, \( T = R \oplus (R \mathcal{F}/R) \). Since Gen(\( T \)) = Gen(\( R \mathcal{F}/R \)) is the class of the \( \mathcal{F} \)-divisible modules, by Corollary 2.2, it follows as a by-product that, when p.d. \( R \) \( \leq 1 \), every \( \mathcal{F} \)-divisible is actually \( h \mathcal{F} \)-divisible.

In [10] it was proved that, when dealing with valuation domains, the canonical tilting module is the module \( \hat{\mathcal{S}} \) investigated in [7], where \( S \) is the multiplicative system of elements of \( R \) which generates the principal localizing system of ideals \( \mathcal{F} \). So, for Prüfer domains, it is natural to try to generalize the module \( \hat{\mathcal{S}} \), starting with a finitely generated localizing system \( \mathcal{F} \) which is, in general, not principal.

So let \( R \) be a Prüfer domain and \( \mathcal{F} \) a finitely generated localizing system of ideals of \( R \); denote by \( \mathcal{F}_0 \) the set of the finitely generated (invertible) ideals in \( \mathcal{F} \). Consider the index set

\[ \Lambda = \{ (I_1, \ldots, I_k) \mid k \geq 1, I_i \in \mathcal{F}_0 \} \cup \{ \emptyset \}. \]

For every \( \lambda = (I_1, \ldots, I_k) \in \Lambda \) consider an \( R \)-module \( G_{\lambda} \cong I_1^{-1} \cdot \ldots \cdot I_k^{-1} \) and fix an isomorphism \( \varphi_{\lambda}: I_1^{-1} \cdot \ldots \cdot I_k^{-1} \to G_{\lambda} \). Furthermore, we consider a module \( G_\emptyset \) isomorphic to
$R$, and fix an isomorphism $\varphi_0: R \to G_\emptyset$, setting $w = \varphi_0(1)$. Let $\lambda \in A \setminus \emptyset, \lambda = (I_1, \ldots, I_{k+1})$; if $k \geq 1$ we set $\lambda^- = (I_1, \ldots, I_k)$, and if $k = 0$ we set $\lambda^- = \emptyset$.

We define the module $\partial_\emptyset$ as the quotient module

$$\partial_\emptyset = \bigoplus_{\lambda \in A} G_\lambda / K,$$

where $K$ is the submodule of $\bigoplus_{\lambda \in A} G_\lambda$ generated by all the elements of the form

$$\varphi_\lambda^-(x) = \varphi_\lambda(x),$$

such that $\lambda = (I_1, \ldots, I_{k+1}) \in A$, and $0 \neq x \in I_1^{-1} \cdot \ldots \cdot I_k^{-1}$ if $k \geq 1$, while $0 \neq x \in R$ if $k = 0$. This makes sense because of the inclusions

$$I_1^{-1} \cdot \ldots \cdot I_k^{-1} \subseteq I_1^{-1} \cdot \ldots \cdot I_k^{-1} I_{k+1}^{-1}, \quad R \subseteq I_1^{-1}.$$

**Remark.** The way the module $\partial_\emptyset$ has been defined amounts to make certain identifications in the module $R \bigoplus (\bigoplus_{k \in \mathbb{F}_0} I_1^{-1} \cdot \ldots \cdot I_k^{-1})$. Given ideals $I_1, \ldots, I_k \in \mathbb{F}_0$, for each $J \in \mathbb{F}_0$ the inclusion $I_1^{-1} \cdot \ldots \cdot I_k^{-1} \subseteq I_1^{-1} \cdot \ldots \cdot I_k^{-1} J^{-1}$ holds; we identify the common submodule $I_1^{-1} \cdot \ldots \cdot I_k^{-1}$ in each summand of the direct sum $\bigoplus_{J} I_1^{-1} \cdot \ldots \cdot I_k^{-1} J^{-1}$. Similarly, we identify the common submodule $R$ in each summand of the direct sum $\bigoplus_{J} J^{-1}$.

The next properties of the module $\partial_\emptyset$ are similar to those of the Fuchs’ divisible module $\partial$, but their proofs require some modification.

1. p.d.$\partial_\emptyset \leq 1$.

There is an ascending sequence of submodules of $\partial_\emptyset$

$$\partial_0 \leq \partial_1 \leq \ldots \leq \partial_n \leq \ldots,$$

where $\partial_n = \bigoplus_{k \leq n} (G_\lambda | \lambda = (I_1, \ldots, I_k), I_i \in \mathbb{F}_0) + K / K$ (if $k = 0$ we mean that $\lambda = \emptyset$). We will show that $\partial_0 = G_\emptyset + K / K \cong R, \partial_1 / \partial_0 \cong \bigoplus_{J \in \mathbb{F}_0} J^{-1} / R$ and, for $n \geq 1$:

$$\partial_{n+1} / \partial_n \cong \bigoplus_{I_i \in \mathbb{F}_0} I_1^{-1} \cdot \ldots \cdot I_n^{-1} I_{n+1}^{-1} / I_1^{-1} \cdot \ldots \cdot I_n^{-1}.$$

To prove that $\partial_0 \cong R$ we must show that $G_\emptyset \cap K = 0$. Note that every element $0 \neq g \in K$ can be written as a finite sum of generating elements of $K$:

$$g = (\varphi_{\lambda_1^-}(x_1) - \varphi_{\lambda_1}(x_1)) + \ldots + (\varphi_{\lambda_n^-}(x_n) - \varphi_{\lambda_n}(x_n)).$$

Assume, by way of contradiction, that $0 \neq g \in G_\emptyset \cap K$ is written in this form with $n$ minimal; clearly $n > 1$ and we can assume that $\lambda_1^- = \emptyset$. $\lambda_1$ must appear in another summand of $g$; if this summand is of the form $\varphi_{\lambda_1^-}(x_1) - \varphi_{\lambda_1}(x_1)$, we can write $g$ in a shorter form, since

$$(\varphi_{\lambda_1^-}(x_1) - \varphi_{\lambda_1}(x_1)) + (\varphi_{\lambda_i^-}(x_i) - \varphi_{\lambda_i}(x_i)) = \varphi_{\lambda_i^-}(x_1 + x_i) - \varphi_{\lambda_i}(x_1 + x_i).$$
and this contradicts the minimality of $n$. Hence we can assume that $\lambda_2^* = \lambda_1$. The same argument applies to prove that $g$ has a third summand of the form $\alpha_{3, n}(x_3) - \alpha_{3, n}(x_3)$ with $\lambda_3^* = \lambda_2$. Since this process must finish in a finite number of steps, at the end we find in $g$ the term $\alpha_{n, n}(x_n)$ with $\lambda_n$ appearing only in this summand, a contradiction. Hence $G_0 \cap K = 0$.

Consider now the isomorphisms (writing $\bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)]$) we mean that the $I_i$’s vary in $F_0$:

$$\partial_{n+1}/\partial_n \cong \bigoplus_{k \leq n+1} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)] + K / \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)] + K$$

\begin{equation}
\cong \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_{n+1})] \cap \bigg( \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)] + K \bigg).
\end{equation}

It is enough to verify that, in the isomorphism

$$\bigoplus_{I_i \in F_0} I_i^{-1} \ldots I_n^{-1} \rightarrow \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_{n+1})],$$

the submodule $\bigoplus_{I_i \in F_0} I_i^{-1} \ldots I_n^{-1}$ is mapped on to the submodule

$$\bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_{n+1})] \cap \bigg( \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)] + K \bigg).$$

If $0 \neq x \in I_1^{-1} \ldots I_n^{-1} \subseteq I_1^{-1} \ldots I_{n+1}$ and $\lambda = (I_1, \ldots, I_{n+1})$, then $\alpha_{\lambda}(x) = \alpha_{\lambda}(x) - (\alpha_{\lambda}(x) - \alpha_{\lambda}(x))$, where $\alpha_{\lambda}(x) \in \bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)]$ and $\alpha_{\lambda}(x) - \alpha_{\lambda}(x) \in K$.

Whence $\bigoplus_{I_i \in F_0} I_i^{-1} \ldots I_n^{-1}$ in the desired submodule.

Conversely, assume that $g = \alpha_{I_1}(x_1) + \ldots + \alpha_{I_n}(x_n)$ belongs to the submodule $\bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_{n+1})]$ (with the $\lambda_j$ different indices of the form $\lambda_j = (I_1, \ldots, I_{n+1})$). If $g$ belongs also to the submodule $\bigoplus_{k \leq n} [G_{\lambda} | \lambda = (I_1, \ldots, I_k)] + K$, one can easily check that each $x_j \in I_1^{-1} \ldots I_n^{-1}$, so we are done.

We have seen that the quotient modules $\partial_{n+1}/\partial_n$ have projective dimension $\leq 1$ for all $n$, hence p.d. $\partial_\mathcal{F} \leq 1$ by the Auslander’s lemma (see [7, VI.2.6]).

II) $\partial_\mathcal{F}$ is an $\mathcal{F}$-divisible module.

It is enough to show that every generating submodule $\tilde{G}_\lambda = G_\lambda + K/K$ satisfies the inclusion $\tilde{G}_\lambda \subseteq I \tilde{\partial}_\mathcal{F}$ for each ideal $I \in F_0$. In view of the definition of the submodule $K$ of the relations, this depends on the fact that, if $\lambda = (I_1, \ldots, I_k)$, then $I(I_1^{-1} \ldots I_k^{-1}) = I_{k-1}^{-1} \ldots I_{k+1}^{-1}$.

III) The module $\partial_\mathcal{F}$ is a generator of the category of the $\mathcal{F}$-divisible modules.

It is enough to prove that, if $M$ is an $\mathcal{F}$-divisible module and $a \in M$, then there exists a homomorphism $\eta: \partial_\mathcal{F} \rightarrow M$ such that $a \in \text{Im } \eta$. We imitate the proof of [7, VII.1.1], by building $\eta$ as the union of maps $\eta_n: \partial_n \rightarrow M$. The correspondence $G\eta \rightarrow M$ which sends $w = \alpha_\eta(1)$ into $a \in M$ induces a map $\eta_0: \partial_0 \rightarrow M$. Assume that a map $\eta_n: \partial_n \rightarrow M$
has been defined for a certain \( n \geq 0 \). We claim that, if \( J \subseteq I \) are ideals in \( \mathcal{F}_0 \), then \( \text{Ext}_R^1(J^{-1}/I^{-1}, M) = 0 \). In fact, from the canonical isomorphisms \( \text{Hom}_R(J^{-1}, M) \cong JM \), \( \text{Hom}_R(I^{-1}, M) \cong IM \) and the equalities \( JM = M = IM \) we deduce that in the exact sequence

\[
\text{Hom}_R(J^{-1}, M) \to \text{Hom}_R(I^{-1}, M) \to \text{Ext}_R^1(J^{-1}/I^{-1}, M) \to 0
\]

the map between the two \( \text{Hom} \)'s is epic, hence \( \text{Ext}_R^1(J^{-1}/I^{-1}, M) = 0 \) and the claim follows. This implies that one can extend \( n_n \) from \( \partial_n \) to \( \partial_{n+1} \), since \( \partial_{n+1}/\partial_n \cong \bigoplus_{I \in \mathcal{F}_0} I_1^{-1} \cdots I_n^{-1} \cdot I_{n+1}^{-1} \cdot I_1^{-1} \cdots I_n^{-1} \) (apply the preceding argument to \( J = I_1 \cdots I_n I_{n+1} \) and \( I = I_1 \cdots I_n \)).

(IV) \( \text{Ext}_R^1(\partial_\mathcal{F}, M) = 0 \) for every \( \mathcal{F} \)-divisible module \( M \).

It is enough to apply Eklof’s Lemma to the chain \( \partial_0 \leq \partial_1 \leq \cdots \leq \partial_n \leq \cdots \), noting that \( \text{Ext}_R^1(\partial_{n+1}/\partial_n, M) = 0 \) for each \( n \geq 0 \), since \( M = IM \) for each \( I \in \mathcal{F}_0 \) and \( \text{Ext}_R^1(J^{-1}/I^{-1}, M) = 0 \) for each \( J < I \in \mathcal{F}_0 \), as shown in (III).

(V) \( R_\mathcal{F} \) is an epic image of \( \partial_\mathcal{F} \).

The map \( \phi: \bigoplus_{\lambda \in A} G_\lambda \to R_\mathcal{F} \) defined by setting \( \phi(\alpha_\lambda(x)) = x \), where \( \lambda = (I_1, \ldots, I_k) \) and \( x \in I_1^{-1} \cdot \cdots \cdot I_k^{-1} = (I_1 \cdot \cdots \cdot I_k)^{-1} \subset R_\mathcal{F} \), is surjective. Its kernel \( H \) clearly contains \( K \), hence \( \phi \) induces an epimorphism from \( \partial_\mathcal{F} \) to \( R_\mathcal{F} \).

(VI) \( \partial_\mathcal{F}/\partial_0 \) is isomorphic to a direct summand of \( \partial_\mathcal{F} \).

We imitate Facchini’s proof in [4, p. 69], adapting it to our setting. Fix an invertible ideal \( J \neq R \) and define a map

\[
\xi: \bigoplus_{\lambda \in A} G_\lambda \to \partial_\mathcal{F}
\]

in the following way: \( \xi(G_\emptyset) = 0 \) and, if \( \lambda = (I_1, \ldots, I_n) \) and \( 0 \neq x \in I_1^{-1} \cdot \cdots \cdot I_n^{-1} \),

\[
\xi(\alpha_\lambda(x)) = \alpha_{R,\lambda}(x) - \alpha_{\lambda,J}(x) + K,
\]

where \( \lambda_R = (R, I_1, \ldots, I_n) \) and \( \lambda_J = (J, I_1, \ldots, I_n) \). Note that \( x \in R^{-1} I_1^{-1} \cdot \cdots \cdot I_n^{-1} \leq J^{-1} \cdot \cdots \cdot I_n^{-1} \), so \( \alpha_{R,\lambda}(x) \) and \( \alpha_{\lambda,J}(x) \) make sense. Clearly \( \xi \) is an homomorphism, which sends \( K \) into 0. In fact,

\[
\xi(\alpha_{\emptyset}(1) - \alpha_{(I)}(1)) = -\xi(\alpha_{(I)}(1)) = -\alpha_{(R,I)}(1) + \alpha_{(J,I)}(1) + K = 0,
\]

since \( \alpha_{(R,I)}(1) - \alpha_{\emptyset}(1) \) and \( \alpha_{(J,I)}(1) - \alpha_{(J)}(1) \) belong to \( K \), as well as \( \alpha_{(R)}(1) - \alpha_{\emptyset}(1) \) and \( \alpha_{(J)}(1) - \alpha_{\emptyset}(1) \).
Let now $\mathcal{J} = (I_1, \ldots, I_{n+1})$, $(n \geq 1)$, and $0 \neq x \in I_{1}^{-1} \cdots I_{n}^{-1}$. Then, since $(\mathcal{J})_R = (\mathcal{J}_R)^-$ and $(\mathcal{J})_J = (\mathcal{J}_J)^-$, we get
\[
\zeta(\alpha_{\mathcal{J}}(x) - \alpha_{\mathcal{J}}(x)) = \alpha_{(\mathcal{J})_R}(x) - \alpha_{(\mathcal{J}_R)^-}(x) + \alpha_{(\mathcal{J}_J)^-}(x) + K
\]
where $\zeta = (I_1, \ldots, I_{n+1})$. We will show that $\zeta(K) = 0$. Trivially $\eta(\alpha_{\mathcal{J}_R}(1) - \alpha(1)) = 0$ for all $I_1 \in \mathcal{F}_0$. Assume now that $\mathcal{J} = (R, I_2, \ldots, I_n)$ and $0 \neq x \in R^{-1} \cdot I_2^{-1} \cdots I_{n-1}^{-1}$; then
\[
\eta(\alpha_{\mathcal{J}_R}(x) - \alpha_{\mathcal{J}_R}(x)) = \alpha_{(I_2, \ldots, I_{n-1})}(x) - \alpha_{(I_2, \ldots, I_n)}(x) + K + \hat{c}_0 = 0,
\]
so $\zeta(K) = 0$. Let $\psi: \check{\mathcal{F}} \to \check{\mathcal{F}}/\hat{c}_0$ be the homomorphism induced by $\eta$. In order to conclude, we must only verify that $\psi \cdot \phi$ coincides with $\pi$. Obviously both maps send $\hat{c}_0$ to 0. If $\mathcal{J} = (I_1, \ldots, I_n)$ and $0 \neq x \in I_1^{-1} \cdots I_{n-1}^{-1}$, then we have
\[
\psi(\phi(\alpha_{\mathcal{J}_R}(x) + K)) = \psi(\alpha_{(R, I_1, \ldots, I_n)}(x) + K) = \alpha_{(I_1, \ldots, I_n)}(x) + K + \hat{c}_0 = \pi(\alpha_{\mathcal{J}}(x) + K),
\]
as desired.

(VII) The module $\partial$ is a tilting module.

Condition (T1) is property (I). Condition (T2) follows by properties (II) and (IV). Property (T3) follows from the exact sequence $0 \to R \to \hat{c}_0 \to \partial \to \partial/\hat{c}_0 \to 0$ and property (VI).

We can now prove the announced result which classifies the tilting torsion classes over Prüfer domains.

**Theorem 2.8.** Let $R$ be a Prüfer domain. There is a bijective correspondence between the tilting torsion classes over $R$ and the finitely generated localizing systems of ideals of $R$.

**Proof.** If $T^\perp$ is a tilting torsion class, we associate with $T^\perp$ the finitely generated localizing system of divisibility $\mathcal{D}(T)$. This correspondence is well defined: in fact, given another tilting module $T_1$, the two classes of the $\mathcal{D}(T)$—divisible and $\mathcal{D}(T_1)$—divisible modules coincide if and only if $\mathcal{D}(T) = \mathcal{D}(T_1)$, by [5, 5.1.10], so our claim follows by
Corollary 2.2. Conversely, given a finitely generated localizing system $\mathcal{F}$, we associate with $\mathcal{F}$ the tilting torsion class $(\mathcal{G}_\mathcal{F})^\perp$. We must prove that 

$$(\mathcal{G}_\mathcal{F}(T))^\perp = T^\perp, \quad \mathcal{G}(\mathcal{G}_\mathcal{F}) = \mathcal{F}.$$  

The first equality follows by the property (III) of the module $\mathcal{G}_\mathcal{F}(T)$. The second equality follows by the facts that the overring $R_\mathcal{F}$ is a quotient of $\mathcal{G}_\mathcal{F}$, by property (V), and the localizing system of divisibility of $R_\mathcal{F}$ is $\mathcal{F}$, again by [5, 5.1.10]. □

It is worthwhile to remark that the proof of Theorem 2.8 could keep away from the use of the module $\mathcal{G}_\mathcal{F}$; one can just associate with the finitely generated localizing system $\mathcal{F}$ the tilting torsion class $\mathcal{G}^\perp$, where $\mathcal{G} = \{R/I \mid I \in \mathcal{F}_0\}$. The proof presented here provides a concrete tilting module generating this torsion class.

From the above discussion and the properties of tilting modules recalled above, we derive the following

**Corollary 2.9.** Every tilting module $T$ over a Prüfer domain is a direct summand of a direct sum of copies of $\mathcal{G}_\mathcal{F}(T)$. □

**References**