J. Math. Anal. Appl. 381 (2011) 706–723

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Plurisubharmonic and holomorphic functions relative to the plurifine topology

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ARTICLE INFO

Article history:
Received 23 November 2010
Available online 23 March 2011
Submitted by Richard M. Aron

Keywords:
Plurifinely plurisubharmonic function
Plurifinely holomorphic function
Finely subharmonic function
Finely holomorphic function
Plurifine topology

ABSTRACT

A weak and a strong concept of plurifinely plurisubharmonic and plurifinely holomorphic functions are introduced. Strong will imply weak. The weak concept is studied further. A function f is weakly plurifinely plurisubharmonic if and only if it is locally bounded from above in the plurifine topology and $f \circ h$ is finely subharmonic for all complex affine-linear maps h. As a consequence, the regularization in the plurifine topology of a pointwise supremum of such functions is weakly plurifinely plurisubharmonic, and it differs from the pointwise supremum at most on a pluripolar set. Weak plurifine plurisubharmonicity and weak plurifine holomorphy are preserved under composition with weakly plurifinely holomorphic maps.

1. Introduction

The *plurifine topology* $\mathcal{F}$ on $\mathbb{C}^n$ was briefly introduced in [17] as the weakest topology in which all plurisubharmonic functions are continuous, in analogy with the H. Cartan fine topology on $\mathbb{R}^n$, in particular on $\mathbb{C} \cong \mathbb{R}^2$. For comments on this choice of "fine" topology on $\mathbb{C}^n$, see [17]. The plurifine topology $\mathcal{F}$ is clearly biholomorphically invariant. Furthermore, $\mathcal{F}$ is locally connected, as shown in [9,10], where also further properties of $\mathcal{F}$ are given. Much as in [8,10,11] we begin by considering (in Definition 2.2, resp. 2.6) two concepts of plurifinely plurisubharmonic (resp. plurifinely holomorphic) functions—a strong concept defined by $\mathcal{F}$-local uniform approximation with plurisubharmonic (resp. holomorphic) functions, and a weak concept defined by restriction to complex lines. We thereby draw on the theory of *finely sub- or superharmonic* and *finely holomorphic* functions defined on finely open subsets of $\mathbb{C}$, cf. [12,14,18]. The plurifine topology $\mathcal{F}$ on $\mathbb{C}^n$ induces on each complex line L in $\mathbb{C}^n$ the Cartan fine topology on $L \cong \mathbb{C}$ (Lemma 2.1). In analogy with ordinary plurisubharmonic functions, the weakly $\mathcal{F}$-plurisubharmonic functions f may be characterized as follows. They are $\mathcal{F}$-upper semicontinuous and such that $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic (or identically $-\infty$ in some fine component of its domain of definition) for every $\mathbb{C}$-affine-linear bijection h of $\mathbb{C}^n$ (Theorem 3.1).

The concepts of strongly $\mathcal{F}$-plurisubharmonic and strongly $\mathcal{F}$-holomorphic functions on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ are obviously biholomorphically invariant. We show that the same holds for the weak concepts (Theorem 4.6), cf. [11]. We do not know whether the strong and the weak concepts are actually the same. The weak concepts are closed under $\mathcal{F}$-locally uniform convergence, and altogether seem to be more useful, cf. [10].

The convex cone of all weakly $\mathcal{F}$-plurisubharmonic functions on Ω is stable under taking the pointwise infimum for lower directed families and under taking the pointwise supremum for finite families. The above characterization of weakly

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.03.041

$\mathcal{F}$-plurisubharmonic functions allows us to answer questions posed by the first named author in [8]. Namely, for any $\mathcal{F}$-locally upper bounded family of weakly $\mathcal{F}$-plurisubharmonic functions f_α on Ω, the $\mathcal{F}$-upper semicontinuous regularization f^* of the pointwise supremum $f = \sup_\alpha f_\alpha$ is likewise weakly $\mathcal{F}$-plurisubharmonic (Theorem 3.9), and the exceptional set $\{f < f^*\}$ is pluripolar, as expected from a theorem of Bedford and Taylor [2, Theorem 7.1]. Furthermore, there is a removable singularity theorem for weakly $\mathcal{F}$-plurisubharmonic functions (Theorem 3.7), and likewise for $\mathcal{F}$-holomorphic functions (Corollary 3.8).

In the final Section 4 we show that the concepts of weakly $\mathcal{F}$-plurisubharmonic function and weakly $\mathcal{F}$-holomorphic function are biholomorphically invariant, even in a plurifine sense. In fact, composition with weakly $\mathcal{F}$-holomorphic maps preserves weak $\mathcal{F}$-plurisubharmonicity and weak $\mathcal{F}$-holomorphy (Theorem 4.6).

2. Definitions and first properties of strongly and weakly $\mathcal{F}$-plurisubharmonic and $\mathcal{F}$-holomorphic functions

The $\mathcal{F}$-interior (plurifine interior) of a set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, is denoted by K'. It is known that every $\mathcal{F}$-neighborhood of a point of $\mathbb{C}^n$ contains an $\mathcal{F}$-neighborhood which is compact in the Euclidean topology—an easy consequence of [10, Theorem 2.3], plurisubharmonic functions being upper semicontinuous. Henceforth, topological properties not explicitly referring to the plurifine topology $\mathcal{F}$ or the Cartan fine topology are tacitly understood to refer to the Euclidean topology. Generalizing known properties of the fine topology, cf. [19], we have

Lemma 2.1.

(a) *The plurifine topology $\mathcal{F}$ on $\mathbb{C}^n$ induces on every $\mathbb{C}$-affine subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$ the plurifine topology on L. Explicitly, for any $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ the intersection $L \cap \Omega$ is $\mathcal{F}$-open in L, and so is the orthogonal projection of Ω on L.*
(b) *A set $\omega \subset \mathbb{C}^k$ is $\mathcal{F}$-open in $\mathbb{C}^k$ if and only if $\omega \times \mathbb{C}^{n-k}$ is $\mathcal{F}$-open in $\mathbb{C}^n$.*

Proof. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ write

$$z' = (z_1, \ldots, z_k), \qquad z'' = (z_{k+1}, \ldots, z_n).$$

For (a) it suffices to consider the particular subspace $L_0 = \{(z', 0''):\ z' \in \mathbb{C}^k\}$ which we identify with $\mathbb{C}^k$. For any $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ denote by ω the part of Ω in L_0. Consider a point $a' \in \omega$. According to [9, Theorem 2.3], there exist a plurisubharmonic function ψ on $\mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$ and neighborhoods U' of a' in $\mathbb{C}^k$ and U'' of $0''$ in $\mathbb{C}^{n-k}$ such that

$$\left(a', 0''\right) \in \left\{\left(z', z''\right) \in U' \times U'':\ \psi\left(z', z''\right) > 0\right\} \subset \Omega. \tag{2.1}$$

Define $\varphi : \mathbb{C}^k \to [-\infty, +\infty[$ by $\varphi(z') = \psi(z', 0'')$. Then φ is plurisubharmonic and

$$a' \in \left\{z' \in U':\ \varphi\left(z'\right) > 0\right\} \subset \omega. \tag{2.2}$$

Thus ω is indeed an $\mathcal{F}$-neighborhood of a' in $\mathbb{C}^k$.

For each $t \in \mathbb{C}^{n-k}$ the translate $\Omega_t = \Omega - (0', t)$ of Ω is $\mathcal{F}$-open in $\mathbb{C}^n$. It follows that $\Omega_t \cap L_0$ is $\mathcal{F}$-open in L_0, and therefore so is the union of the $\Omega_t \cap L_0$, that is, the projection of Ω on L_0.

For (b) we have just shown, in particular, that if $\Omega := \omega \times \mathbb{C}^{n-k}$ is $\mathcal{F}$-open in $\mathbb{C}^n$ then ω is $\mathcal{F}$-open in $\mathbb{C}^k$. To establish the converse, suppose that ω is $\mathcal{F}$-open in $\mathbb{C}^k$ and let us prove that every point $a = (a', a'')$ of $\omega \times \mathbb{C}^{n-k}$ is an $\mathcal{F}$-inner point of that set. Since ω is an $\mathcal{F}$-neighborhood of a' in $\mathbb{C}^k$ there exist (again by [9, Theorem 2.3]) a plurisubharmonic function φ on $\mathbb{C}^k$ and a neighborhood U' of a' in $\mathbb{C}^k$ such that (2.2) holds. The function ψ defined on $\mathbb{C}^n$ by $\psi(z', z'') = \varphi(z')$ is plurisubharmonic—an easy and well-known consequence of the definition of plurisubharmonicity [24, p. 306], cf. [21, p. 62]. Furthermore, (2.1) holds (with $\Omega = \omega \times \mathbb{C}^{n-k}$ and with $(a', 0'')$ replaced by a) for any neighborhood U'' of a'' in $\mathbb{C}^{n-k}$. Thus $\omega \times \mathbb{C}^{n-k}$ is indeed an $\mathcal{F}$-neighborhood of a in $\mathbb{C}^n$. □

For a compact set $K \subset \mathbb{C}^n$ we denote by $S_0(K)$ the convex cone of all restrictions to K of *finite continuous* plurisubharmonic functions defined on open subsets of $\mathbb{C}^n$ containing K. By $S(K)$ we denote the closure of $S_0(K)$ in $C(K, \mathbb{R})$ (the continuous functions $K \to \mathbb{R}$ with the uniform norm); $S(K)$ is likewise a convex cone.

Definition 2.2 *(Plurifinely plurisubharmonic function).* Let Ω denote an $\mathcal{F}$-open (i.e., plurifinely open) subset of $\mathbb{C}^n$:

(i) A function $f : \Omega \to \mathbb{R}$ is said to be *$\mathcal{F}$-cpsh* if every point of Ω has a compact $\mathcal{F}$-neighborhood K in Ω such that $f|K \in S(K)$.
(ii) A function $f : \Omega \to [-\infty, +\infty[$ is said to be *strongly $\mathcal{F}$-plurisubharmonic* if f is the pointwise limit of a decreasing net of $\mathcal{F}$-cpsh functions on Ω.
(iii) (Cf. [8, Section 5], [10, Definition 5.1].) A function $f : \Omega \to [-\infty, +\infty[$ is said to be *weakly $\mathcal{F}$-plurisubharmonic* if f is $\mathcal{F}$-upper semicontinuous and, for every complex line L in $\mathbb{C}^n$, the restriction of f to the finely open subset $L \cap \Omega$ of L is *finely hypoharmonic*.

See [12, Definition 8.2 and §10.4] for finely hypoharmonic (resp. finely sub- or superharmonic) functions, and recall that a function f is finely hypoharmonic on a finely open subset U of $\mathbb{C}$ (or of $\mathbb{R}^N$) if and only if f is finely subharmonic on every fine component of U in which $f \not\equiv -\infty$. The concepts strongly and weakly $\mathcal{F}$-plurisubharmonic are both $\mathcal{F}$-local ones (that is, these have the sheaf property).

The concept of $\mathcal{F}$-cpsh functions, defined in (i), is an auxiliary one. Every strongly $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-upper semicontinuous (even $\mathcal{F}$-continuous, see Theorem 2.4(c) and Proposition 2.5) because every $\mathcal{F}$-cpsh function is $\mathcal{F}$-continuous. The class of all strongly, resp. weakly, $\mathcal{F}$-plurisubharmonic functions on Ω is clearly a convex cone which is stable under taking of the pointwise supremum of finite families. The latter class is furthermore stable under taking of the pointwise infimum for lower directed (possibly infinite) families, and is closed under $\mathcal{F}$-locally uniform convergence in view of [12, Lemma 9.6]. For upper directed families of weakly $\mathcal{F}$-plurisubharmonic functions, see Theorem 3.9 below.

If f is strongly, resp. weakly, $\mathcal{F}$-plurisubharmonic on Ω ($\mathcal{F}$-open in $\mathbb{C}^n$) then the restriction of f to $L \cap \Omega$ (L is a $\mathbb{C}$-affine subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$) has the same property in $L \cap \Omega$. This follows easily from Lemma 2.1(a) above.

For $n = 1$, f is strongly, resp. weakly, $\mathcal{F}$-plurisubharmonic on Ω (finely open in $\mathbb{C}$) if and only if f is finely hypoharmonic on Ω. This is obvious in the weak case. In the strong case, suppose first that f is finite and finely hypoharmonic on Ω. By the Brelot property [18, p. 284], every point of Ω has a compact fine neighborhood K in Ω such that $f|K \in C(K,\mathbb{R})$ (f being finely continuous by [12, Theorem 9.10]). Because f is finite and finely hypoharmonic in the fine interior K' of K we have $f \in S(K)$ according to [1, Theorem 4.7], cf. [16, Theorem 4], and so f is $\mathcal{F}$-cpsh on Ω. For a general finely hypoharmonic function f on Ω write $f = \inf_{n\in\mathbb{N}} \max\{f, -n\}$ and note that $\max\{f, -n\}$ is finite and finely hypoharmonic, cf. [12, Corollary 2, p. 84]. Conversely, if f is strongly $\mathcal{F}$-plurisubharmonic we may assume by the same corollary that f is even $\mathcal{F}$-cpsh. For any compact set $K \subset \mathbb{C}$, every function of class $S(K)$ is finite and finely hypoharmonic on K' according to [12, Lemma 9.6]. With K as in (i) this shows that f indeed is finite and finely hypoharmonic on Ω.

In the following two theorems we collect some properties of weakly finely plurisubharmonic functions recently obtained by the third author in collaboration with S. El Marzguioui. By an *$\mathcal{F}$-domain* we understand an $\mathcal{F}$-connected $\mathcal{F}$-open set.

Theorem 2.3. *(See [10].) Let f be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$:*

(a) *If $f \not\equiv -\infty$ then $\{z \in \Omega\colon f(z) = -\infty\}$ has no $\mathcal{F}$-interior point.*
(b) *If $f \not\equiv -\infty$ then, for any $\mathcal{F}$-closed set $E \subset \{z \in \Omega\colon f(z) = -\infty\}$, $\Omega \setminus E$ is an $\mathcal{F}$-domain.*
(c) *If $f \leqslant 0$ then either $f < 0$ or $f \equiv 0$.*

Theorem 2.4. *(See [11].) Let f be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$:*

(a) *Every point $z_0 \in \Omega$ such that $f(z_0) > -\infty$ has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which f can be represented as the difference $f = \varphi_1 - \varphi_2$ between two bounded plurisubharmonic functions φ_1 and φ_2 defined on some open ball $B(z_0, r)$ containing O.*
(b) *If f maps Ω into a fixed bounded interval $]a, b[$, then r, O, and φ_2 will depend on $]a, b[$, but can be chosen independently of f.*
(c) *f is $\mathcal{F}$-continuous.*
(d) *If Ω is $\mathcal{F}$-connected and $f \not\equiv -\infty$ then $\{z \in \Omega\colon f(z) = -\infty\}$ is an $\mathcal{F}$-closed, pluripolar subset of $\mathbb{C}^n$.*

Assertion (d) states that pluripolar sets and weakly $\mathcal{F}$-pluripolar sets (in the obvious sense) are the same. The proofs of (a), (b), and (c) given below are essentially taken from [11].

Proof of Theorem 2.4. (a) To begin with, suppose that f is *bounded*. We may then assume that $-1 < f < 0$, for f maps Ω into a bounded interval $]a, b[$, and hence $\frac{f-b}{b-a}$ maps Ω into $]-1, 0[$ and is likewise weakly $\mathcal{F}$-plurisubharmonic. Let $V \subset \Omega$ be a compact $\mathcal{F}$-neighborhood of z_0. Since the complement $\complement V$ of V is pluri-thin at z_0, there exist $0 < r < 1$ and a plurisubharmonic function φ on $B(z_0, r)$ such that

$$\limsup_{z\to z_0,\, z\in\complement V} \varphi(z) < \varphi(z_0).$$

Without loss of generality we may suppose that φ is negative on $B(z_0, r)$ and

$$\varphi(z) = -1 \quad \text{on } B(z_0, r) \setminus V \quad \text{and} \quad \varphi(z_0) = -1/2.$$

Hence

$$f(z) + \lambda\varphi(z) \leqslant -\lambda \quad \text{for } z \in \Omega \cap B(z_0, r) \setminus V \text{ and } \lambda > 0. \tag{2.3}$$

Now define a function u_λ on $B(z_0, r)$ by

$$u_\lambda(z) = \begin{cases} \max\{-\lambda, f(z) + \lambda\varphi(z)\} & \text{for } z \in \Omega \cap B(z_0, r), \\ -\lambda & \text{for } z \in B(z_0, r) \setminus V. \end{cases} \tag{2.4}$$

This definition makes sense because $(\Omega \cap B(z_0, r)) \cup (B(z_0, r) \setminus V) = B(z_0, r)$, and the two definitions agree on $\Omega \cap B(z_0, r) \setminus V$ in view of (2.3).

Clearly, u_λ is weakly $\mathcal{F}$-plurisubharmonic on $\Omega \cap B(z_0, r)$ and on $B(z_0, r) \setminus V$, hence on all of $B(z_0, r)$ in view of the sheaf property, cf. [10]. Since u_λ is bounded on $B(z_0, r)$, it follows from [12, Theorem 9.8] that u_λ is subharmonic on each complex line where it is defined. It is well known that a bounded function, which is subharmonic on each complex line where it is defined, is plurisubharmonic, cf. [23] or [24, p. 24]. Thus, u_λ is plurisubharmonic on $B(z_0, r)$.

Since $\varphi(z_0) = -1/2$, the set $O = \{z \in \Omega\colon \varphi(z) > -3/4\}$ is an $\mathcal{F}$-neighborhood of z_0, and because $\varphi = -1$ on $B(z_0, r) \setminus V$ it is clear that $O \subset V \subset \Omega$.

Observe now that $-4 \leqslant f(z) + 4\varphi(z)$ for every $z \in O$. Hence $f = \varphi_1 - \varphi_2$ on O, with $\varphi_1 = u_4$ and $\varphi_2 = 4\varphi$, both plurisubharmonic on $B(z_0, r)$. Thus f is weakly $\mathcal{F}$-plurisubharmonic on O, which is an $\mathcal{F}$-neighborhood of z_0. It follows that f is $\mathcal{F}$-continuous on O along with φ_1 and φ_2, provided that f is bounded.

Without assuming that f be bounded, f remains $\mathcal{F}$-continuous on O according to (c), proven below. It follows that f is bounded on some $\mathcal{F}$-neighborhood U of z_0 in Ω, and we therefore have a decomposition of f as required, on some $\mathcal{F}$-neighborhood (replacing the above O) of z_0 on $U \subset \Omega$.

(b) Again we may assume that $-1 < f < 0$. The set V and the plurisubharmonic function φ in the proof of (a) then do not depend on f, and that applies to $\varphi_2 = 4\varphi$ as well.

(c) In the remaining case where f may be unbounded (cf. the proof of (a) above), note that f is $\mathcal{F}$-upper semicontinuous and $< +\infty$. Choose $c, d \in \mathbb{R}$ with $d < c$. Then the set $\Omega_c = \{z \in \Omega\colon f(z) < c\}$ is $\mathcal{F}$-open. The function $\max\{f, d\}$ is bounded and weakly $\mathcal{F}$-plurisubharmonic on Ω_c, hence $\mathcal{F}$-continuous there. The set

$$\{z \in \Omega\colon d < f(z) < c\} = \{z \in \Omega_c\colon d < \max\{f(z), d\} < c\}$$

therefore is $\mathcal{F}$-open, and hence f is $\mathcal{F}$-continuous.

For (d) we refer to the proof given in [11, Theorem 4.1]. □

Proposition 2.5. *Every strongly $\mathcal{F}$-plurisubharmonic function $f : \Omega \to [-\infty, +\infty[$ is weakly $\mathcal{F}$-plurisubharmonic.*

Proof. We may assume that f is even $\mathcal{F}$-cpsh. Let (f_ν) be a sequence of finite continuous plurisubharmonic functions on open sets Ω_ν containing K from Definition 2.2(i) such that $f_\nu|K \to f|K$ uniformly. For any complex line L in $\mathbb{C}^n$, $f_\nu|L \cap K'$ is finely hypoharmonic. This uses [12, Theorem 8.7] and the fact that the intersection of any $\mathcal{F}$-open subset of $\mathbb{C}^n$ with any complex line L is finely open, by Lemma 2.1. It follows by [12, Lemma 9.6] that $f|L \cap K'$ is finely hypoharmonic, and in particular finely continuous, by [12, Theorem 9.10]. Consequently, f is indeed weakly $\mathcal{F}$-plurisubharmonic. □

We now pass to concepts of $\mathcal{F}$-holomorphic functions. For a compact set $K \subset \mathbb{C}^n$ we denote by $H_0(K)$ the algebra of all restrictions to K of holomorphic functions defined on open subsets of $\mathbb{C}^n$ containing K. By $H(K)$ we denote the closure of $H_0(K)$ in $C(K, \mathbb{C})$ (the continuous functions $K \to \mathbb{C}$ with the uniform norm). Then $H(K)$ is likewise an algebra.

Definition 2.6 *(Plurifinely holomorphic function).* Let Ω denote an $\mathcal{F}$-open subset of $\mathbb{C}^n$:

(i) (Cf. [10, Definition 6.1].) A function $f : \Omega \to \mathbb{C}$ is said to be *strongly $\mathcal{F}$-holomorphic* if every point of Ω has a compact $\mathcal{F}$-neighborhood K in Ω such that $f|K \in H(K)$.

(ii) A function $f : \Omega \to \mathbb{C}$ is said to be *weakly $\mathcal{F}$-holomorphic* if f is $\mathcal{F}$-continuous and if, for every complex line L in $\mathbb{C}^n$, the restriction $f|L \cap \Omega$ is *finely holomorphic*.

For finely holomorphic functions see [14,18]. The concepts strongly and weakly $\mathcal{F}$-holomorphic are $\mathcal{F}$-local ones. The class of all strongly, resp. weakly, $\mathcal{F}$-holomorphic functions on Ω is an algebra, and the latter class is closed under $\mathcal{F}$-locally uniform convergence, in view of [14, Théorème 4]. Clearly, every strongly $\mathcal{F}$-holomorphic function is $\mathcal{F}$-continuous (on K' from Definition 2.6(i), and so on all of Ω).

If f is strongly, resp. weakly, $\mathcal{F}$-holomorphic on Ω ($\mathcal{F}$-open in $\mathbb{C}^n$) then the restriction of f to $L \cap \Omega$ (L is a $\mathbb{C}$-affine subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$) has the same property on $L \cap \Omega$. This follows easily from Lemma 2.1 above.

For $n = 1$, f is strongly (resp. weakly) $\mathcal{F}$-holomorphic on Ω (finely open in $\mathbb{C}$) if and only if f is finely holomorphic on Ω. This is obvious in the weak case. In the strong case, suppose first that f is finely holomorphic on Ω. By [14, Corollaire, p. 75], every point of Ω has a compact fine neighborhood K in Ω such that $f|K \in R(K)$ ($= H(K)$ in the 1-dimensional case). Consequently, f is indeed strongly $\mathcal{F}$-holomorphic on Ω. Conversely, if f is strongly $\mathcal{F}$-holomorphic then, for any compact set $K \subset \mathbb{C}$, every function of class $H(K)$ is finely holomorphic on K', see [14, p. 63]. With K as in Definition 2.6(i) this shows that f indeed is finely holomorphic on Ω.

Proposition 2.7. *Every strongly $\mathcal{F}$-holomorphic function $f : \Omega \to \mathbb{C}$ is weakly $\mathcal{F}$-holomorphic, and in particular $\mathcal{F}$-continuous.*

Proof. For any K as in Definition 2.6(i) there exists a sequence of holomorphic functions f_ν defined on open sets containing K such that $f_\nu|K \to f|K$ uniformly. For every complex line L in $\mathbb{C}^n$ this shows that the finely holomorphic functions

$f_\nu|L \cap K'$ converge uniformly to $f|L \cap K'$, which therefore is finely holomorphic, see again [14, p. 63]. Consequently $f|L \cap \Omega$ is finely holomorphic, and so f is indeed weakly $\mathcal{F}$-holomorphic, being also $\mathcal{F}$-continuous. $\square$

The concept of weakly $\mathcal{F}$-holomorphic function can be characterized in terms of weakly $\mathcal{F}$-*pluriharmonic* functions (that is, functions $f : \Omega \to \mathbb{C}$ such that $\pm \operatorname{Re} f$ and $\pm \operatorname{Im} f$ are weakly $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$):

Lemma 2.8. *A function $f : \Omega \to \mathbb{C}$ is weakly $\mathcal{F}$-holomorphic if and only if f and each of the functions $z \mapsto z_j f(z)$ ($j \in \{1, \dots, n\}$) are weakly $\mathcal{F}$-pluriharmonic on Ω.*

Proof. This reduces right away to the case $n = 1$ which is due to Lyons [25], cf. [14, Section 3], and which asserts that a function $h : U \to \mathbb{C}$, defined on a finely open set $U \subset \mathbb{C}$, is finely holomorphic if and only if h and $z \mapsto zh(z)$ are (complex) finely harmonic. $\square$

For any $\mathcal{F}$-open set $U \subset \mathbb{C}^m$, an n-tuple $(h_1, \dots, h_n)$ of strongly (resp. weakly) $\mathcal{F}$-holomorphic functions $h_j : U \to \mathbb{C}$ will be termed a strongly (resp. weakly) $\mathcal{F}$-*holomorphic map* $U \to \mathbb{C}^n$.

Assertion (b) of the following proposition provides two slight strengthenings of [10, Lemma 6.2].

Proposition 2.9. *Let $U \subset \mathbb{C}^m$ be $\mathcal{F}$-open and let $h = (h_1, \dots, h_n) : U \to \mathbb{C}^n$ be a strongly (resp. weakly) $\mathcal{F}$-holomorphic map. Then:*

(a) *The map $h : U \to \mathbb{C}^n$ is continuous from U with the $\mathcal{F}$-topology on $\mathbb{C}^m$ to $\mathbb{C}^n$ with the Euclidean topology.*
(b) *For any plurisubharmonic function f on an open set Ω in $\mathbb{C}^n$, the function $f \circ h$ is strongly (resp. weakly) $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $h^{-1}(\Omega) = \{z \in U:\ h(z) \in \Omega\} \subset \mathbb{C}^m$.*
(c) *For any holomorphic function f on an open set Ω in $\mathbb{C}^n$, the function $f \circ h$ is strongly (resp. weakly) $\mathcal{F}$-holomorphic on the $\mathcal{F}$-open set $h^{-1}(\Omega) \subset \mathbb{C}^m$.*

Proof. Assertion (a) holds because each h_j (whether strongly or weakly $\mathcal{F}$-holomorphic) is $\mathcal{F}$-continuous and because the Euclidean topology on $\mathbb{C}^n$ is the product of the Euclidean topology on each of n copies of $\mathbb{C}$.

For (b) with each h_j *strongly* $\mathcal{F}$-holomorphic we begin by showing that, if the plurisubharmonic function f on Ω is finite and continuous, then $f \circ h$ is even $\mathcal{F}$-*cpsh* (cf. Definition 2.2(i)) on $h^{-1}(\Omega)$, which is $\mathcal{F}$-open according to (a). Every point $a \in h^{-1}(\Omega)$ has a compact $\mathcal{F}$-neighborhood K_j in $h^{-1}(\Omega)$ ($\subset U \subset \mathbb{C}^m$) such that $h_j|K_j \in H(K_j)$. Thus there exists a sequence $(h_j^\nu)_{\nu\in\mathbb{N}}$ of holomorphic functions h_j^ν on open sets U_j^ν in $\mathbb{C}^m$ containing K_j such that $h_j^\nu|K_j \to h_j|K_j$ uniformly as $\nu \to \infty$. Write $K = K_1 \cap \dots \cap K_n$ and $h^\nu = (h_1^\nu, \dots, h_n^\nu)$ on $U^\nu = U_1^\nu \cap \dots \cap U_n^\nu$. Then $h_j|K_j \in C(K_j, \mathbb{C})$ and hence $h|K \in C(K, \mathbb{C}^n)$. It follows that $h(K)$ is a compact subset of $\Omega \subset \mathbb{C}^n$. Denoting by $\|\cdot\|$ the Euclidean norm on $\mathbb{C}^n$ and by B the closed unit ball in $\mathbb{C}^n$, there accordingly exists $\delta > 0$ such that $h(K) + \delta B \subset \Omega$. We may assume that $\|h^\nu(z) - h(z)\| < \delta$ for any ν and any $z \in K$. Under the present extra hypothesis, f is finite and uniformly continuous on the compact set $h(K) + \delta B$ containing any $h^\nu(K)$, and it follows that $f \circ h^\nu|K \to f \circ h|K$ uniformly as $\nu \to \infty$. Because $f \circ h^\nu$ is finite, continuous, and plurisubharmonic, on the open set $U^\nu \supset K$, we have $f \circ h|K \in S(K)$. By varying $a \in h^{-1}(\Omega)$ and hence the $\mathcal{F}$-neighborhood K of a in $h^{-1}(\Omega)$ we infer that $f \circ h$ is $\mathcal{F}$-cpsh on $h^{-1}(\Omega)$.

If we drop the extra hypothesis that f be finite and continuous, f is the pointwise limit of a decreasing net of finite continuous plurisubharmonic functions f_ν on Ω, and $f \circ h$ is then the pointwise limit of the decreasing net of functions $f_\nu \circ h$ on $h^{-1}(\Omega)$ which we have just shown are $\mathcal{F}$-cpsh, and so $f \circ h$ is indeed strongly $\mathcal{F}$-plurisubharmonic, cf. Definition 2.2(ii) (with Ω replaced by $h^{-1}(\Omega)$).

Next suppose instead that each h_j is *weakly* $\mathcal{F}$-holomorphic on U, and consider a complex line L in $\mathbb{C}^m$; then $L \cap U$ is finely open in L. According to Definition 2.6(ii), $h_j|L \cap U$ is then finely holomorphic, which is the same as strongly $\mathcal{F}$-holomorphic (see above for $n = 1$). As shown above (now with $m = 1$ and with U replaced by $L \cap U$) it follows that $f \circ h|L \cap h^{-1}(\Omega)$ is strongly $\mathcal{F}$-plurisubharmonic, which is the same as finely hypoharmonic (because the dimension is 1). According to Definition 2.2(iii) this means that $f \circ h$ indeed is weakly $\mathcal{F}$-plurisubharmonic on $h^{-1}(\Omega)$, noting that $f \circ h$ is $\mathcal{F}$-upper semicontinuous in view of (a) because f is upper semicontinuous.

For (c), suppose first that each h_j is *strongly* $\mathcal{F}$-holomorphic on U. Proceeding as in the first part of the proof of (b) we arrange that $f \circ h^\nu|K \to f \circ h|K$ uniformly as $\nu \to \infty$; but now $f \circ h^\nu$ is holomorphic on U^ν. We therefore conclude that $f \circ h|K \in H(K)$, and so $f \circ h$ is indeed strongly $\mathcal{F}$-holomorphic according to Definition 2.6(i).

If instead each h_j is *weakly* $\mathcal{F}$-holomorphic on U then, for every complex line L in $\mathbb{C}^m$, each $h_j|L \cap U$ is again *strongly* $\mathcal{F}$-holomorphic. As just established, this implies that $f \circ h|L \cap h^{-1}(\Omega)$ is strongly $\mathcal{F}$-holomorphic, or equivalently finely holomorphic. We conclude that indeed $f \circ h$ is *weakly* $\mathcal{F}$-holomorphic, according to Definition 2.6(ii), noting that $f \circ h$ is $\mathcal{F}$-continuous in view of (a). $\square$

In the version of Proposition 2.9 with 'weakly' in each of the three cases one may allow f in (b) to be just *weakly $\mathcal{F}$-plurisubharmonic* (in place of plurisubharmonic), and similarly f in (c) to be *weakly $\mathcal{F}$-holomorphic* (in place of holomorphic), see Theorem 4.6 at the end of the paper. At this point we merely show that we may allow f in (b) (of Proposition 2.9) to be *strongly* $\mathcal{F}$-plurisubharmonic, and f in (c) to be strongly $\mathcal{F}$-holomorphic:

Theorem 2.10. *Let $U \subset \mathbb{C}^m$ be $\mathcal{F}$-open and let $h = (h_1, \dots, h_n) : U \to \mathbb{C}^n$ be a weakly $\mathcal{F}$-holomorphic map. Then*:

(a) *The map $h : U \to \mathbb{C}^n$ is continuous from U with the $\mathcal{F}$-topology on $\mathbb{C}^m$ to $\mathbb{C}^n$ with the $\mathcal{F}$-topology there.*
(b) *For any strongly $\mathcal{F}$-plurisubharmonic function f defined on an $\mathcal{F}$-open set Ω in $\mathbb{C}^n$, the function $f \circ h$ is weakly $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $h^{-1}(\Omega) = \{z \in U\colon\ h(z) \in \Omega\} \subset \mathbb{C}^m$.*
(c) *For any strongly $\mathcal{F}$-holomorphic function f defined on an $\mathcal{F}$-open set Ω in $\mathbb{C}^n$, the function $f \circ h$ is weakly $\mathcal{F}$-holomorphic on the $\mathcal{F}$-open set $h^{-1}(\Omega) \subset \mathbb{C}^m$.*

Proof. For the present weakly $\mathcal{F}$-holomorphic functions h_j assertion (a) is stronger than Proposition 2.9(a). We shall prove that $h^{-1}(\Omega)$ is $\mathcal{F}$-open in $\mathbb{C}^m$ for any $\mathcal{F}$-open set Ω in $\mathbb{C}^n$. Fix a point $a \in h^{-1}(\Omega)$ and write $h(a) = b$ $(\in \Omega)$. According to [10, Theorem 2.3], there exist a plurisubharmonic function φ on an open ball $B(b, r)$ in $\mathbb{C}^n$ and a number $c < \varphi(b)$ such that the basic $\mathcal{F}$-neighborhood

$$W = \{w \in B(b, r)\colon\ \varphi(w) > c\}$$

of b in $\mathbb{C}^n$ is a subset of the $\mathcal{F}$-open set Ω in $\mathbb{C}^n$. Then $h^{-1}(W) \subset h^{-1}(\Omega)$ $(\subset U)$, and

$$h^{-1}(W) = \{z \in U\colon\ h(z) \in B(b, r) \text{ and } (\varphi \circ h)(z) > c\}$$

is $\mathcal{F}$-open in $\mathbb{C}^m$ because $h : U \to \mathbb{C}^n$ is $\mathcal{F}$-continuous by Proposition 2.9(a). Moreover, $\varphi \circ h : h^{-1}(B(b, r)) \to [-\infty, +\infty[$ is weakly $\mathcal{F}$-plurisubharmonic by Proposition 2.9(b) (applied with Ω, f replaced by $B(b, r), \varphi$), and in particular *$\mathcal{F}$-continuous*, by Theorem 2.4(c). By varying $a \in h^{-1}(\Omega)$ we infer that indeed $h^{-1}(\Omega)$ is $\mathcal{F}$-open.

For (b) we may assume that f is even *$\mathcal{F}$-cpsh* on Ω. Let $K \subset \Omega$ be as in Definition 2.2(i), and let (f^ν) be a sequence of finite continuous plurisubharmonic functions on open sets $\Omega^\nu \supset K$ such that $f^\nu|K \to f|K$ uniformly as $\nu \to \infty$. According to Proposition 2.9(a), (b) each $f^\nu \circ h$ is weakly $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $h^{-1}(\Omega^\nu)$. By (a), $h^{-1}(K')$ is $\mathcal{F}$-open, and it follows that each $f^\nu \circ h|h^{-1}(K')$ likewise is weakly $\mathcal{F}$-plurisubharmonic, in particular $\mathcal{F}$-upper semicontinuous. Hence so is its uniform limit $f \circ h|h^{-1}(K')$ in view of [12, Lemma 9.6]. By varying $K \subset \Omega$ and hence K' we conclude that indeed $f \circ h$ is weakly $\mathcal{F}$-plurisubharmonic on $h^{-1}(\Omega)$.

Finally, the proof of (c) is quite parallel to that of (b) in view of Proposition 2.9(a), (c), using [14, Théorème 4] in place of [12, Lemma 9.6]. □

Theorem 2.10 has two corollaries for $m = 1$ and $n = 1$, respectively. In either corollary ‘strongly’ can be replaced by ‘weakly’ according to Theorem 4.6. For $m = 1$ we have

Corollary 2.11. *Let $h_j : U \to \mathbb{C}$ $(j \in \{1, \dots, n\})$ be finely holomorphic functions defined on a finely open set $U \subset \mathbb{C}$, and write $h = (h_1, \dots, h_n)$. For any strongly $\mathcal{F}$-plurisubharmonic* (*resp. strongly $\mathcal{F}$-holomorphic*) *function f defined on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$, the function $f \circ h$ is finely hypoharmonic* (*resp. finely holomorphic*) *on the finely open set $h^{-1}(\Omega) = \{z \in U\colon\ h(z) \in \Omega\} \subset \mathbb{C}$.*

Remark 2.12. Suppose now that one can prove that the cone of $\mathcal{F}$-cpsh, resp. the algebra of strongly $\mathcal{F}$-holomorphic, functions on an $\mathcal{F}$-open subset Ω of $\mathbb{C}^n$ is closed under uniform convergence, then the proofs of Theorem 2.10(b), (c) easily show that ‘weakly’ can be replaced throughout the theorem by ‘strongly’. Indeed, with f $\mathcal{F}$-cpsh (resp. strongly $\mathcal{F}$-holomorphic) and h strongly $\mathcal{F}$-holomorphic, let $K \subset \Omega$ denote a compact $\mathcal{F}$-neighborhood of a point $a \in \Omega$, and let (f_ν) denote a sequence of finite continuous plurisubharmonic (resp. a sequence of holomorphic) functions, defined on open subsets Ω_ν of $\mathbb{C}^n$ containing K, and such that $f_\nu \to f$ uniformly on K. Then $f_\nu \circ h$ is strongly $\mathcal{F}$-plurisubharmonic and even $\mathcal{F}$-cpsh (resp. strongly $\mathcal{F}$-holomorphic) on the $\mathcal{F}$-open set $h^{-1}(K') \subset h^{-1}(\Omega_\nu) \subset U \subset \mathbb{C}^m$, by Proposition 2.9 (cf. the beginning of the proof of Theorem 2.10(b)). Under the hypothesis at the beginning of this remark it will follow that the uniform limit $f \circ h$ of $(f_\nu \circ h)$ on $h^{-1}(K')$ likewise is $\mathcal{F}$-cpsh (resp. strongly $\mathcal{F}$-holomorphic) on $h^{-1}(K')$, and therefore on $h^{-1}(\Omega)$, by varying K and hence K'. If f is merely strongly *$\mathcal{F}$-plurisubharmonic* (rather than $\mathcal{F}$-cpsh), it follows as usual that indeed $f \circ h$ is likewise strongly $\mathcal{F}$-plurisubharmonic.

In the case $n = 1$ the hypothesis stated in the above remark is always fulfilled in view of [12, Lemma 9.6] (resp. [14, Théorème 4]). We therefore have the following corollary of Theorem 2.10 for that case:

Corollary 2.13. *Let $h : U \to \mathbb{C}$ be a strongly $\mathcal{F}$-holomorphic function defined on an $\mathcal{F}$-open set $U \subset \mathbb{C}^m$. Then for any finely hypoharmonic* (*resp. finely holomorphic*) *function f defined on a finely open set $\Omega \subset \mathbb{C}$, the function $f \circ h$ is strongly $\mathcal{F}$-plurisubharmonic* (*resp. strongly $\mathcal{F}$-holomorphic*) *on the $\mathcal{F}$-open set $h^{-1}(\Omega) \subset \mathbb{C}^m$.*

The same statement with ‘strongly’ replaced throughout by ‘weakly’ is simply the case $n = 1$ of Theorem 2.10 as it stands.

Proposition 2.14. *Let now Ω be a Euclidean open subset of $\mathbb{C}^n$. For a function $f:\Omega\to[-\infty,+\infty[$ the following are equivalent:*

(i) *f is plurisubharmonic (in the ordinary sense).*
(ii) *f is strongly $\mathcal{F}$-plurisubharmonic and not identically $-\infty$ on any component of Ω.*
(iii) *f is weakly $\mathcal{F}$-plurisubharmonic and not identically $-\infty$ on any component of Ω.*

Proof. Every finite continuous plurisubharmonic function on Ω is of course $\mathcal{F}$-cpsh. It follows that any plurisubharmonic function on Ω is strongly $\mathcal{F}$-plurisubharmonic (being the pointwise limit of a decreasing sequence of finite continuous plurisubharmonic functions). Conversely, if f is weakly $\mathcal{F}$-plurisubharmonic on Ω then f is plurisubharmonic on every connectivity component ω of Ω on which f is not identically $-\infty$. To see this, we can assume without loss of generality that Ω is convex. First observe that f is $\mathcal{F}$-locally bounded from above, so every point $a\in\Omega$ has an $\mathcal{F}$-neighborhood $U\subset\Omega$ on which $f<M_a$, say. According to [10, Proposition 4.1] we may further arrange that there exists $\delta>0$ such that, for every complex line L passing through a, the intersection $U\cap L$ contains a circle C_L about a with radius at least δ. Let D_L be the disc in L bounded by C_L. Then $D_L\subset\Omega$, and the C_L is the fine boundary of D_L. By the maximum principle for finely subharmonic functions on a planar domain [13, Theorem 2.3], it follows that $f<M_a$ on the discs D_L, hence in particular on the ball $B(a,\delta)\subset\cup_L D_L$. For functions that are locally bounded from above in the Euclidean topology, the statement alternatively follows from [12, Theorem 9.8(a)] in view of [23, Définition 1, p. 306]. □

Remark 2.15. Similarly to Proposition 2.14, a function $f:\Omega\to\mathbb{C}$ (with Ω Euclidean open) is holomorphic if and only if f is strongly, or equivalently weakly, $\mathcal{F}$-holomorphic; the 'if part' follows from [14, p. 63] in view of Hartogs' theorem.

We close this section with an application to pluripolar hulls. Recall that the *pluripolar hull* P^*_Ω of a pluripolar set P relative to an open set Ω containing P is defined as the following set ($\mathcal{F}$-closed relatively to Ω):

$$P^*_\Omega=\bigcap_u\bigl\{z\in\Omega:\ u(z)=-\infty\bigr\}.$$

Here the intersection is taken over all plurisubharmonic functions u defined on Ω and such that $u|P\equiv-\infty$. A pluripolar set E has empty plurifine interior E' [10, Theorem 5.2]. (More generally, a polar set is a Lebesgue null set and therefore has empty fine interior.)

For any set $E\subset\mathbb{C}^m$, $m\in\mathbb{N}$, and any function $h:E\to\mathbb{C}$ we denote by $\Gamma_h(E)=\{(z,h(z)):\ z\in E\}$ the graph of $h|E$ and by $\Gamma_h(E)^*_{\mathbb{C}^{m+1}}$ the pluripolar hull of $\Gamma_h(E)$.

Proposition 2.16. *Let h be a weakly $\mathcal{F}$-holomorphic function on an $\mathcal{F}$-domain $U\subset\mathbb{C}^m$:*

(a) *If $h\not\equiv 0$, the set $h^{-1}(0)$ of zeros of h is pluripolar in $\mathbb{C}^m$. In particular, the graph $\Gamma_h(U)$ of h is pluripolar in $\mathbb{C}^{m+1}$.*
(b) *If E is a non-pluripolar subset of U then $\Gamma_h(U)$ is pluripolar, and $\Gamma_h(U)\subset\Gamma_h(E)^*_{\mathbb{C}^{m+1}}$.*

With h supposed *strongly* $\mathcal{F}$-holomorphic on U, this proposition was obtained in [11, Corollary 4.4 and Theorem 4.5], extending [10, Theorem 6.4], and [7, Theorem 3.5].

Proof of Proposition 2.16. (a) It follows from Proposition 2.9(b) that the function $\log|h(\cdot)|$ is weakly $\mathcal{F}$-plurisubharmonic on U. Since $\log|h(z)|=-\infty$ for $z\in h^{-1}(0)$, but $\log|h(\cdot)|\not\equiv-\infty$, we conclude from Theorem 2.4(d) that $h^{-1}(0)$ is pluripolar.

(b) The function $(z,w)\mapsto w-h(z)$ is weakly $\mathcal{F}$-holomorphic and $\not\equiv 0$ on the $\mathcal{F}$-open set $U\times\mathbb{C}\subset\mathbb{C}^{m+1}$. Again by Proposition 2.9(b) it follows that the function $(z,w)\mapsto\log|w-h(z)|$ is weakly $\mathcal{F}$-plurisubharmonic and $\not\equiv-\infty$ on $U\times\mathbb{C}$. Since this function equals $-\infty$ on $\Gamma_h(E)$ we conclude that $\Gamma_h(E)$ is pluripolar. By Josefson's theorem [20] there exists a plurisubharmonic function f on all of $\mathbb{C}^{m+1}$ such that $f(z,h(z))=-\infty$ for every $z\in E$. It follows by Theorem 2.3(a) that $f(z,h(z))=-\infty$ even for every $z\in U$, and hence $\Gamma_h(U)$ is pluripolar in $\mathbb{C}^{m+1}$. By the definition of the pluripolar hull of $\Gamma_h(E)$ we conclude that indeed $\Gamma_h(U)\subset\Gamma_h(E)^*_{\mathbb{C}^{m+1}}$. □

3. A characterization of weakly $\mathcal{F}$-plurisubharmonic functions

By the prefix '$\mathbb{R}^{2n}$-fine' we denote concepts relative to the Cartan fine topology on $\mathbb{C}^n\cong\mathbb{R}^{2n}$. Recall that this topology is finer than the plurifine topology $\mathcal{F}$ [17]. It is well known that a plurisubharmonic function f on a domain $\Omega\subset\mathbb{C}^n$ is subharmonic when considered as a function on $\Omega\subset\mathbb{R}^{2n}$, because the average of f over a sphere can be expressed in terms of the average of f over the circles that are intersection of the sphere with complex lines passing through the center. While this approach does not work in the fine setting, the analogous result nevertheless remains valid. Indeed, a well-known characterization of plurisubharmonic functions (see [24, Théorème 1, p. 18] or [21, Theorem 2.9.12]) may be adapted as follows. This will lead to further properties of weakly $\mathcal{F}$-plurisubharmonic functions (Theorems 3.7 and 3.9).

Theorem 3.1. *A function $f : \Omega \to [-\infty, +\infty[$ (Ω is $\mathcal{F}$-open in $\mathbb{C}^n$) is weakly $\mathcal{F}$-plurisubharmonic if and only if f is $\mathcal{F}$-locally bounded from above and $f \circ h$ is $\mathbb{R}^{2n}$-finely hypoharmonic on the $\mathcal{F}$-open set $h^{-1}(\Omega)$ for every $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$.*

For the proof of the 'only if part' of Theorem 3.1 we need the following

Lemma 3.2. *Let u_1, u_2 be bounded subharmonic functions on an open set $B \subset \mathbb{R}^n$, and consider the function $f = u_1 - u_2$ on B. Let U be a finely open Borel subset of B. Then $f|U$ is finely subharmonic if and only if the signed Riesz measure Δf on B has a positive restriction to U.*

Proof. Suppose first that $(\Delta f)|U \geqslant 0$, and let us prove that f then is finely subharmonic on U.

Recall that the base $b(X)$ in B of $X \subset B$ consists of the points of B at which X is not thin. Denote by $\complement X$ the complement of X relative to B. For a finely open set U, its regularization equals $r(U) = \complement b(\complement U)) = U \cup i(\complement U)$ where $i(X)$ is the polar set consisting of the points of X at which X is thin. We may assume that U is *regular*, i.e., $U = r(U)$ and hence an F_σ-set, for u_1 and u_2 are bounded, and therefore Δu_1 and Δu_2 do not charge the polar set by which U differs from $r(U)$.

Writing $\Delta f := \mu = \mu_+ - \mu_-$ on B we have, by hypothesis, $\mu_-|U = 0$. Proceeding as in the proof of the former (and easier) 'if part' of [12, Theorem 8.10], consider any bounded finely open set V of compact closure $\overline{V} \subset U$. Then $(\mu_-)^{\complement V} = \mu_-$ because μ_- is carried by $b(\complement V) \supset \complement U$. For any $x \in V$ we obtain in terms of the Green kernel G on B according to [6, Theorems 1.X.3 and 1.X.5] applied within B

$$\int G\mu_- \, d\epsilon_x^{\complement V} = \widehat{R}_{G\mu_-}^{\complement V}(x) = G\big((\mu_-)^{\complement V}\big)(x) = G\mu_-(x) < +\infty,$$
$$\int G\mu_+ \, d\epsilon_x^{\complement V} = \widehat{R}_{G\mu_+}^{\complement V}(x) \leqslant G\mu_+(x) < +\infty,$$

whence by subtraction $\int G\mu \, d\epsilon_x^{\complement V} \leqslant G\mu(x)$, showing that the finely continuous function $G\mu$ is finely hyperharmonic, and indeed (being also bounded) finely superharmonic on U [12, Theorem 8.10 and §10.4]. By the Riesz representation theorem, $f = -G\mu + h$ on B, with h harmonic on B, in particular finely harmonic on U, and therefore f is likewise finely subharmonic on U.

Conversely, suppose that $f|U$ is finely subharmonic. Recall the corollary in [6, 1.XI.18] that if two subharmonic functions g_1 and g_2, defined on some open set, coincide on a set A, then their Riesz masses satisfy $\Delta g_1 = \Delta g_2$ on the fine interior of A. Hence the Riesz measure $\Delta f = \Delta u_1 - \Delta u_2$ on U is independent of the choice of u_1 and u_2. In fact, if $f = w_1 - w_2$ (with w_1 and w_2 subharmonic on U) then $u_1 + w_2 = u_2 + w_1$ on U, hence $\Delta u_1 + \Delta w_2 = \Delta u_2 + \Delta w_1$ on (the fine interior of) U, that is, $\Delta w_1 - \Delta w_2 = \Delta u_1 - \Delta u_2$ on U.

Since f is finely subharmonic on U it follows by the proof of [12, Theorem 9.9] that every point $x \in U$ has a fine neighborhood $V_x \Subset U$ in which we can write $f = v_1 - v_2$, where v_1 and v_2 are superharmonic functions on some Euclidean neighborhood B_0 of x in B. Moreover, v_2 is the swept-out on $B_0 \setminus V_x$ of a certain superharmonic function $\geqslant 0$ on B_0. The Riesz mass of v_2 is concentrated on the fine boundary of the complement of V_x, cf. e.g. [6, Theorem 1.XI.14(b)], hence on the fine boundary of V_x. It follows that the Riesz mass Δf of f is positive on V_x for every x. By the quasi-Lindelöf property, we can find countably many $x_j \in U$ such that $U = \bigcup_{j=1}^{\infty} V_{x_j} \cup E$, where E is polar. Clearly Δf is positive on $\bigcup V_{x_i}$. Because E is polar, the Riesz mass of a bounded subharmonic function does not charge E, so we have $\Delta u_i(E) = 0$ $(i = 1, 2)$. We conclude that the measure Δf is positive on U. □

Proof of the 'only if part' of Theorem 3.1. We may assume that Ω is $\mathcal{F}$-connected and that $f \not\equiv -\infty$. Let $E = \{f = -\infty\}$. According to Theorem 2.4(a), every point $a \in \Omega \setminus E$ has a bounded $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which f is representable as $f = u_1 - u_2$, where u_1 and u_2 are bounded plurisubharmonic functions, defined on an open ball B in $\mathbb{C}^n$ containing O. For every $j \in \{1, \dots, n\}$ the distributions

$$M_{i,j} = M_{i,j}(z) = \frac{\partial^2 u_i(z_1, \dots, z_n)}{\partial z_j \partial \bar{z}_j}, \quad i = 1, 2,$$

are well-defined positive measures on B. Below we show that $(M_{1,j} - M_{2,j})|O \geqslant 0$ and hence $\Delta(u_1 - u_2) = 4\sum_{j=1}^{n}(M_{1,j} - M_{2,j}) \geqslant 0$ on O, where Δ denotes the Laplacian on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. According to Lemma 3.2 this will imply that $u_1 - u_2$ indeed is $\mathbb{R}^{2n}$-finely subharmonic on O, and hence actually on $\Omega \setminus E$, by varying a and O. By Theorem 2.4(c), f is $\mathcal{F}$-continuous, in particular $\mathbb{R}^{2n}$-finely continuous; and since $f < +\infty$, E is pluripolar and hence $\mathbb{R}^{2n}$-polar. It follows by the removable singularity property [12, Theorem 9.14] that f is $\mathbb{R}^{2n}$-finely hypoharmonic on all of Ω.

For the proof that $(M_{1,j} - M_{2,j})|O \geqslant 0$ it is convenient to write points of $\mathbb{C}^n$ as (z, w), now with $z \in \mathbb{C}$ and $w \in \mathbb{C}^{n-1}$. Each of the above measures $M_{i,j}$ on B then takes the form

$$M_i = M_i(z, w) = \frac{\partial^2 u_i(z, w)}{\partial z \partial \bar{z}}, \quad i = 1, 2,$$

and we shall prove that $(M_1 - M_2)|O \geqslant 0$.

For each $w \in \mathbb{C}^{n-1}$ define

$$B(w) = \{z \in \mathbb{C}\colon (z, w) \in B\}$$

(and similarly with B replaced by other subsets of $\mathbb{C}^n$). The functions $u_i(\cdot, w)$, $i = 1, 2$, induce subharmonic functions $u_i(\cdot, w)$ on the open subset $B(w)$ of $\mathbb{C}$, and therefore the distributions

$$\mu_{i,w} = \mu_{i,w}(z) = \frac{\partial^2 u_i(z, w)}{\partial z \partial \bar{z}}, \quad i = 1, 2,$$

are positive measures on the open set $B(w)$ (if non-empty). Being weakly $\mathcal{F}$-plurisubharmonic on O, $f = u_1 - u_2$ induces the finely subharmonic function $f(\cdot, w) = u_1(\cdot, w) - u_2(\cdot, w)$ on the finely open set $O(w)$. According to the planar version of Lemma 3.2, applied to the induced bounded subharmonic functions $u_i(\cdot, w)$ on $B(w)$, $i = 1, 2$, the Riesz measure $\mu_{1,w} - \mu_{2,w}$ of $f(\cdot, w)$ is positive on the finely open set $O(w) \subset B(w)$.

Let V_z, V_w denote Lebesgue measure on $\mathbb{C}$, $\mathbb{C}^{n-1}$, respectively. For any test function $\varphi \in C_0^\infty(B)$ we have by Fubini's theorem

$$\begin{aligned}
\int_B \varphi \, dM_i &= \int_B \frac{\partial^2 \varphi(z, w)}{\partial z \partial \bar{z}} u_i(z, w) \, dV_z \, dV_w \\
&= \int_{\mathbb{C}^{n-1}} \left(\int_{B(w)} \frac{\partial^2 \varphi(z, w)}{\partial z \partial \bar{z}} u_i(z, w) \, dV_z \right) dV_w \\
&= \int_{\mathbb{C}^{n-1}} \left(\int_{B(w)} \varphi(z, w) \, d\mu_{i,w}(z) \right) dV_w.
\end{aligned}$$

Choose a compact $\mathcal{F}$-neighborhood K of the given point $a \in O \subset \Omega$ so that $K \subset O$. There exists a decreasing sequence of functions $\varphi_k \in C_0^\infty(B)$ with $0 \leqslant \varphi_k \leqslant 1$ so that $\varphi_k = 1$ on K and $\varphi_k \searrow \chi_K$ (the characteristic function of K) as $k \nearrow \infty$. Since M_i and $\mu_{i,w}$ are locally finite positive measures and B and φ_k are bounded, we obtain by the monotone convergence theorem

$$\begin{aligned}
M_i(K) &= \int_B \chi_K \, dM_i = \lim_{k \to \infty} \int_B \varphi_k \, dM_i \\
&= \lim_{k \to \infty} \int_{\mathbb{C}^{n-1}} \left(\int_{B(w)} \varphi_k(\cdot, w) \, d\mu_{i,w} \right) dV_w \\
&= \int_{\mathbb{C}^{n-1}} \left(\int_{B(w)} \chi_{K(w)} \, d\mu_{i,w} \right) dV_w \\
&= \int_{\mathbb{C}^{n-1}} \mu_{i,w}\big(K(w)\big) \, dV_w.
\end{aligned}$$

It follows that

$$M_1(K) - M_2(K) = \int_{\mathbb{C}^{n-1}} \big(\mu_{1,w}\big(K(w)\big) - \mu_{2,w}\big(K(w)\big)\big) \, dV_w \geqslant 0$$

because $\mu_{1,w} - \mu_{2,w} \geqslant 0$ on $O(w) \supset K(w)$. Thus $M_1(K) \geqslant M_2(K)$ for every compact $\mathcal{F}$-neighborhood K of a in O.

The proof of Theorem 2.4(a) shows that we may take $O = \{z \in B(z_0, r)\colon \Phi^*(z) \geqslant -\frac{1}{4}\}$, where Φ^* is plurisubharmonic on the open ball $B(z_0, r)$, and in particular upper semicontinuous there. It follows that $O = \bigcup_{p \in \mathbb{N}} F_p$ with

$$F_p = \left\{z \in B\left(z_0, \left(1 - \frac{1}{p}\right) r\right)\colon \Phi^*(z) \geqslant -\frac{1}{4} + \frac{1}{p}\right\},$$

a bounded closed and hence compact subset of $\mathbb{C}^n$. Defining $K_p = F_p \cup K$ we find that K_p is a compact $\mathcal{F}$-neighborhood of a. We infer that $K_p \nearrow O$ as $p \nearrow \infty$, and consequently

$$M_1(O) = \sup_{p \in \mathbb{N}} M_1(K_p) \geqslant \sup_{p \in \mathbb{N}} M_2(K_p) = M_2(O).$$

By Lemma 3.2, this completes the proof of the 'only if part' of Theorem 3.1. $\square$

For the proof of the 'if part' of Theorem 3.1 we will need the following lemma, and some results of Bedford and Taylor on slicing of currents.

Lemma 3.3. *Let f be a bounded finely subharmonic function on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ and suppose that for every $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$ the function $f \circ h$ is finely subharmonic on $h^{-1}(\Omega)$. Then every $z_0 \in \Omega$ admits a* (compact) *$\mathcal{F}$-neighborhood K_{z_0} such that f can be written as*

$$f = f_1 - f_2 \quad \text{on } K_{z_0},$$

where f_1, f_2 are plurisubharmonic functions defined on a ball $B(z_0, r) \supset K_{z_0}$.

Proof. As in the proof of (a) of Theorem 2.4, we can assume that $-1 < f < 0$, and find a compact $\mathcal{F}$-neighborhood V of z_0 and a negative plurisubharmonic function φ on a ball $B(z_0, r) \supset V$ such that $\varphi(z_0) = -1/2$ and $\varphi = -1$ on $B(z_0, r) \setminus V$. For every $\lambda > 0$ we can form the function

$$u_\lambda(z) = \begin{cases} \max\{-\lambda, f(z) + \lambda\varphi(z)\} & \text{for } z \in \Omega \cap B(z_0, r), \\ -\lambda & \text{for } z \in B(z_0, r) \setminus V. \end{cases}$$

It is a bounded finely subharmonic function on $B(z_0, r)$, hence u_λ is subharmonic on $B(z_0, r)$. Similarly, for every $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$ the function $u_\lambda \circ h$ is finely subharmonic, hence subharmonic on $h^{-1}(B(z_0, r))$. From this we conclude that u_λ is in fact plurisubharmonic. Taking $\lambda = 4$, we see that

$$u_4(z) = f(z) + 4\varphi(z)$$

on the closed $\mathcal{F}$-neighborhood $K_{z_0} = \{z \in \Omega\colon\ \varphi(z) \geqslant -2/3\} \subset V \cap B(z_0, r)$, and K_{z_0} is compact along with V. This proves the lemma. □

Corollary 3.4. *We keep the notation as above. Then for every $z_0 \in \Omega$, f is $\mathcal{F}$-continuous on K_{z_0}, hence on Ω.*

We recall from [4] the concept of slice of an $(n-1, n-1)$-current, now on a domain D in $\mathbb{C}^n$. As usual we will write $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$ so that $dd^c u = 2i\partial\bar\partial u$. Let T be an $(n-1, n-1)$-current on D.

The *slice* of T with respect to a hyperplane $z_1 = a$ is the current

$$\langle T, z_1, a\rangle(\psi) = \lim_{\epsilon \to 0} \frac{1}{\pi\epsilon^2} \int_{\{|z_1 - a| \leqslant \epsilon\} \cap D} \psi(z_2, \dots, z_n) \frac{1}{4} dd^c|z_1|^2 \wedge T.$$

Here ψ is a C_0^∞ test form on $z_1 = a$, extended to D independently of z_1.

Now let $u_1, \dots, u_{n-1}$ and w be bounded plurisubharmonic functions on D, and put $T = w\, dd^c u_1 \wedge \dots \wedge dd^c u_{n-1}$. Then by [4, Proposition 4.1], $\langle T, z_1, a\rangle$ exists for every $a \in \mathbb{C}$ and

$$\langle T, z_1, a\rangle = \frac{1}{2\pi} w\big(a, z'\big)\, dd^c u_1\big(a, z'\big) \wedge \dots \wedge dd^c u_{n-1}\big(a, z'\big).$$

Here $z' = (z_2, \dots, z_n)$.

Finally, if F is holomorphic on D and $M = \{z \in D\colon\ F(z) = 0\}$, then by changing variables and since only regular points of M have to be taken into account, one gets

$$\big\langle w\big(dd^c u\big)^{n-1}, F, 0\big\rangle = w|_M \big(dd^c u\big)^{n-1}.$$

We write $\epsilon' = (\epsilon_2, \dots, \epsilon_n)$, $\epsilon'^2 = \prod_{j=2}^n \epsilon_j^2$, and $|z'| < \epsilon'$ for $|z_j| < \epsilon_j$, $j = 2, \dots, n$.

Lemma 3.5. *Let $\psi = \psi(z_1)$ be a test function on $\{z \in D\colon\ z' = 0'\}$, and let w and u be bounded plurisubharmonic functions on a bounded domain $D \subset \mathbb{C}^n$. Then*

$$\int_{\{z_2 = 0, \dots, z_n = 0\}} \psi(z_1) w\big(z_1, 0'\big)\, dd^c u\big(z_1, 0'\big) = \lim_{\epsilon' \downarrow 0} \frac{1}{2^{n-1}\epsilon'^2} \int_{\{|z'| < \epsilon'\}} \psi(z_1) w(z)\, dd^c|z_2|^2 \wedge \dots \wedge dd^c|z_n|^2 \wedge dd^c u. \tag{3.1}$$

Proof. Apply slicing with respect to $z_2 = 0$ to the current $T = w\, dd^c|z_3|^2 \wedge \dots \wedge dd^c|z_n|^2 \wedge dd^c u$ to obtain $\langle T, z_2, 0\rangle = \frac{1}{2\pi} w(z_1, 0, z_3, \dots, z_n)\, dd^c|z_3|^2 \wedge \dots \wedge dd^c|z_n|^2 \wedge dd^c u$. Next in $\{z_2 = 0\}$, apply slicing with respect to $z_3 = 0$ to the current $w\, dd^c|z_4|^2 \wedge \dots \wedge dd^c|z_n|^2 \wedge dd^c u$. Continuing in this fashion we obtain (3.1). □

Proof of the 'if part' of Theorem 3.1. Suppose for the moment that the 'if part' of the theorem holds under the extra hypothesis that f is bounded. If instead f is merely bounded from above then each of the functions $\max\{f, -p\}$ ($p \in \mathbb{N}$) is bounded and has the properties required in the 'if part'. It follows that $\max\{f, -p\}$ is weakly $\mathcal{F}$-plurisubharmonic, and so is therefore f.

Having thus reduced the proof of the 'if part' of Theorem 3.1 to the case where f is bounded, we proceed as follows, keeping our notation from Lemmas 3.3 and 3.5. First we will show that $dd^c f \geqslant 0$ on the compact neighborhood $K = K_{z_0}$ of z_0 provided by Lemma 3.3. Next we apply Lemma 3.5 to show that the restriction of f to any complex line passing through z_0 is finely subharmonic on a fine neighborhood of z_0.

Let v be any plurisubharmonic function on a ball B in $\mathbb{C}^n$, let h be a $\mathbb{C}$-affine bijection of $\mathbb{C}^n$, and let $\varphi \in C_0^\infty(B)$ be a test function. Then the action of the Riesz measure $\Delta(v \circ h)$ on $\varphi \circ h$ can be expressed as follows

$$4^{n-1}(n-1)! \int\limits_{h^{-1}(B)} \varphi \circ h(z) \Delta(v \circ h) = \int\limits_{h^{-1}(B)} \varphi \circ h(z)\, dd^c(v \circ h) \wedge \left(dd^c \|z\|^2\right)^{n-1}$$
$$= \int\limits_{B} \varphi(\zeta)\, dd^c v(\zeta) \wedge \left(dd^c \left\|h^{-1}(\zeta)\right\|^2\right)^{n-1}.$$

Returning to f, we have by Lemma 3.2 that the Riesz measure $\Delta(f \circ h)$ is positive on $h^{-1}(K)$, hence (with $h^{-1} = g$) we obtain that

$$dd^c f(\zeta) \wedge \left(dd^c \left\|g(\zeta)\right\|^2\right)^{n-1} \tag{3.2}$$

is a positive measure on K for every $\mathbb{C}$-affine bijection g of $\mathbb{C}^n$, and by continuity also for every $\mathbb{C}$-affine map $g : \mathbb{C}^n \to \mathbb{C}^n$.

To finish the proof we want to show that f restricted to a complex line L passing through z_0 is finely subharmonic in a fine neighborhood of z_0 relative to L. We write $z = (z_1, z')$ and can assume that $z_0 = 0$ and that L is given by $z' = 0'$. Because K is an $\mathcal{F}$-neighborhood of 0, there exists a bounded non-negative plurisubharmonic function w defined on a ball B_0 about 0 such that $w = 0$ on $B_0 \setminus K$, while $w(0) > 0$. Then $\{z \in B_0 \colon w(z) > 0\}$ is an $\mathcal{F}$-open subset of K that contains 0. On K we have $f = f_1 - f_2$ where f_1, f_2 are plurisubharmonic on a ball containing K, say on B_0. We apply Lemma 3.5 to f_1 and f_2 separately and subtract to obtain from (3.2) (with $g(z) = g(z_1, z') = (0, z')$)

$$\int\limits_{\{z_2=0,\dots,z_n=0\}} \psi(z_1) w\left(z_1, 0'\right) dd^c f\left(z_1, 0'\right) = \lim_{\epsilon' \downarrow 0} \frac{1}{2^{n-1}\epsilon'^2} \int\limits_{\{|z'|<\epsilon'\}} \psi(z_1) w(z)\, dd^c|z_2|^2 \wedge \cdots \wedge dd^c|z_n|^2 \wedge dd^c f.$$

If ϵ' is sufficiently small then the integrals occurring in the limit on the right-hand side are non-negative for every non-negative test function ψ on $B_0 \cap \{z' = 0'\}$.

We conclude that the Riesz measure of $f|_L$ is positive on a neighborhood of 0. By Lemma 3.2, $f|_L$ is finely subharmonic on this neighborhood. Varying z_0 over L and using the sheaf-property, we find that $f|L$ is finely subharmonic. □

Corollary 3.6. *Let f be a bounded weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$ such that f admits the representation $f = f_1 - f_2$ of Lemma* 3.3 *on Ω, and let χ_K denote the characteristic function of a compact set K in Ω. Then for $\mathbb{C}$-affine functions $l_1, \dots, l_{n-1}$ on $K = K_{z_0}$ from Lemma* 3.3

$$\int\limits_{\Omega} \chi_K(z)\, dd^c|l_1|^2 \wedge \cdots \wedge dd^c|l_{n-1}|^2 \wedge dd^c f \geqslant 0.$$

Proof. This follows from (3.2) with $g(z) = (0, l_1(z), \dots, l_{n-1}(z))$. □

From Theorem 3.1 we derive the following two results, one about removable singularities for weakly $\mathcal{F}$-plurisubharmonic functions, and the other about the supremum of a family of such functions.

Theorem 3.7. *Let $f : \Omega \to [-\infty, +\infty[$ be $\mathcal{F}$-locally bounded from above on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$, and let E be an $\mathcal{F}$-closed pluripolar subset of Ω. If f is weakly $\mathcal{F}$-plurisubharmonic on $\Omega \setminus E$ then f has a unique extension to a weakly $\mathcal{F}$-plurisubharmonic function on all of Ω, and this extension f^* is given by*

$$f^*(z) = \mathcal{F}\text{-}\limsup_{\substack{\zeta \to z \\ \zeta \in \Omega \setminus E}} f(\zeta), \quad z \in \Omega.$$

Proof. The function f^* (the $\mathcal{F}$-upper semicontinuous regularization of f) equals f on the $\mathcal{F}$-open set $\Omega \setminus E$ because f is $\mathcal{F}$-upper semicontinuous on $\Omega \setminus E$. Furthermore, f^* is $\mathcal{F}$-upper semicontinuous and $< +\infty$ on all of Ω (finiteness because f is $\mathcal{F}$-locally bounded from above). Therefore, by the 'only if part' of Theorem 3.1, for any $\mathbb{C}$-affine bijection h of $\mathbb{C}^n$, the function $f^* \circ h$ is $\mathbb{R}^{2n}$-finely hypoharmonic on $h^{-1}(\Omega \setminus E) = h^{-1}(\Omega) \setminus h^{-1}(E)$ and $\mathcal{F}$-upper semicontinuous $< +\infty$ on $h^{-1}(\Omega)$. In particular, $f^* \circ h < +\infty$ is $\mathbb{R}^{2n}$-finely upper semicontinuous on $h^{-1}(\Omega)$. Because E is pluripolar so is $h^{-1}(E)$, which thus is $\mathbb{R}^{2n}$-polar. According to [12, Theorem 9.14], $f \circ h$ is therefore $\mathbb{R}^{2n}$-finely hypoharmonic on all of $h^{-1}(\Omega)$, and so f^* is indeed weakly $\mathcal{F}$-plurisubharmonic on Ω, by the 'if part' of Theorem 3.1. Because the pluripolar set E has empty $\mathcal{F}$-interior, f^* is the only weakly $\mathcal{F}$-plurisubharmonic and hence $\mathcal{F}$-continuous extension of f to Ω. $\square$

In view of Lemma 2.8 there is a similar result about removable singularities for weakly $\mathcal{F}$-holomorphic functions:

Corollary 3.8. *Let $h : \Omega \to \mathbb{C}$ be $\mathcal{F}$-locally bounded on Ω ($\mathcal{F}$-open in $\mathbb{C}^n$). If h is weakly $\mathcal{F}$-holomorphic on $\Omega \setminus E$ (E is $\mathcal{F}$-closed and pluripolar in $\mathbb{C}^n$) then h extends uniquely to a weakly $\mathcal{F}$-holomorphic function $h^* : \Omega \to \mathbb{C}$, given by*

$$h^*(z) = \mathcal{F}\text{-}\lim_{\substack{\zeta \to z \\ \zeta \in \Omega \setminus E}} h(\zeta), \quad z \in \Omega.$$

Theorem 3.9. *Let Ω denote an $\mathcal{F}$-open subset of $\mathbb{C}^n$. For any uniformly $\mathcal{F}$-locally upper bounded family of weakly $\mathcal{F}$-plurisubharmonic functions f_α on Ω, the least $\mathcal{F}$-upper semicontinuous majorant f^* of the pointwise supremum $f = \sup_\alpha f_\alpha$ is likewise weakly $\mathcal{F}$-plurisubharmonic on Ω, and $\{z \in \Omega\colon\ f(z) < f^*(z)\}$ is pluripolar.*

Proof. We may assume that the set A of indices α is upper directed and that the net $(f_\alpha)_{\alpha \in A}$ is increasing; furthermore that Ω is $\mathcal{F}$-connected and that $f_\alpha \not\equiv -\infty$ for some $\alpha \in A$. For any function $f : \Omega \to [-\infty, +\infty[$ which is $\mathcal{F}$-locally bounded from above, write

$$f^*(z) = \mathcal{F}\text{-}\limsup_{\zeta \to z} f(\zeta), \qquad \check{f}(z) = \mathbb{R}^{2n}\text{-}\operatorname{fine}\limsup_{\zeta \to z} f(\zeta).$$

Then $\check{f}(z) \leqslant f^*(z) < +\infty$, the former inequality because the $\mathbb{R}^{2n}$-fine topology is finer than the $\mathcal{F}$-topology.

As in Theorem 3.1, let $h : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-affine bijection, and note that

$$f \circ h = \sup_\alpha (f_\alpha \circ h), \qquad (f \circ h)\check{} = \check{f} \circ h, \quad \text{on } h^{-1}(\Omega),$$

the latter equation because h is an $\mathbb{R}^{2n}$-fine homeomorphism. By Theorem 3.1, $f_\alpha \circ h$ is $\mathbb{R}^{2n}$-finely hypoharmonic. Now $f_\alpha \circ h \leqslant f^* \circ h$. Furthermore, f^* and hence $f^* \circ h$ and $\check{f} \circ h$ are $\mathbb{R}^{2n}$-finely locally bounded from above. It follows by [12, Lemma 11.2] that $\check{f} \circ h = (f \circ h)\check{}$ is $\mathbb{R}^{2n}$-finely hypoharmonic.

We proceed to show that $\check{f} = f^*$ on Ω, and hence that $\check{f}$ is $\mathcal{F}$-upper semicontinuous there. Invoking also Theorem 3.1 we shall thus altogether find that $\check{f} = f^*$ becomes $\mathcal{F}$-plurisubharmonic on Ω, and in particular $\mathcal{F}$-continuous there, by Theorem 2.4(c).

Consider a point $z_0 \in \Omega$ such that $f(z_0) > -\infty$. Fix $\beta \in A$ with $f_\beta(z_0) > -\infty$, and choose an $\mathcal{F}$-open $\mathcal{F}$-neighborhood U of z_0 so that $U \subset \Omega$ and

$$f_\beta(z_0) - 1 < f_\beta \leqslant f^* < f^*(z_0) + 1 \quad \text{on } U,$$

noting that the weakly $\mathcal{F}$-plurisubharmonic function f_β is $\mathcal{F}$-continuous and that f^* is $\mathcal{F}$-upper semicontinuous and $< +\infty$. Since $f_\beta \leqslant f_\alpha \leqslant f$ for every $\alpha \succcurlyeq \beta$ in A, any such f_α maps U into some fixed bounded interval. According to Theorem 2.4(a), (b) there exist $r > 0$, an $\mathcal{F}$-open set O such that $z_0 \in O \subset B(z_0, r)$, and locally bounded ordinary plurisubharmonic functions φ_α and ψ on $B(z_0, r)$ such that $f_\alpha = \varphi_\alpha - \psi$ on O for every $\alpha \succcurlyeq \beta$ in A. The net (φ_α) is increasing, along with the given net (f_α). The plurisubharmonic functions φ_α and ψ are $\mathcal{F}$-continuous, in particular $\mathbb{R}^{2n}$-finely continuous. Writing $\sup_\alpha \varphi_\alpha = \varphi$ and denoting by $\bar{\varphi}$ the Euclidean $\mathbb{R}^{2n}$-subharmonic regularization of φ in $B(z_0, r)$, we therefore have $\check{\varphi} = \bar{\varphi}$ there, by Brelot's fundamental convergence theorem, see e.g. [6, 1.XI.7]. Because $\check{\varphi} \leqslant \varphi^* \leqslant \bar{\varphi}$ it follows that $\check{\varphi} = \varphi^*$ in $B(z_0, r)$, and consequently

$$\check{f} = (\varphi - \psi)\check{} = \check{\varphi} - \psi = \varphi^* - \psi = (\varphi - \psi)^* = f^* \quad \text{on } O$$

since ψ is $\mathcal{F}$-continuous and hence $\mathbb{R}^{2n}$-finely continuous on the $\mathcal{F}$-open, hence $\mathbb{R}^{2n}$-finely open set $O \subset B(z_0, r)$.

Next, the set $\{z \in O\colon\ f(z) < f^*(z)\} = \{z \in O\colon\ \varphi(z) < \varphi^*(z)\}$ is pluripolar, by the deep theorem of Bedford and Taylor [2], or see [21, Theorem 4.7.6]. Writing

$$E = \bigl\{z \in \Omega\colon\ f(z) < f^*(z)\bigr\}, \qquad e = \bigl\{z \in \Omega\colon\ f(z) = -\infty\bigr\},$$

we have thus found that every point $z_0 \in \Omega \setminus e$ has an $\mathcal{F}$-neighborhood $O \subset \Omega \setminus e$ for which $O \cap E$ is pluripolar. Because $e = \bigcap_{\alpha \in A}\{z \in \Omega\colon\ f_\alpha(z) = -\infty\}$ is $\mathcal{F}$-closed relative to Ω, and pluripolar (some f_α being $\not\equiv -\infty$), we infer by the quasi-Lindelöf principle [3, Theorem 2.7] that indeed E is pluripolar. Finally, we have found that f^* is $\mathcal{F}$-plurisubharmonic on each $\mathcal{F}$-open set O as above (as z_0 varies), and hence on their union $\Omega \setminus e$, by the sheaf property. Because f^* is $\mathcal{F}$-upper semicontinuous and $< +\infty$ on Ω, and e is pluripolar, we conclude from Theorem 3.7 above that indeed f^* is weakly $\mathcal{F}$-plurisubharmonic on all of Ω. $\square$

Taking for Ω a *Euclidean* open set we obtain in particular the following

Corollary 3.10. *For any family $\{f_\alpha\}$ of ordinary plurisubharmonic functions on a Euclidean open set $\Omega \subset \mathbb{C}^n$ such that $f := \sup_\alpha f_\alpha$ is locally bounded from above, the least plurisubharmonic majorant of f exists and can be expressed as the upper semicontinuous regularization of f in the Euclidean topology on $\mathbb{C}^n$, as well as in the $\mathcal{F}$-topology and in the $\mathbb{R}^{2n}$-fine topology; that is, $\bar{f} = f^* = \check{f}$.*

The version of this involving the *Euclidean* topology is due to Lelong [23], or see [24, p. 26] or [21, Theorem 2.9.10]. Being locally bounded from above, f is in particular $\mathcal{F}$-locally bounded from above, and hence so is f^*, which is $\mathcal{F}$-plurisubharmonic by Theorem 3.9. Because Ω is Euclidean open, it follows by Proposition 2.14 that f^* even is an ordinary plurisubharmonic function. From $f \leqslant f^* \leqslant \bar{f}$ it therefore follows that $f^* = \bar{f}$. Similarly, $\check{f} = \bar{f}$ in view of [12, Theorem 9.8(a)].

The identity $f^* = \bar{f}$ is perhaps new even for ordinary plurisubharmonic functions on a Euclidean open set.

We close this section with an alternative proof of the 'only if part' of Theorem 3.1. It is a bit shorter than the proof given above. On the other hand it draws substantially on the theory of functions of Beppo Levi and Deny, cf. [5], and its connection to fine potential theory, cf. [15]. We will need this approach again in Section 4.

Following Deny [5] and subsequently [15] we consider for a given Greenian domain D (denoted Ω in [5] and [15]) of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ the complex Hilbert space

$$\widehat{\mathcal{D}}^1(D),$$

the completion of $\mathcal{D}(D) = C_0^\infty(D, \mathbb{C})$ in the Dirichlet norm $\|u\|_1 = \|\nabla u\|_{L^2(D,\mathbb{C})}$. (For $n \geqslant 2$ we may thus take $D = \mathbb{C}^n$. For $n = 1$, any bounded domain D will do.) Note that $\widehat{\mathcal{D}}^1(D)$ is a space of distributions [5, Théorème 2.1, p. 350]. Elements of $\widehat{\mathcal{D}}^1(D)$ may be represented by quasi-continuous functions that are finite quasi-everywhere. For an $\mathbb{R}^{2n}$-finely open set $\Omega \subset D$ denote by $\widehat{\mathcal{D}}^1(D, \Omega)$ the Hilbert subspace consisting of all $\varphi \in \widehat{\mathcal{D}}^1(D)$ such that some (and hence any) $\mathbb{R}^{2n}$-quasi-continuous representative of φ satisfies $\varphi = 0$ $\mathbb{R}^{2n}$-quasi-everywhere on $D \setminus \Omega$, cf. [5, Théorème 5.1, pp. 358 f.]. The positive cone in for example $\widehat{\mathcal{D}}^1(D, \Omega)$ is denoted by $\widehat{\mathcal{D}}^1_+(D, \Omega)$. Let V_l denote Lebesgue measure on $\mathbb{C}^l$, and write $V_n = V$.

According to [15, Théorème 11] an $\mathbb{R}^{2n}$-finely continuous (hence quasi-continuous) function $f \in \widehat{\mathcal{D}}^1(D)$ is finely subharmonic quasi-everywhere (hence actually everywhere by [12, Theorem 9.14]) on Ω, if and only if $f < +\infty$ and the inequality sign holds in (3.3):

$$\frac{1}{4}\int_D \nabla f \cdot \nabla\varphi \, dV = \sum_{j=1}^n \int_D (\partial_j f)(\bar{\partial}_j \varphi)\, dV \leqslant 0 \tag{3.3}$$

for every $\varphi \in \widehat{\mathcal{D}}^1_+(D, \Omega)$. (It suffices of course to integrate over Ω.)

Alternative proof of the 'only if part' of Theorem 3.1. Consider a weakly $\mathcal{F}$-plurisubharmonic function f on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$; hence f is $\mathcal{F}$-continuous and $< +\infty$. We leave out the trivial case $n = 1$. We may assume that $f > -\infty$ on Ω (otherwise replace f by $\max\{f, -p\}$ and let $p \to +\infty$). It suffices to prove that f is $\mathbb{R}^{2n}$-finely hypoharmonic.

Write $z = (z_1, \ldots, z_n) = (z_1, z') \in \mathbb{C}^n$. According to Theorem 2.4(a), every point $z_0 \in \Omega$ then has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which $f = f_1 - f_2$, f_1 and f_2 being bounded plurisubharmonic $> -\infty$ on some open ball $B = B(z_0, r)$ containing O. In particular, f_1 and f_2 are $\mathbb{R}^{2n}$-subharmonic on B. We may further assume that $-f_1$ and $-f_2$ are $\mathbb{R}^{2n}$-*potentials* on B, for otherwise we may replace $-f_i$ for $i = 1, 2$ by its swept-out (relative to B) $\widehat{R}^A_{-f_i}$ on $A = B(z_0, r/2)$ (and O by $O \cap A$). In terms of the Green kernel G on B we therefore may write $-f_i = G\mu_i$ on B for some bounded positive measure μ_i of compact support in B. Since $-f_i$ is bounded, its G-energy $\int G\mu_i \, d\mu_i$ is finite, and hence $G\mu_i$ is of Sobolev class $W_0^{1,2}(B) \subset \widehat{\mathcal{D}}^1(\mathbb{C}^n, B)$ [22, pp. 91–99], cf. [5, Théorème 3.1, p. 315].

For every $z' \in \mathbb{C}^{n-1}$ we have the $\mathbb{C}$-finely open set

$$O(z') = \{z_1 \in \mathbb{C}\colon (z_1, z') \in O\}.$$

Because f is weakly $\mathcal{F}$-plurisubharmonic and $> -\infty$ on O, $f|_{L\cap O}$ is finely subharmonic for every complex line L in $\mathbb{C}^n$. It follows that (3.3) holds with z replaced by z_1 and with O replaced by $O(z')$ for each $z' \in \mathbb{C}^{n-1}$:

$$\int_{O(z')} \nabla_1 f(z_1, z') \cdot \nabla_1 \varphi(z_1, z')\, dV_1 \leqslant 0. \tag{3.4}$$

Here $\nabla_1 = (\partial/\partial x_1, \partial/\partial y_1)$. Integrating (3.4) with respect to V_{n-1} leads by Fubini's theorem to

$$\int_O \nabla_1 f(z_1, z') \cdot \nabla_1 \varphi(z_1, z')\, dV \leqslant 0.$$

Similarly with the subscript 1 replaced by any $j \in \{1, \ldots, n\}$. After addition this leads to

$$\int_O \nabla f \cdot \nabla \varphi \, dV \leqslant 0.$$

According to [15, Théorème 11] and [12, Theorem 9.14], this shows that f indeed is $\mathbb{R}^{2n}$-finely subharmonic on O, and hence, by varying z_0, on all of Ω. □

4. Biholomorphic invariance

The sigma-algebra QB of *quasi Borel sets* in $\mathbb{C}^n$ is generated by the Borel sets and the sets of capacity 0 (see [3]). QB contains the finely open sets. All currents originating from wedge products of dd^c of bounded plurisubharmonic functions have measure coefficients that are Borel measures and put no mass on pluripolar sets, hence they extend naturally to QB.

First let us extend Corollary 3.6 to holomorphic functions $g_1, \ldots, g_{n-1}$ in place of affine functions $l_1, \ldots, l_{n-1}$.

Proposition 4.1. *Let f be a bounded weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$ such that f admits the representation $f = f_1 - f_2$ of Lemma* 3.3 *on Ω, and let χ_K denote the characteristic function of a compact set $K \subset \Omega$. Then for holomorphic functions $g_1, \ldots, g_{n-1}$ on $K = K_{z_0}$ from Lemma* 3.3

$$\int_\Omega \chi_K(z) \, dd^c|g_1|^2 \wedge \cdots \wedge dd^c|g_{n-1}|^2 \wedge dd^c f \geqslant 0. \tag{4.1}$$

Proof. Corollary 3.6 yields that (4.1) is valid for compact sets $\tilde{K} \subset K_{z_0}$ and $\mathbb{C}$-affine functions g_i. For arbitrary holomorphic functions g_j we have

$$\int_\Omega \chi_K(z) \, dd^c|g_1|^2 \wedge \cdots \wedge dd^c|g_{n-1}|^2 \wedge dd^c f = \lim_{N \to \infty} \sum_{j=1}^N \int_\Omega \chi_{E_j^N}(z) \, dd^c \left|l_1^{j,N}\right|^2 \wedge \cdots \wedge dd^c \left|l_{n-1}^{j,N}\right|^2 \wedge dd^c f \tag{4.2}$$

for suitable quasi Borel sets E_j^N and complex affine approximants $l_k^{j,N}$ of g_k on E_j^N $(k = 1, \ldots, n)$. Hence the right-hand side of (4.2) is indeed non-negative. □

Next we give two characterizations of functions of class $\widehat{\mathcal{D}}^1(\mathbb{C}^n)$ (see the fourth paragraph following Corollary 3.10) which are $\mathcal{F}$-plurisubharmonic on an $\mathcal{F}$-open set.

Theorem 4.2. *Let Ω be $\mathcal{F}$-open in $\mathbb{C}^n$ $(n \geqslant 2)$. Given an $\mathcal{F}$-continuous function $f \in \widehat{\mathcal{D}}^1(\mathbb{C}^n)$ with values in $[-\infty, +\infty[$, the following are equivalent*:

(a) *f is weakly $\mathcal{F}$-plurisubharmonic on Ω,*

(b) *for every $\varphi \in \widehat{\mathcal{D}}^1_+(\mathbb{C}^n, \Omega)$ and every $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$,*

$$\sum_{j,k=1}^n \lambda_j \bar{\lambda}_k \int_\Omega (\partial_j f)(\bar{\partial}_k \varphi) \, dV \leqslant 0,$$

(c) *for every regular holomorphic map $h : \omega \to \mathbb{C}^n$ (ω is open in $\mathbb{C}^n$), $f \circ h$ is weakly $\mathcal{F}$-plurisubharmonic on $h^{-1}(\Omega)$ $(\subset \omega)$.*

Proof. (a) ⇒ (b). Using the characterization of weakly $\mathcal{F}$-plurisubharmonic functions given in Theorem 3.1, one may adapt the proof of the ‘only if part’ of [21, Theorem 2.9.12] as follows. Suppose $f \in \widehat{\mathcal{D}}^1(\mathbb{C}^n)$ is weakly $\mathcal{F}$-plurisubharmonic on Ω, and so $f \circ T$ is $\mathbb{R}^{2n}$-finely subharmonic on $T^{-1}(\Omega)$ for any $\mathbb{C}$-affine bijection T of $\mathbb{C}^n$. To prove (b) with constant $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, take

$$T_\epsilon(z) = z_1 \lambda + \epsilon \sum_{l=2}^n z_l e_l, \quad \epsilon > 0,$$

where $(e_1, \ldots, e_n)$ denotes the canonical base of $\mathbb{C}^n$. From (3.3) we obtain (with integrations over $\mathbb{C}^n$), replacing Ω and φ, as we may, by $T_\epsilon^{-1}(\Omega)$ and $\varphi \circ T_\epsilon \in \widehat{\mathcal{D}}^1_+(\mathbb{C}^n, T_\epsilon^{-1}(\Omega))$,

$$
\begin{aligned}
0 &\geqslant \sum_{l=1}^{n} \int \partial_l (f \circ T_\epsilon) \bar{\partial}_l (\varphi \circ T_\epsilon)\, dV \\
&= \sum_{j,k=1}^{n} \int \big[(\partial_j f) \circ T_\epsilon\big]\big[(\bar{\partial}_k \varphi) \circ T_\epsilon\big]\big(\lambda_j \bar{\lambda}_k + O(\epsilon)\big)\, dV \\
&= |\det T_\epsilon|^2 \left(\sum_{j,k=1}^{n} \int (\partial_j f)(\bar{\partial}_k \varphi) \lambda_j \bar{\lambda}_k\, dV + O(\epsilon) \right).
\end{aligned}
$$

Here $\det T_\epsilon$ denotes the Jacobian of T_ϵ. This leads to (b) after division by $|\det T_\epsilon|^2$ when we make $\epsilon \to 0$.

(b) $\Rightarrow$ (a). Consider any $\mathbb{C}$-affine bijection $T = (T_1, \dots, T_n)$ of $\mathbb{C}^n$, say

$$
T_l(z) = \sum_{j=1}^{n} c_{lj} z_j + d_l, \quad l \in \{1, \dots, n\},\ z \in \mathbb{C},
$$

with $c_{lj}, d_l \in \mathbb{C}$ and $\det T \neq 0$. We obtain

$$
\begin{aligned}
\int \partial_l (f \circ T) \bar{\partial}_l (\varphi \circ T)\, dV &= \sum_{j,k=1}^{n} \int \big[(\partial_j f) \circ T\big]\big[(\bar{\partial}_k \varphi) \circ T\big] c_{lj} \bar{c}_{lk}\, dV \\
&= |\det T|^2 \sum_{j,k=1}^{n} \int c_{lj} \bar{c}_{lk} (\partial_j f)(\bar{\partial}_k \varphi)\, dV \leqslant 0
\end{aligned}
$$

by (b) with $\lambda_j = c_{lj}$. After division by $|\det T|^2$ and summation over l this shows according to (3.3) and Theorem 3.1 that the $\mathcal{F}$-continuous function $f < +\infty$ indeed is $\mathcal{F}$-plurisubharmonic on Ω.

(c) $\Rightarrow$ (a). This is contained in the ‘if part’ of Theorem 3.1 (even with h in (c) just a $\mathbb{C}$-affine bijection and with $f \circ h$ just $\mathbb{R}^{2n}$-finely subharmonic).

(a) $\Rightarrow$ (c). We may assume that $f > -\infty$ on Ω (otherwise replace f by $f^p := \max\{f, -p\}$, $p \in \mathbb{N}$, and let $p \to +\infty$). According to Theorem 2.4(a), every point $z_0 \in h^{-1}(\Omega)$ then has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset h^{-1}(\Omega)$ on which $f = f_1 - f_2$, f_1 and f_2 being bounded plurisubharmonic on some open set $D \subset \mathbb{C}^n$ containing O. In particular, Ω and O are $\mathbb{R}^{2n}$-finely open, and f_1 and f_2 are $\mathbb{R}^{2n}$-subharmonic on D. We may further assume that the Jacobian matrix $(\partial_j h_k)$ of the regular holomorphic map $h : \omega \to \mathbb{C}^n$ is bounded with determinant bounded away from 0.

Denoting by $\mathcal{S}(D, O)$ the convex cone of all functions of class $\widehat{\mathcal{D}}^1(D)$ which are $\mathbb{R}^{2n}$-finely superharmonic quasi-everywhere on O, we have by [15, p. 129] that $-f \in \mathcal{S}(D, O)$ and hence by [15, Théorème 11(b)]

$$
\int_O \nabla f \cdot \nabla \varphi\, dV \leqslant 0 \quad \text{for } \varphi \in \widehat{\mathcal{D}}^1_+(D, O).
$$

For any $\psi \in \widehat{\mathcal{D}}^1(\mathbb{C}^n)$ we have (by the properties of h required in (c)) $\psi \circ h \in \widehat{\mathcal{D}}^1(\omega)$. According to Theorem 3.1 it suffices to show that the $\mathcal{F}$-continuous function $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic on $h^{-1}(O)$. For this it suffices by (3.3) to prove that, for every $j \in \{1, \dots, n\}$,

$$
\int_{h^{-1}(O)} \big(\partial_j (f \circ h)\big)(\bar{\partial}_j \psi)\, dV \leqslant 0 \quad \text{for every } \psi \in \widehat{\mathcal{D}}^1_+\big(\mathbb{C}^n, h^{-1}(O)\big),
$$

and here ψ may be replaced equivalently by $\varphi \circ h$ with $\varphi \in \widehat{\mathcal{D}}^1_+(D, O)$ (or just as well with $\varphi \in \mathcal{D}_+(D, O)$). We take $j = 1$ and write

$$
dV = (i/2)^n dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n = (1/4)^n dd^c |z_1|^2 \wedge \dots \wedge dd^c |z_n|^2.
$$

By abuse of notation we will write $h^{-1} = (h_1^{-1}, h_2^{-1}, \dots, h_n^{-1})$ for the components of the inverse h^{-1} of the map h. Then by the chain rule we obtain

$$
\begin{aligned}
&\int_{h^{-1}(O)} \big(\partial_1 (f \circ h)\big)\big(\bar{\partial}_1 (\varphi \circ h)\big)\, dd^c |z_1|^2 \wedge \dots \wedge dd^c |z_n|^2 \\
&\quad = \int d(f \circ h) \wedge d^c(\varphi \circ h) \wedge dd^c |z_2|^2 \wedge \dots \wedge dd^c |z_n|^2 \\
&\quad = \int d(f \circ h) \wedge d^c(\varphi \circ h) \wedge dd^c \big|h_2^{-1} \circ h\big|^2 \wedge \dots \wedge dd^c \big|h_n^{-1} \circ h\big|^2
\end{aligned}
$$

$$= \int df \wedge d^c\varphi \wedge dd^c|h_2^{-1}|^2 \wedge \cdots \wedge dd^c|h_n^{-1}|^2 \tag{4.3}$$

$$= \int \varphi d\big(d^c f \wedge dd^c|h_2^{-1}|^2 \wedge \cdots \wedge dd^c|h_n^{-1}|^2\big) \tag{4.4}$$

$$= -\int \varphi\, dd^c f \wedge dd^c|h_2^{-1}|^2 \wedge \cdots \wedge dd^c|h_n^{-1}|^2.$$

The last three lines are in h-coordinates. Equality (4.3) is justified by approximating f and φ in $\widehat{\mathcal{D}}^1$ with functions in $\mathcal{D}$ and applying Stokes' theorem to the approximants. The final expression is non-positive because of Proposition 4.1, and we are done. □

Now we wish to consider the case where h is just some sort of plurifinely holomorphic map. Recall from the text preceding Proposition 2.9 that an n-tuple $(h_1, \dots, h_n)$ of strongly/weakly $\mathcal{F}$-holomorphic functions $h_j : U \to \mathbb{C}$ (U is $\mathcal{F}$-open in some $\mathbb{C}^m$) is termed a *strongly/weakly $\mathcal{F}$-holomorphic map* (or *curve* if $m = 1$).

The following concept of a strongly $\mathcal{F}$-biholomorphic map is an auxiliary one.

Definition 4.3 *(Plurifinely biholomorphic map).* A *strongly $\mathcal{F}$-biholomorphic map* h from an $\mathcal{F}$-open set $U \subset \mathbb{C}^n$ onto its image in $\mathbb{C}^n$ is an $\mathcal{F}$-homeomorphism with the property that there exist for every $z \in U$ a compact $\mathcal{F}$-neighborhood K_z of z in U and a C^∞-diffeomorphism Φ_z from an open neighborhood of K_z to its image in $\mathbb{C}^n$ such that $\Phi_z|_{K_z} = h|_{K_z}$ and that $\Phi_z|_{K_z}$ is a C^2-limit of biholomorphic maps defined on open sets containing K_z.

Proposition 4.4. *The composition $f \circ h$ of a weakly $\mathcal{F}$-plurisubharmonic function f on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ with a strongly $\mathcal{F}$-biholomorphic map $h : U \to \Omega$ (U is $\mathcal{F}$-open in $\mathbb{C}^n$) is weakly $\mathcal{F}$-plurisubharmonic on $h^{-1}(\Omega)$ ($\subset \mathbb{C}^n$).*

Proof. For $n = 1$ this is contained in [14, §4 and Théorème 11(c)] (in which h is any finely holomorphic function on U). Therefore suppose that $n \geqslant 2$. We may assume that Ω is $\mathcal{F}$-connected and that $f \not\equiv -\infty$, and so f is in particular $\mathbb{R}^{2n}$-finely subharmonic. As shown in the beginning of the alternative proof of Theorem 3.1 given at the end of Section 3 we may further suppose that Ω is bounded in $\mathbb{C}^n$ and that f is bounded and of class $\widehat{\mathcal{D}}^1(D)$ for some bounded domain $D \subset \mathbb{C}^n$ containing Ω. Fix $z \in U$ and let K_z be a compact $\mathcal{F}$-neighborhood of z in U on which h has the properties described in Definition 4.3. It will be sufficient to see that the expression (4.3) is non-positive if $\varphi \in \widehat{\mathcal{D}}^1_+(D, O)$ for some $\mathcal{F}$-open set $O \subset D$ with $z \in O \subset K_z$. Notice that $df \wedge d^c\varphi$ is a form with L^1 coefficients that is supported on K_z. Thus let (h_m) be a sequence of bi-holomorphic maps on open sets containing K_z that converge in C^2 to h on K_z. Then

$$\lim_{m\to\infty} dd^c|h_{m,2}^{-1}|^2 \wedge \cdots \wedge dd^c|h_{m,n}^{-1}|^2 = dd^c|h_2^{-1}|^2 \wedge \cdots \wedge dd^c|h_n^{-1}|^2,$$

uniformly on K_z. Now the expression (4.4) is non-positive when we replace h by h_m. By Lebesgue's dominated convergence theorem we conclude that (4.4) is also non-positive for h a strongly $\mathcal{F}$-biholomorphic map. □

It is reasonable to expect that the concept of weakly $\mathcal{F}$-plurisubharmonic function is invariant even under composition with suitable *weakly* $\mathcal{F}$-biholomorphic mappings. Currently, we do not know of a fine inverse function theorem for weakly $\mathcal{F}$-holomorphic maps of several variables. In fact we don't even know if weakly $\mathcal{F}$-holomorphic functions have weakly $\mathcal{F}$-holomorphic partial derivatives. However, we can handle the special case of a map of the form

$$G(z) = g(z_1) + (0, z_2, \dots, z_n), \tag{4.5}$$

where $g = (g_1, \dots, g_n)$ is a finely holomorphic curve in $\mathbb{C}^n$, and that turns out to be sufficient.

Theorem 4.5. *The composition $f \circ g$ of a weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) function f on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ with a finely holomorphic curve g in $\mathbb{C}^n$ defined on a finely open set $U \subset \mathbb{C}$, is finely hypoharmonic (resp. finely holomorphic) on the finely open pre-image $g^{-1}(\Omega)$ ($\subset \mathbb{C}$).*

Proof. The theorem is known for $n = 1$, cf. [14, §4 and Théorème 13(a)], so we suppose that $n \geqslant 2$. According to Theorem 2.10(a), g is continuous from U with the fine topology to $\mathbb{C}^n$ with the plurifine topology. The pre-image $g^{-1}(\Omega)$ therefore is finely open in $\mathbb{C}$, and $f \circ g$ is finely continuous. We may of course assume that f is bounded, that U is finely connected, and that g is non-constant, for example that g_1 is non-constant, hence a fine-to-fine open map [14, p. 64].

Given a point $z_0 \in g^{-1}(\Omega)$ ($\subset U \subset \mathbb{C}$) with $g_1'(z_0) \neq 0$, cf. [14, Corollaire 11], there exists a finely open set $O \subset \mathbb{C}$ such that $z_0 \in O \subset g^{-1}(\Omega)$ and that g_1 is injective on O with $g_1'(z) \neq 0$ for every $z \in O$, hence $(g_1|_O)^{-1}$ is finely holomorphic on the finely open set $g_1(O)$ [14, Théorème 13]. We may further assume after diminishing O that there exists a C^∞-map $\varphi : \mathbb{C} \to \mathbb{C}^n$ such that $\varphi = g$ on O and hence $\partial\varphi = g'$, $\bar\partial\varphi = 0$ on O [14, Théorème 11(c)]. Since $\partial\varphi_1(z_0) = g_1'(z_0) \neq 0$ we may arrange (by further diminishing O) that φ_1 is injective on some open set $\omega \subset \mathbb{C}$ containing the closure of O in $\mathbb{C}$, and hence that $\varphi_1|_\omega$ is a C^∞-diffeomorphism of ω onto $\varphi_1(\omega)$. Likewise, we may achieve that there exists a sequence of curves $\varphi^{(\nu)}$ such that each coordinate $\varphi_j^{(\nu)}$ of $\varphi^{(\nu)}$, $j \in \{1,\dots,n\}$, is a rational function defined on some open set $O^{(\nu)} \subset \omega$ containing O, and that $\varphi^\nu \to \varphi$, $(\varphi^{(\nu)})' \to \varphi'$ $(= \partial\varphi)$, uniformly on O as $\nu \to \infty$, cf. [14, Theorem 11(a)]. We also require that $\varphi_1^{(\nu)}$ is injective with $\partial_1 \varphi_1^{(\nu)} \neq 0$. With this final choice of O define $G : O \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ by (4.5), writing now $t \in \mathbb{C}^n$ in place of $z \in \mathbb{C}^n$.

For a given point $z = (z_1, z') \in O \times \mathbb{C}^{n-1}$ choose a compact fine neighborhood L_{z_1} of z_1 in $\mathbb{C}$ so that $L_{z_1} \subset O$ $(\subset \omega)$, and a number $c > \max\{|z_2|,\dots,|z_n|\}$. In analogy with (4.5) define

$$\Phi_z(t) = \varphi(t_1) + (0, t_2, \dots, t_n) \quad \text{for } t \in \omega \times \mathbb{C}^{n-1}.$$

It is easily verified that Φ_z is a C^∞-diffeomorphism of $\omega \times \mathbb{C}^{n-1}$ onto its image in $\mathbb{C}^n$. We have $\Phi_z = G$ on $O \times \mathbb{C}^{n-1}$, and in particular on the compact $\mathcal{F}$-neighborhood $K_z := L_{z_1} \times [-c, c]^{n-1}$ of z in $O \times \mathbb{C}^{n-1}$, and consequently $G : O \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ is a strongly $\mathcal{F}$-biholomorphic map.

Suppose first that f is weakly *$\mathcal{F}$-plurisubharmonic* on Ω. According to Proposition 4.4, $f \circ G$ is a weakly $\mathcal{F}$-plurisubharmonic map defined on the $\mathcal{F}$-open set $G^{-1}(\Omega) \subset O \times \mathbb{C}^{n-1}$. Now

$$G(t_1, 0, \dots, 0) = g(t_1) = \big(g_1(t_1), g_2(t_1), \dots, g_n(t_1)\big) \quad \text{for } t_1 \in O,$$

and hence $(f \circ G)(t_1, 0, \dots, 0) = (f \circ g)(t_1)$ for $t_1 \in O$. It follows that $f \circ g$ is finely hypoharmonic on O, hence so (by varying z_0) on $\{z \in g^{-1}(\Omega)\colon\ g_1'(z) \neq 0\}$, which differs only by a countable and hence polar set from $g^{-1}(\Omega)$, cf. [14, Théorème 15]. Because $f \circ g$ is finely continuous and $< +\infty$ on $g^{-1}(\Omega)$ $(\subset U)$ we conclude by the removable singularity theorem [12, Theorem 9.15] that indeed $f \circ g$ is finely hypoharmonic on $g^{-1}(\Omega)$.

Finally, let instead f be weakly *$\mathcal{F}$-holomorphic* on Ω, in particular weakly (complex) $\mathcal{F}$-harmonic, by [14, Définition 3]. As shown in the alternative proof of the 'only if part' of Theorem 3.1 given at the end of Section 3 we may suppose that Ω is contained in some bounded domain $D \subset \mathbb{C}^n$ and that f is bounded and of class $\widehat{\mathcal{D}}^1(D)$. Each of the functions $z \mapsto z_j f(z)$, $j \in \{1,\dots,n\}$, therefore is bounded and of class $\widehat{\mathcal{FD}}^1(D)$.

By the former part of the theorem, the bounded functions $\pm\mathrm{Re}(f \circ g)$ and $\pm\mathrm{Im}(f \circ g)$ are finely subharmonic on $g^{-1}(\Omega)$, and hence $f \circ g$ is (complex) finely harmonic there. Similarly, $(z_j f) \circ g = g_j \cdot (f \circ g)$ is finely harmonic on $g^{-1}(\Omega)$ $(\subset U \subset \mathbb{C})$, $j \in \{1,\dots,n\}$. We therefore may further assume that U is contained in a bounded domain $D_1 \subset \mathbb{C}$ and that $f \circ g$ and each g_j are bounded and of class $\widehat{\mathcal{D}}^1(D_1)$. Some component of g, say g_1, is non-constant, and it therefore again follows from [14, Théorème 13] that every point $z_0 \in g^{-1}(\Omega)$ $(\subset U)$ with $g_1'(z_0) \neq 0$ has a finely open fine neighborhood $O \subset g^{-1}(\Omega)$ on which g_1 is injective with $g_1' \neq 0$; and the inverse $h_1 := (g_1|_O)^{-1}$ is finely holomorphic on the finely open set $g_1(O)$. Hence, so is the re-parametrized curve $g \circ h_1 : g_1(O) \to \mathbb{C}^n$. The function h on $g_1(O) \times \mathbb{C}^{n-1}$ defined by

$$h(z) = h(z_1, \dots, z_n) = h_1(z_1)$$

is weakly $\mathcal{F}$-holomorphic. Therefore, so is hf, and consequently $(hf) \circ g$ is finely harmonic on O, like $f \circ g$ above. Note that $h \circ g = h_1 \circ g_1$. For $z \in O$,

$$\big[(hf) \circ g\big](z) = \big[(h \circ g)(z)\big]\big[(f \circ g)(z)\big] = \big[(h_1 \circ g_1)(z)\big]\big[(f \circ g)(z)\big] = t \cdot (f \circ g)(z),$$

and since this function $(hf) \circ g = z \cdot (f \circ g)$ of $z \in O$ is finely harmonic it follows according to the result of Lyons [25], cf. [14, §3], which was utilized in Lemma 2.8, that $f \circ g$ is finely holomorphic on O, hence (by varying z_0) quasi-everywhere on $\{t \in g^{-1}(\Omega)\colon\ g_1'(t) \neq 0\}$, and indeed everywhere on $g^{-1}(\Omega)$ by the removable singularity theorem for finely holomorphic functions [14, Corollaire 3]. $\square$

The following extension of Theorem 4.5 from finely holomorphic curves g to $\mathcal{F}$-holomorphic *maps* h is a strengthening of Theorem 2.10(b), (c), in which f was required to be *strongly* $\mathcal{F}$-plurisubharmonic (resp. strongly $\mathcal{F}$-holomorphic). Likewise, Theorem 4.6 (for a weakly $\mathcal{F}$-plurisubharmonic function f) extends Proposition 4.4 (in which $m = n$, and h is strongly biholomorphic).

Theorem 4.6. *The composition $f \circ h$ of a weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) function f on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ with a weakly $\mathcal{F}$-holomorphic map $h : U \to \mathbb{C}^n$ (U is $\mathcal{F}$-open in $\mathbb{C}^m$) is weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) on $h^{-1}(\Omega)$ $(\subset \mathbb{C}^m)$.*

Proof. According to Theorem 2.10(a), h is continuous from $U \subset \mathbb{C}^m$ to $\mathbb{C}^n$ with their respective plurifine topologies. It follows that $h^{-1}(\Omega)$ is $\mathcal{F}$-open in $\mathbb{C}^m$, and that $f \circ h$ is $\mathcal{F}$-continuous. Next, we restrict $f \circ h$ to a complex line L in $\mathbb{C}^m$, and observe that $h|_{L \cap U}$ is a finely holomorphic curve. By Theorem 4.5, $f \circ h$ restricted to $L \cap U$ therefore is finely hypoharmonic (resp. finely holomorphic), and we are done. $\square$

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