Controllability of stochastic semilinear functional differential equations in Hilbert spaces

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Abstract

In this paper approximate and exact controllability for semilinear stochastic functional differential equations in Hilbert spaces is studied. Sufficient conditions are established for each of these types of controllability. The results are obtained by using the Banach fixed point theorem. Applications to stochastic heat equation are given.

Keywords: Approximate controllability; Exact controllability; Semilinear stochastic functional differential equations; Banach fixed point theorem

1. Introduction

Stochastic partial functional differential equations with finite delays are very important as stochastic models of biological, chemical, physical and economical systems. The qualitative properties (existence, stability, invariant measures, controllability and others) of these systems have not been studied in great detail (see [1,4,13,15,16] and references therein). As a matter of fact, there exist extensive literature on the related topics for deterministic partial differential equations with finite delays (for example, see [2,7,8,18] and references therein). We would also like to mention that controllability questions for stochastic differential equations have already been investigated by Balasubramaniam and Dauer [3] and Mahmudov [11,12].

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The paper is organized as follows. In Section 2 we give definitions, preliminary results and prove needed lemmas. In Section 3, by using semigroup methods we discuss approximate controllability of mild solutions for a class of stochastic partial differential equations with finite delays,

\[ dX(t) = \left[-AX(t) + Bu(t) + f(t, X_t)\right] dt + g(t, X_t) dW(t), \quad t \geq 0, \]

\[ X_0 = \phi \in L_p(\Omega, C_\alpha), \]

where \( \phi \) is \( \mathcal{F}_0 \)-measurable and \(-A\) is a closed, densely defined linear operator generating an analytic semigroup \( S(t) \), \( t > 0 \), on a separable Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). Define the Banach space \( D(A^\alpha) \) with the norm \( \| x \|_\alpha := \| A^\alpha x \| \) for \( x \in D(A^\alpha) \), where \( D(A^\alpha) \) denotes the domain of the fractional power operator \( A^\alpha : H \to H \) (we refer the reader to [14] for a detailed presentation of the definition and relevant properties of \( A^\alpha \)).

Let \( H_\alpha := D(A^\alpha) \) and \( C_\alpha = C([-r, 0], H_\alpha) \) be the space of all continuous functions from \([-r, 0]\) into \( H_\alpha \), where \( 0 < r < \infty \). Let \( K, U \) be another separable Hilbert spaces. Suppose \( W(t) \) is given \( K \)-valued Wiener process with a finite trace nuclear covariance operator \( Q \geq 0 \). Assume \( f : [0, \infty) \times C_\alpha \to H \) and \( g : [0, \infty) \times C_\alpha \to L^2_0(K, H) \) are two measurable mappings such that \( f(t, 0) \) and \( g(t, 0) \) are locally bounded in \( H \)-norm and \( L^2_0(K, H) \)-norm, respectively. Here \( L^2_0(K, H) \) denotes the space of all \( Q \)-Hilbert–Schmidt operators from \( K \) into \( H \). We also employ the same notation \( \| \cdot \| \) for the norm of \( L(K, H) \), where \( L(K, H) \) denotes the space of all bounded linear operators from \( K \) into \( H \).

In Section 3 we state and prove our main result on approximate controllability of stochastic functional differential equations in Hilbert spaces. Assuming the approximate controllability of the corresponding deterministic system under some conditions we prove the approximate controllability of system (1). Apparently the result is also new for deterministic functional differential equations.

Notice that when the semigroup \( S(t) \), \( t > 0 \), is compact an infinite-dimensional linear deterministic system is cannot be exactly controllable [17]. So, the analogue for exact controllability of the results of Section 3 cannot hold in infinite-dimensional space.

In Section 4 exact controllability of the system (1) is investigated. The compactness of the semigroup \( S(t) \) is not assumed, and conditions are obtained for exact controllability of the system (1). In Section 5 an application to the stochastic heat equation is given.

2. Preliminaries

Let \( (\Omega, \mathcal{F}, P) \) be a probability space on which an increasing and right continuous family \( \{\mathcal{F}_t : t \geq 0\} \) of complete sub-\( \sigma \)-algebras of \( \mathcal{F} \) is defined. Suppose \( X(t) : \Omega \to H_\alpha, t \geq -r \), is a continuous \( \mathcal{F}_t \)-adapted, \( H_\alpha \)-valued stochastic process we can associate with another process \( X_t : \Omega \to C_\alpha, t \geq 0 \), by setting \( X_t = \{X(t + s)(\omega) : s \in [-r, 0]\} \). This is regarded as a \( C_\alpha \)-valued stochastic process. Let \( \beta_n(t) (n = 1, 2, \ldots) \) be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over \( (\Omega, \mathcal{F}, P) \). Set

\[ W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad t \geq 0, \]
where $\lambda_n \geq 0$ ($n = 1, 2, \ldots$) are nonnegative real numbers and $\{e_n\}$ ($n = 1, 2, \ldots$) is a complete orthonormal basis in $K$. Let $Q \in L(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $\text{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$. Then the above $K$-valued stochastic process $W(t)$ is called a $Q$-Wiener process. We assume that $\mathcal{F}_t = \sigma(W(s): 0 \leq s \leq t)$ is the $\sigma$-algebra generated by $W$ and $\mathcal{F}_T = \mathcal{F}_t$. Let $\psi \in L(K, H)$ and define

$$
\|\psi\|^2_Q := \text{tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \|\lambda_n \psi e_n\|^2.
$$

If $\|\psi\|^2_Q < \infty$, then $\psi$ is called a $Q$-Hilbert–Schmidt operator. Let $L^Q_2(K, H)$ denote the space of all $Q$-Hilbert–Schmidt operators $\psi : K \to H$.

Recall that $f$ is said to be $\mathcal{F}_t$-adapted if $f(t, \cdot) : \Omega \to H$ is $\mathcal{F}_t$-measurable, a.e. $t \in [0, T]$ and $\mathcal{F}_0$-measurable, a.e. $t \in [-r, 0]$. Let $MC_\alpha(0, p)$, $p > 2$, denote the space of all $\mathcal{F}_0$-measurable functions that belong to $L_p(\Omega, C_\alpha)$, that is, $MC_\alpha(0, p)$, $p > 2$, is the space of all $\mathcal{F}_0$-measurable $C_\alpha$-valued functions $\psi : \Omega \to C_\alpha$ with the norm

$$
E\|\psi\|_{C_\alpha}^p = E\left\{ \sup_{-r \leq t \leq 0} \|A^\alpha \psi(s)\|^p \right\} < \infty.
$$

Let $L^Q_p(0, T; H)$ be the closed subspace of $L_p([0, T] \times \Omega, H)$ consisting of $\mathcal{F}_t$-adapted processes. Let $C([-r, T], L_p(\Omega, \mathcal{F}, P; H))$ be the Banach space of all continuous maps from $[-r, T]$ into $L_p(\Omega, \mathcal{F}, P; H)$ satisfying the condition $\sup_{t \in [-r, T]} E\|X(t)\|^p < \infty$. Let $\mathcal{F}_p$ be the closed subspace of all continuous processes $X$ that belong to the space $C([-r, T], L_p(\Omega, \mathcal{F}, P; H))$ consisting of measurable and $\mathcal{F}_t$-adapted processes $X = \{X(t): t \in [-r, T]\}$ with $\|X\|_{\mathcal{F}_p} < \infty$, where

$$
\|X\|_{\mathcal{F}_p} = \left( \sup_{t \in [-r, T]} E\|X_t\|^p_C \right)^{1/p} = \left( \sup_{t \in [0, T]} E\|X_t\|^p \right)^{1/p}.
$$

**Assumption A.** $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t > 0$, on the separable Hilbert space $H$ and $0 \in \rho(A)$.

Under Assumption A, the following results relating $A^\alpha$ and the analytic semigroup $S(t)$ generated by $-A$ hold, see [14].

**Lemma 1.** Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$. If $0 \in \rho(A)$, then:

1. There exist a constant $M > 0$ and a real number $a > 0$ such that
   $$
   \|S(t)h\| \leq Me^{-at}\|h\|, \quad t \geq 0, \text{ for any } h \in H. \tag{2}
   $$
2. The fractional power $A^\alpha$ satisfies that
   $$
   \|A^\alpha S(t)h\| \leq M_\alpha e^{-at}t^{-\alpha}\|h\|, \quad t > 0, \tag{3}
   $$
   for any $h \in H$, where $M_\alpha > 0$.
3. Let $0 < \alpha \leq 1$ and $h \in D(A^\alpha)$, then
   $$
   \|S(t)h - h\| \leq N_\alpha t^\alpha\|A^\alpha h\|, \quad N_\alpha > 0. \tag{4}
   $$
Assumption B. For arbitrary \( \gamma, \xi \in C_\alpha \) and \( 0 \leq t \leq T \), suppose that there exists positive real constant \( N_1 > 0 \) such that
\[
\| f(t, \gamma) - f(t, \xi) \|_p + \| g(t, \gamma) - g(t, \xi) \|_Q \leq N_1 \| \gamma - \xi \|_{C_\alpha}^p.
\]
\[
\| f(t, \xi) \|_p + \| g(t, \xi) \|_Q \leq N_1 (1 + \| \xi \|_{C_\alpha}^p).
\]

Assumption B1. For arbitrary \( \gamma, \xi \in C_\alpha \) and \( 0 \leq t \leq T \), suppose that there exists positive real constant \( N_1 > 0 \) such that
\[
\| f(t, \gamma) - f(t, \xi) \|_p + \| g(t, \gamma) - g(t, \xi) \|_Q \leq N_1 \| \gamma - \xi \|_{C_\alpha}^p.
\]
\[
\| f(t, \xi) \|_p + \| g(t, \xi) \|_Q \leq N_1.
\]

Assumption C. For each \( 0 \leq s < T \) the operator \( \lambda (\lambda I + \Gamma_s^T)^{-1} \rightarrow 0 \) in the strong operator topology as \( \lambda \rightarrow 0^+ \), where \( \Gamma_s^T = \int_s^T S(T - r)BB^*S(T - r)dr \) is the controllability Grammian.

Notice that the deterministic linear system corresponding to (1) is approximately controllable on \([s, T]\) if and only if the operator \( \lambda (\lambda I + \Gamma_s^T)^{-1} \rightarrow 0 \) strongly as \( \lambda \rightarrow 0^+ \), see [9,10].

Definition 2. A stochastic process \( X \) is said to be a mild solution of Eq. (1) if the following conditions are satisfied:

1. \( X(t, \omega) \) is measurable as a function from \([0, T] \times \Omega \) to \( H \) and \( X(t) \) is \( \mathcal{F}_t \)-adapted;
2. \( \mathbb{E} \| X(t) \|_p < \infty \) for each \( t \in [-r, T] \);
3. For each \( u \in L^F_p(0, T; U) \) the process \( X \) satisfies the following integral equation:
\[
X(t) = S(t)\phi(0) + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)f(s, X_s)ds + \int_0^t S(t-s)g(s, X_s)dW(s),
\]
\[
X_0 = \phi \in MC_\alpha(0, p), \quad t \geq 0.
\] (5)

Definition 3. System (1) is approximately (exactly) controllable \([0, T]\) if
\[
\overline{R}(T) = L_p(\Omega, \mathcal{F}, P; H) \quad (R(T) = L_p(\Omega, \mathcal{F}, P; H)),
\]
where \( R(T) = \{ X(T) = X(T, u) : u \in L^F_p(0, T; U) \} \).

We also need the following lemmas (see [6, Proposition 4.15 and Lemma 7.2], respectively).
Lemma 4. If $\Phi \in L_{\alpha}^{2}(0, T; L_{2}^{0}(K, H))$, $A^{\alpha} \Phi \in L_{\alpha}^{2}(0, T; L_{2}^{0}(K, H))$ and $\Phi(t)k \in H_{\alpha}$, $t \geq 0$, for arbitrary $k \in K$, then

$$A^{\alpha} \int_{0}^{t} \Phi(s) dW(s) = \int_{0}^{t} A^{\alpha} \Phi(s) dW(s).$$

Lemma 5. For any $p > 2$, $\Phi \in L_{\alpha}^{p}(\Omega; L_{2}^{0}(K, H)))$ we have

$$E \left( \sup_{0 \leq s \leq t} \left\| \int_{0}^{s} \Phi(r) dW(r) \right\|^{p} \right) \leq c_{p} \sup_{0 \leq s \leq t} E \left( \int_{0}^{s} \left\| \Phi(r) dW(r) \right\|^{p} \right) \leq C_{p} \left( \int_{0}^{t} \left\| \Phi(r) \right\|_{Q}^{2} dr \right)^{p/2}, \quad t \in [0, T],$$

where

$$c_{p} = \left( \frac{p}{p-1} \right)^{p}, \quad C_{p} = \left( \frac{p}{2(p-1)} \right)^{p/2} \left( \frac{p}{p-1} \right)^{p^2/2}.$$

Lemma 6. For any $h \in L_{p}(\Omega, \mathcal{F}, P; H)$ there exists $\varphi \in L_{\alpha}^{p}(\Omega; L_{2}^{0}(K, H)))$ such that

$$h = Eh + \int_{0}^{T} \varphi(s) dW(s).$$

Proof. The proof for the case $p = 2$ is given in [9]. The general case can be proved by approximation argument. Define $h^{n} = h_{x_{[0,n]}(\|h\|)}$ in $L_{p}(\Omega, \mathcal{F}, P; H)$. Then for any $n \geq 1$ there exists $\varphi^{n} \in L_{2}^{\alpha}(0, T; L_{2}^{0}(K, H))$ such that $h^{n} = Eh^{n} + \int_{0}^{T} \varphi^{n}(s) dW(s)$. Let $I_{r}^{n} = \int_{0}^{r} \varphi^{n}(s) dW(s)$. Then by Burkholder–Davis–Gundy inequality and Lemma 5 we have

$$E \left( \sup_{0 \leq s \leq t} \left\| \int_{0}^{s} \varphi^{n}(s) dW(s) \right\|_{Q}^{2} ds \right)^{p/2} \leq l_{1} E \left( \sup_{0 \leq s \leq t} \left\| I_{r}^{n} - I_{r}^{m} \right\|_{Q}^{p} \right) \leq l_{2} \left\| h^{n} - h^{m} - Eh^{n} + Eh^{m} \right\|_{Q}^{p},$$

where $l_{1}, l_{2}$ are constants. Hence, there exists $\varphi \in L_{\alpha}^{2}(\Omega; L_{2}(0, T; L_{2}^{0}(K, H)))$ such that $\varphi^{n} \rightharpoonup \varphi$ in $L_{\alpha}^{2}(\Omega; L_{2}(0, T; L_{2}^{0}(K, H)))$ and consequently, taking the limit in the representation for $h^{n}$, we obtain desired representation. The lemma is proven.

Lemma 7. Let $p > 2$ and let $g \in L_{\alpha}^{p}(0, T; L_{2}^{0}(K, H))$. There exist a constant $N_{3} > 0$ and $\varepsilon > 0$ such that

$$E \sup_{-\varepsilon \leq \theta \leq 0} \left\| A^{\alpha} S(t + \theta - \tau) g(\tau) dW(\tau) \right\|_{Q}^{p} \leq N_{3} E \int_{0}^{t} \left\| g(\tau) \right\|_{Q}^{p} d\tau.$$
where
\[ N_3 = M^p_\alpha \left( \Gamma \left( 1 + q(\beta - 1 - \alpha) \right) (aq)^q (1+\alpha-\beta)^{p/q} C_p \frac{t^{p(1-2\beta)/2}}{(1-2\beta)^{p/2}} \right). \]

**Proof.** We use the factorization method introduced in [5], which is based on the following elementary identity:
\[ \int_{\tau}^{t} (t - s)^{\beta-1} (s - \tau)^{-\beta} ds = \frac{\pi}{\sin \pi \alpha}, \quad \tau \leq s \leq t, \quad 0 < \beta < 1. \quad (6) \]

By using identity (6) we obtain
\[ \int_0^{t+\theta} A^\alpha S(t + \theta - \tau) g(\tau) dW(\tau) \]
\[ = \frac{\sin \pi \alpha}{\pi} \int_0^{t+\theta} A^\alpha S(t + \theta - \tau) \left[ \int_{\tau}^{t+\theta} (t + \theta - s)^{\beta-1} (s - \tau)^{-\beta} ds \right] g(\tau) dW(\tau) \]
\[ = \frac{\sin \pi \alpha}{\pi} \int_0^{t+\theta} A^\alpha S(t + \theta - \tau) \left[ \int_{\tau}^{t+\theta} (t + \theta - s)^{\beta-1} (s - \tau)^{-\beta} ds \right] g(\tau) dW(\tau) \]
\[ = \frac{\sin \pi \alpha}{\pi} \int_0^{t+\theta} A^\alpha (t + \theta - s)^{\beta-1} S(t + \theta - s) R(s) ds, \]

where \( R(s) = \int_s^t S(s - \tau)(s - \tau)^{-\beta} g(\tau) dW(\tau) \). Fix \( 1/p + \alpha < \beta < 1/2 \). Applying the Hölder inequality we obtain that
\[ \mathbb{E} \sup_{-r \leq \theta \leq 0} \left\| \int_0^{t+\theta} A^\alpha S(t + \theta - \tau) g(\tau) dW(\tau) \right\|^p \]
\[ \leq \mathbb{E} \sup_{-r \leq \theta \leq 0} \left\| \int_0^{t+\theta} (t + \theta - s)^{\beta-1} A^\alpha S(t + \theta - s) R(s) ds \right\|^p \]
\[ \leq M^p_\alpha \mathbb{E} \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} (t + \theta - s)^{\beta-1-\alpha} e^{-a(t+\theta-s)} R(s) ds \right)^p \]
\[ \leq M^p_\alpha \mathbb{E} \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} (t + \theta - s)^{\beta-1-\alpha} e^{-aq(t+\theta-s)} ds \right)^{p/q} \left( \int_0^{t+\theta} \| R(s) \|^p ds \right) \]
\[ \leq M^p_\alpha \left( \Gamma \left( 1 + q(\beta - 1 - \alpha) \right) (aq)^q (1+\alpha-\beta)^{p/q} \mathbb{E} \int_0^t \| R(s) \|^p ds, \right. \quad (7) \]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). Moreover, by Lemma 5, there exists a constant \( C_p > 0 \) such that
\[
E \| R(s) \|^p \leq C_p E \left( \int_0^t (s - \tau)^{-2\beta} \| g(\tau) \|^2_Q d\tau \right)^{p/2},
\]
from which, using the Young inequality, we obtain
\[
\int_0^t E \| R(s) \|^p ds \leq C_p \int_0^t E \left( \int_0^t (s - \tau)^{-2\beta} \| g(\tau) \|^2_Q d\tau \right)^{p/2} ds
\]
\[
\leq C_p \left( \int_0^t (s - \tau)^{-2\beta} d\tau \right)^{p/2} E \int_0^t \| g(\tau) \|^p_Q d\tau
\]
\[
\leq C_p \frac{t^{p(1-2\beta)/2}}{(1-2\beta)^{p/2}} E \int_0^t \| g(\tau) \|^p_Q d\tau. \tag{8}
\]
From (7) and (8) it follows that
\[
E \sup_{-\theta \leq \tau \leq 0} \left\| \int_0^t \lambda A^\alpha S(t + \theta - \tau) dW(\tau) \right\|^p \leq N_1 E \int_0^t \| g(\tau) \|^p_Q d\tau,
\]
and the lemma is proven. \( \square \)

For any \( \lambda > 0 \) and \( h \in L_p(\Omega, \mathfrak{F}; P; H) \) define the control
\[
\hat{u}(t, X) = B^* S^*(T - t) \left( \lambda I + I_{S^*}^T \right)^{-1} (Eh - S(T)\phi(0))
\]
\[
- B^* S^*(T - t) \int_0^t \left( \lambda I + I_{S^*}^T \right)^{-1} S(T - s) f(s, X_s) ds
\]
\[
- B^* S^*(T - t) \int_0^t \left( \lambda I + I_{S^*}^T \right)^{-1} \left[ S(T - s) g(s, X_s) - \phi(s) \right] dW(s). \tag{9}
\]
where \( I_{S^*}^T = \int_0^T S(T - r) BB^* S^*(T - r) dr \) is the controllability Grammian and \( h = Eh + \int_0^T \phi(s) dW(s) \) by Lemma 6.

**Lemma 8.** There exists a positive real constant \( N_2 > 0 \) such that for all \( X, Y \in S_p \)
\[
E \| u^\lambda(t, X) - u^\lambda(t, Y) \|^p \leq \frac{1}{\lambda^p} N_2 \int_0^t E \| X_s - Y_s \|^p_{C_u} ds,
\]
\[
E \| u^\lambda(t, X) \|^p \leq \frac{1}{\lambda^p} N_2 \left( 1 + \int_0^t E \| X_s \|^p_{C_u} ds \right).
\]
**Proof.** We will only prove the first inequality, since the proof of the second is similar.

\[
E \left\| u^\lambda(t,X) - u^\lambda(t,Y) \right\|_p^p \\
\leq 2^{p-1} E \left\| B^* S^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} S(T-s) \left[ f(s,X_s) - f(s,Y_s) \right] ds \right\|_p^p \\
+ 2^{p-1} E \left\| B^* S^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} S(T-s) \right. \\
\times \left. \left[ g(s,X_s) - g(s,Y_s) \right] dW(s) \right\|_p^p \\
\leq \frac{1}{\lambda^p} 2^{p-1} \| B \|^p M^2 \rho N_1 e^{-2 \rho (T-t)} \int_0^t E \left\| X_s - Y_s \right\|_{C^\alpha}^p ds \\
+ 2^{p-1} \| B \|^p M^2 e^{-p \rho (T-t)} \left( \int_0^t \left\| (\lambda I + \Gamma_s^T)^{-1} S(T-s) \right\| g(s,X_s) - g(s,Y_s) \|_Q^2 ds \right) \right\|_p^{p/2} \\
\leq \frac{1}{\lambda^p} 2^{p-1} \| B \|^p M^2 \rho N_1 e^{-2 \rho (T-t)} \int_0^t E \left\| X_s - Y_s \right\|_{C^\alpha}^p ds \\
+ \frac{1}{\lambda^p} 2^{p-1} \| B \|^p M^2 \rho N_1 e^{-2 \rho (T-t)} E \left( \int_0^t \left\| X_s - Y_s \right\|_{C^\alpha}^2 ds \right) \right\|_p^{p/2} \\
\leq \frac{1}{\lambda^p} N_2 \int_0^t E \left\| X_s - Y_s \right\|_{C^\alpha}^p ds,
\]

and the lemma is proven. ☐

### 3. Approximate controllability

In this section we present our main result on approximate controllability of system (1). Assuming the approximate controllability of the corresponding deterministic system under some conditions we prove the approximate controllability of system (1).

Let us fix \( \lambda > 0 \) and introduce the following mapping \( \Phi \) on \( \delta_\rho \):

\[
(\Phi Z)(t) = S(t) A^\alpha \phi(0) + \int_0^t A^\alpha S(t-s) Bu^\lambda(s, A^{-\alpha} Z) ds
\]
\begin{align*}
+ \int_0^t A^\alpha S(t-s) f(s, A^{-\alpha} Z_s) \, ds + \int_0^t A^\alpha S(t-s) g(s, A^{-\alpha} Z_s) \, dW(s),
\end{align*}

\begin{equation}
(\Phi Z)(t) = A^\alpha \phi(t),
\end{equation}

\begin{align*}
 u^\lambda(t, A^{-\alpha} Z) &= B^* S^* (T-t) \left( \lambda I + \Gamma T \right)^{-1} \left( E h - S(T) \phi(0) \right) \\
&- B^* S^* (T-t) \int_0^t \left( \lambda I + \Gamma T \right)^{-1} S(T-s) f(s, A^{-\alpha} Z_s) \, ds \\
&- B^* S^* (T-t) \int_0^t \left( \lambda I + \Gamma T \right)^{-1} S(T-s) g(s, A^{-\alpha} Z_s) \, ds \\
&\times \left[ S(T-s) g(s, A^{-\alpha} Z_s) - \phi(s) \right] \, dW(s).
\end{align*}

Lemma 9. Assume $0 < \alpha < (p - 2)/2p$. For any $Z \in \mathcal{S}_p$, $(\Phi Z)(t)$ is continuous on the interval $[0, T]$ in the $L_p$-sense.

**Proof.** Let $0 \leq t_1 < t_2 < T$. Then for any fixed $Z \in \mathcal{S}_p$,

\begin{align*}
E \left\| (\Phi Z)(t_1) - (\Phi Z)(t_2) \right\|^p &\leq 4^{p-1} E \left\| (S(t_1) - S(t_2)) A^\alpha \phi(0) \right\|^p \\
&+ 4^{p-1} E \left\| \int_0^{t_2} A^\alpha S(t_2-s) u^\lambda(s, A^{-\alpha} Z) \, ds - \int_0^{t_1} A^\alpha S(t_1-s) u(s) \, ds \right\|^p \\
&+ 4^{p-1} E \left\| \int_0^{t_2} A^\alpha S(t_2-s) f(s, A^{-\alpha} Z_s) \, ds - \int_0^{t_1} A^\alpha S(t_1-s) f(s, A^{-\alpha} Z_s) \, ds \right\|^p \\
&+ 4^{p-1} E \left\| \int_0^{t_2} A^\alpha S(t_2-s) g(s, A^{-\alpha} Z_s) \, dW(s) \\
&\quad - \int_0^{t_1} A^\alpha S(t_1-s) g(s, A^{-\alpha} Z_s) \, dW(s) \right\|^p \\
&= I_1 + I_2 + I_3 + I_4.
\end{align*}

Thus, we obtain by Lemma 1 that

\begin{align*}
I_1 &= 4^{p-1} E \left\| S(t_2-t_1) S(t_1) A^\alpha \phi(0) - S(t_1) A^\alpha \phi(0) \right\|^p \\
&\leq 4^{p-1} N_{\alpha}^{p} (t_2-t_1)^{p\alpha} E \left\| A^\alpha S(t_1) A^\alpha \phi(0) \right\|^p
\end{align*}

and
\[ I_2 \leq 8^{p-1} E \left( \int_{t_1}^{t_2} A^\alpha S(t_2 - s)u^\lambda(s, A^{-\alpha} Z) \, ds \right)^p \]

\[ + 8^{p-1} E \left( \int_{0}^{t_1} A^\alpha S(t_2 - s) - S(t_1 - s))u^\lambda(s, A^{-\alpha} Z) \, ds \right)^p \]

\[ \leq 8^{p-1} E \left( \int_{t_1}^{t_2} \| A^\alpha S(t_2 - s)u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \right) \]

\[ + 8^{p-1} E \left( \int_{0}^{t_1} \| A^\alpha S(t_2 - s) - S(t_1 - s))u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \right) \]

\[ = I_{21} + I_{22}. \]

Therefore, there exist positive constants \( I_{21}, I_{22} > 0 \) and \( \epsilon_1 = p(1 - \alpha) > 0 \) such that

\[ I_{21} \leq 8^{p-1} M_a E \left( \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} e^{-a(t_2-s)} \| u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \right)^{p/q} \]

\[ \leq 8^{p-1} M_a \left( \frac{1}{\lambda p} \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} e^{-a(t_2-s)} \, ds \right)^{p/q} \left( t_2 - t_1 \right)^{\epsilon_1} (1 + \| Z \|_{H_p}^p) \]

and

\[ I_{22} = 8^{p-1} E \left( \int_{0}^{t_1} \| A^\alpha S \left( \frac{t_1 - s}{2} \right) S(t_2 - t_1) - I \right) S \left( \frac{t_1 - s}{2} \right) u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \]

\[ \leq 8^{p-1} M_a^p E \left( \int_{0}^{t_1} \left( \frac{t_1 - s}{2} \right)^{-\alpha} e^{-a(t_1-s)/2} \right) \]

\[ \times \left( \| S(t_2 - t_1) - I \right) S \left( \frac{t_1 - s}{2} \right) u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \]

\[ \leq 8^{p-1} M_a^p N_a^p E \left( \int_{0}^{t_1} \left( \frac{t_1 - s}{2} \right)^{-2\alpha} e^{-a(t_1-s)}(t_2 - t_1)^{\alpha} \| u^\lambda(s, A^{-\alpha} Z) \|^p \, ds \right). \]
\[ \leq 8^{p-1} M_{a}^{p} N_{a}^{p} (t_{2} - t_{1})^{pa} \left( \int_{0}^{t_{1}} \left( \frac{t_{1} - s}{2} \right)^{-2a} e^{-aq(t_{1} - s)} ds \right)^{p/q} \times E \int_{0}^{t_{1}} \| u_{\lambda}(s, A^{-\alpha} Z) \|^{p} ds \]

\[ \leq l_{22} (t_{2} - t_{1})^{pa} \left( 1 + \| Z \|_{\mathcal{B}_{p}}^{p} \right). \]

In a similar way, there exist positive constants \( l_{31}, l_{33} > 0 \) such that

\[ I_{3} \leq (l_{31}(t_{2} - t_{1})^{\varepsilon_{1}} + l_{33}(t_{2} - t_{1})^{pa}) \left( 1 + \| Z \|_{\mathcal{B}_{p}}^{p} \right). \]

Now by using Lemma 5 for some \( C_{p}' \), we have

\[ I_{4} \leq 8^{p-1} E \left\| \int_{t_{1}}^{t_{2}} A^{\alpha} S(t_{2} - s) g(s, A^{-\alpha} Z_{s}) dW(s) \right\|^{p} + 8^{p-1} E \left\| \int_{0}^{t_{1}} A^{\alpha} S(t_{2} - s) g(s, A^{-\alpha} Z_{s}) dW(s) \right\|^{p} \leq 8^{p-1} C_{p}' E \left( \int_{t_{1}}^{t_{2}} \left\| A^{\alpha} S(t_{2} - s) g(s, A^{-\alpha} Z_{s}) \right\|^{2} ds \right)^{p/2} + 8^{p-1} C_{p}' E \left( \int_{0}^{t_{1}} \left\| A^{\alpha} S(t_{2} - s) g(s, A^{-\alpha} Z_{s}) \right\|^{2} ds \right)^{p/2} \]

\[ = I_{41} + I_{42}. \]

Then, it follows that there exist positive constants \( l_{41} > 0 \) and \( \varepsilon_{2} = (p - 2 - 2pa)/2 > 0 \) such that

\[ I_{41} \leq 8^{p-1} C_{p}' M_{a}^{p} \left( \int_{t_{1}}^{t_{2}} (t_{2} - s)^{-2a} e^{-2a(t_{2} - s)} \left\| g(s, A^{-\alpha} Z_{s}) \right\|^{2} ds \right)^{p/2} \leq 8^{p-1} C_{p}' M_{a}^{p} \left( \int_{t_{1}}^{t_{2}} (t_{2} - s)^{-2a} e^{-2a(t_{2} - s)} \left\| g(s, A^{-\alpha} Z_{s}) \right\|^{2} ds \right)^{p/2} \times E \int_{t_{1}}^{t_{2}} \left\| g(s, A^{-\alpha} Z_{s}) \right\|^{2} ds \]

\[ \leq 8^{p-1} C_{p}' M_{a}^{p} \left( 1 + \| Z \|_{\mathcal{B}_{p}}^{p} \right) \int_{t_{1}}^{t_{2}} \left( 1 + \| Z \|_{\mathcal{B}_{p}}^{p} \right) ds \]

\[ \leq l_{41}(t_{2} - t_{1})^{\varepsilon_{2}} \left( 1 + \| Z \|_{\mathcal{B}_{p}}^{p} \right). \]
Let \( \{ e_n \}, n \geq 1, \) be a complete orthonormal basis of the separable Hilbert space \( K \) such that \( Q^{1/2} e_n = \sqrt{\lambda_n} e_n, \) where \( Q \) is the covariance operator of Wiener process \( W. \) Then we obtain that there exists a positive constant \( l_{42} > 0 \) such that

\[
I_{42} = 8^{p-1} C_p^* E \left( \int_0^{t_1} A^a S \left( \frac{t_1 - s}{2} \right) (S(t_2 - t_1) - I) \right)
\times S \left( \frac{t_1 - s}{2} \right) g(s, A^{-\alpha} Z_s) \right) \int_{Q} ds \right)^{p/2}
\leq 8^{p-1} C_p^* M_\alpha^{\alpha} E \left( \int_0^{t_1} \left( \frac{t_1 - s}{2} \right)^{-\alpha} e^{-a(t_1 - s)/2} \right)
\times \sum_{n=1}^\infty \left( \| S(t_2 - t_1) - I \| S(t_1 - s) \right) \int_{Q} ds \right)^{p/2}
\leq 8^{p-1} C_p^* M_\alpha^{\alpha} N_\alpha \left( \int_0^{t_1} \left( \frac{t_1 - s}{2} \right)^{-2\alpha} e^{-a(t_1 - s)} \right) \left( \| S(t_1 - s) \| Q ds \right)^p
\leq l_{42} (t_2 - t_1) \alpha \left( 1 + \| Z \|_{H^p} \right).
\]

Since \( Z \in H^p, \) it follows that \( I_1, I_2, I_3, I_4 \) tend to zero as \( t_2 \to t_1. \) Hence \( (\Phi Z)(t) \) is continuous from the right in \([0, T)\). A similar reasoning shows that it is also continuous from the left in \( (0, T)\). Therefore, the proof of the lemma is complete. \( \square \)

**Lemma 10.** The operator \( \Phi \) sends \( \mathcal{S}_p \) into itself: \( \Phi(\mathcal{S}_p) \subset \mathcal{S}_p. \)

**Proof.** Let \( Z \in \mathcal{S}_p. \) Then we have

\[
E \| (\Phi Z)(t) \|^p \leq 8^{p-1} C_p^* \sup_{0 \leq t_1 \leq T} \left\| S(t_1 + \theta) A^a \phi(0) \right\|^p
\]

\[
+ 4^{p-1} E \sup_{0 \leq t_1 \leq T} \left\| \int_0^{t_1} A^\alpha S(t + \theta - s) B u^\lambda (s, A^{-\alpha} Z_s) \right\|^p
\]

\[
+ 4^{p-1} E \sup_{0 \leq t_1 \leq T} \left\| \int_0^{t_1} A^\alpha S(t + \theta - s) f (s, A^{-\alpha} Z_s) \right\|^p
\]

\[
+ 4^{p-1} E \sup_{0 \leq t_1 \leq T} \left\| \int_0^{t_1} A^\alpha S(t + \theta - s) g(s, A^{-\alpha} Z_s) \right\|^p
\]

\[
= I_5 + I_6 + I_7 + I_8.
\]

Let \( q = p/(p - 1). \) Then we obtain that (recall \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \)
Theorem 11. Assume 

Proof. We prove the theorem through the classical Banach fixed point theorem that for each fixed \( \lambda > C_{\alpha} \parallel \), therefore we obtain that \( H \rightarrow \) into \( p \). By Lemma 7 for any \( \beta \) satisfying the inequality \( 1/p + \alpha < \beta < 1/2, \)

\[
I_8 \leq 4^{p-1} M_p E \sup_{\theta \leq 0} \left( \int_0^{t+\theta} e^{-\alpha(t+\theta-s)} \left\| B\alpha (s, A^{-\alpha} Z) \right\| ds \right)^p
\]

\[
\leq \frac{1}{\lambda} 4^{p-1} M_p N_2 T ( \Gamma (1 - \alpha q)(aq) )^{p/q} \left( 1 + \| Z \|_{S_p}^p \right),
\]

\[
I_7 \leq 4^{p-1} M_p E \sup_{\theta \leq 0} \left( \int_0^{t+\theta} e^{-\alpha(t+\theta-s)} \right) \left\| f (s, A^{-\alpha} Z_s) \right\| ds \right)^p
\]

\[
\leq 4^{p-1} M_p N_1 T ( \Gamma (1 - \alpha q)(aq) )^{p/q} \left( 1 + \| Z \|_{S_p}^p \right).
\]

By Lemma 7 for any \( \beta \) satisfying the inequality \( 1/p + \alpha < \beta < 1/2, \)

\[
I_8 \leq 4^{p-1} M_p N_t ( \Gamma (1 + (\rho - 1 - \alpha q)(aq)) )^{p/q} \left( 1 + \| Z \|_{S_p}^p \right),
\]

\[
\times c_p M_p T^{1/2} (1 + \| Z \|_{S_p}^p).
\]

Therefore, we obtain that \( \| F Z \|_{S_p} < \infty. \) By Lemma 9, \( (FZ)(t) \) is continuous on \( [0, T] \) and so \( F \) maps \( S_p \) into \( S_p \). Thus this completes the proof. \( \square \)

**Theorem 11.** Assume \( 0 < \alpha < (p - 2)/2p \) and let \( f : [0, \infty) \times C_{\alpha} \rightarrow H, \ g : [0, \infty) \times C_{\alpha} \rightarrow L_2^0 (E, H) \) satisfy Assumptions A and B. Then the operator \( F \) has a unique fixed point in \( S_p \).

**Proof.** We prove the theorem through the classical Banach fixed point theorem that for each fixed \( \lambda > 0 \) the operator \( F \) has a unique fixed point in \( S_p \). By Lemma 10, \( F \) maps \( S_p \) into \( S_p \). To show that there exists a natural \( n \) such that \( F^n \) is contraction, let \( X, Y \in S_p \), then for any fixed \( t \in [0, T], \)

\[
E \left\| (FX)_t - (FY)_t \right\|_C^p
\]

\[
\leq E \sup_{\theta \leq 0} \left\| (FX)(t + \theta) - (FY)(t + \theta) \right\|_C^p
\]

\[
\leq 3^{p-1} E \sup_{\alpha \leq \theta \leq 0} \left( \int_0^{t+\theta} A^\alpha S(t + \theta - s) B (u(s, A^{-\alpha} X) - u(s, A^{-\alpha} Y)) ds \right)^p
\]

\[
+ 3^{p-1} E \sup_{\alpha \leq \theta \leq 0} \left( \int_0^{t+\theta} A^\alpha S(t + \theta - s) (f(s, A^{-\alpha} X_s) - f(s, A^{-\alpha} Y_s)) ds \right)^p
\]

\[
+ 3^{p-1} E \sup_{\alpha \leq \theta \leq 0} \left( \int_0^{t+\theta} A^\alpha S(t + \theta - s) (g(s, A^{-\alpha} X_s) - g(s, A^{-\alpha} Y_s)) dW(s) \right)^p
\]

\[
= J_1 + J_2 + J_3
\]
and

\[ J_1 \leq 3^{\rho-1} M_1^p E \sup_{-\rho \leq \theta \leq 0} \left( \int_0^{t+\theta} e^{-a(t+\theta-s)}(t+\theta-s)^{-\alpha} \right) \times \left\| B(u(s, A^{-\alpha} X) - u(s, A^{-\alpha} Y)) \right\| ds \right)^p \]
\[ \leq 3^{\rho-1} M_1^p E \sup_{-\rho \leq \theta \leq 0} \left( \int_0^{t+\theta} e^{-q\alpha(t+\theta-s)}(t+\theta-s)^{-q\alpha} ds \right)^{p/q} \times \left( \int_0^{t+\theta} \|B\|^p \left\| u(s, A^{-\alpha} X) - u(s, A^{-\alpha} Y) \right\|^p ds \right) \]
\[ \leq \frac{1}{\lambda^p} 3^{\rho-1} M_1^p N_2 \left\| B \right\|^p t \left( \Gamma(1-\alpha q)(aq)^{\alpha} \right)^{p/q} E \int_0^t \| X_s - Y_s \|^p_C ds, \]

\[ J_2 \leq 3^{\rho-1} M_2^p E \sup_{-\rho \leq \theta \leq 0} \left( \int_0^{t+\theta} e^{-a(t+\theta-s)}(t+\theta-s)^{-\alpha} \right) \times \left\| f(s, A^{-\alpha} X) - f(s, A^{-\alpha} Y) \right\| ds \right)^p \]
\[ \leq 3^{\rho-1} M_2^p N_1 \left( \Gamma(1-\alpha q)(aq)^{\alpha} \right)^{p/q} E \int_0^t \| X_s - Y_s \|^p_C ds, \]

Next, let \( 1/p + \alpha < \beta < 1/2 \). Then by Lemma 7 we have

\[ J_3 \leq 3^{\rho-1} M_2^p N_1 \left( \Gamma(1-\alpha)\Gamma(1+\alpha - \rho)q \right)^{p/q} \times C_p M^p \frac{t^{-p\beta + p/2}}{(1-2\beta)^{p/2}} E \int_0^t \| X_s - Y_s \|^p_C ds. \]

Hence, we obtain a positive real number \( B(\lambda) > 0 \) such that

\[ E \left\| (\Phi X)_{I} - (\Phi Y)_{I} \right\|^p_C \leq B(\lambda) E \int_0^t \| X_s - Y_s \|^p_C ds \quad (12) \]

for any \( X, Y \in \mathfrak{H}_p \).

For any integer \( n \geq 1 \), by iteration, it follows from (12) that

\[ \| \Phi^n X - \Phi^n Y \|^p_{\mathfrak{H}_p} \leq \frac{(TB(\lambda))^n}{n!} \| X - Y \|^p_{\mathfrak{H}_p}. \]
Since for sufficiently large \(n\), \((TB(\lambda))^n/n! < 1\), \(\Phi^n\) is a contraction map on \(\mathcal{H}_P\) and therefore \(\Phi\) itself has a unique fixed point \(Z\) in \(\mathcal{H}_P\). The theorem is proved.

Thus, by Theorem 11 for any \(\lambda > 0\) the operator \(\Phi_\lambda\) has a unique fixed point \(Z_\lambda \in \mathcal{H}_P\) which setting \(X_\lambda(t) = A^{-\alpha}Z_\lambda(t)\) immediately yields

\[
X_\lambda(t) = S(t)\phi(0) + \int_0^t \left[ I - \Gamma_\lambda S(T - t) \left( \lambda I + \Gamma_T \right)^{-1} (\mathbf{E} h - S(T)\phi(0)) \right] S(t - s) f(s, X_\lambda^s) ds \\
+ \int_0^t \left[ I - \Gamma_\lambda S(T - t) \left( \lambda I + \Gamma_T \right)^{-1} S(T - s) g(s, X_\lambda^s) dW(s) \right] \\
+ \int_0^T \Gamma_\lambda S(T - t) \left( \lambda I + \Gamma_T \right)^{-1} \phi(s) dW(s).
\]

\[X(t) = \phi(t), \quad -r \leq t \leq 0\]  

(13)

Our main result in this paper can now be stated as follows.

**Theorem 12.** Under Assumptions A, B1 and C the system (1) is approximately controllable on \([0, T]\).

**Proof.** Let \(X_\lambda\) be a solution of Eq. (5). Then writing Eq. (13) at \(t = T\) yields

\[
X_\lambda(T) = h - \lambda \left( \lambda I + \Gamma_T \right)^{-1} (\mathbf{E} h - S(T)\phi(0)) \\
- \lambda \int_0^T \left( \lambda I + \Gamma_T \right)^{-1} S(T - \tau) f(\tau, X_\lambda^\tau) d\tau \\
- \lambda \int_0^T \left( \lambda I + \Gamma_T \right)^{-1} \left[ S(T - \tau) g(\tau, X_\lambda^\tau) + \phi(\tau) \right] dW(\tau).
\]

(14)

By Assumption B1,

\[
\|f(\tau, X_\lambda^\tau)\|_P + \|g(\tau, X_\lambda^\tau)\|_Q \leq N_1
\]

in \([0, T] \times \Omega\). Then there is a subsequence, still denoted by \(\{f(\tau, X_\lambda^\tau), g(\tau, X_\lambda^\tau)\}\), weakly converging to, say, \((f(\tau, \omega), g(\tau, \omega))\) in \(H \times L_2^0(K, H)\). The compactness of \(S(t), t > 0\), implies that

\[
\begin{align*}
S(T - \tau) f(\tau, X_\lambda^\tau) &\to S(T - \tau) f(\tau), \\
S(T - \tau) g(\tau, X_\lambda^\tau) &\to S(T - \tau) g(\tau)
\end{align*}
\]

in \([0, T] \times \Omega\).
On the other hand, by Assumption C, for all \(0 \leq \tau < T\)
\[
\lambda (\lambda I + \Gamma_T^\tau)^{-1} \rightarrow 0 \quad \text{strongly as } \lambda \rightarrow 0^+,
\]
and moreover
\[
\|\lambda (\lambda I + \Gamma_T^\tau)^{-1}\| \leq 1.
\]
Thus from (14)–(17) by the Lebesgue dominated convergence theorem it follows that
\[
\mathbb{E} \|X_\lambda(T) - h\|^p
\leq 6^{p-1} \|\lambda (\lambda I + \Gamma_T^0)^{-1}(\mathbb{E}h - S(T)\phi(0))\|^p
+ 6^{p-1}\mathbb{E} \left( \int_0^T \|\lambda (\lambda I + \Gamma_T^\tau)^{-1}\| \|S(T - \tau)[f(\tau, X_\lambda^\tau) - f(\tau)]\| d\tau \right)^p
+ 6^{p-1}\mathbb{E} \left( \int_0^T \|\lambda (\lambda I + \Gamma_T^\tau)^{-1}S(T - \tau)f(\tau)\| d\tau \right)^p
+ 6^{p-1}\mathbb{E} \left( \int_0^T \|\lambda (\lambda I + \Gamma_T^\tau)^{-1}\|^2 \|S(T - \tau)[g(\tau, X_\lambda^\tau) - g(\tau)]\|^2_Q d\tau \right)^{p/2}
+ 6^{p-1}\mathbb{E} \left( \int_0^T \|\lambda (\lambda I + \Gamma_T^\tau)^{-1}S(T - \tau)g(\tau)\|^2_Q d\tau \right)^{p/2}
+ 6^{p-1}\mathbb{E} \left( \int_0^T \|\lambda (\lambda I + \Gamma_T^\tau)^{-1}\|^2 \|\varphi(\tau)\|^2_Q d\tau \right)^{p/2} \to 0 \quad \text{as } \lambda \rightarrow 0^+.
\]
This gives the approximate controllability. Theorem is proved. \(\square\)

4. Exact controllability

In this section we study the exact controllability of the mild solution for the stochastic functional differential equation (1),
\[
X(t) = S(t)\phi(0) + \int_0^t S(t - s)Bu(s) ds + \int_0^t S(t - s)f(s, X_s) ds
+ \int_0^t S(t - s)g(s, X_s) dW(s),
\]
\[
X_0 = \phi \in MC_\alpha(0, p), \quad t \geq 0,
\]
without a compactness assumption on the semigroup $S(t)$. We formulate conditions under which the semilinear system (18) is exactly controllable. No compactness assumption on semigroup $S(t)$ is made.

We impose the following assumptions on the data of the problem.

**Assumption D.** $S(t)$, $t \geq 0$, is the strongly continuous semigroup of the linear bounded operators generated by $A$:

$$\max_{0 \leq t \leq T} \|S(t)\| \leq M.$$

**Assumption L.** The linear operator $L^T_0$ from $L^\beta_p(0, T; U)$ into $L^p(\Omega, \mathcal{F}, P; U)$, defined by

$$L^T_0 = \int_0^T S(T - s)Bu(s) \, ds,$$

induces a boundedly invertible operator $\tilde{L}$ defined on $L^\beta_p(0, T; U)/\ker L^T_0$.

**Assumption B.** For arbitrary $\gamma, \xi \in C_\alpha$ and $0 \leq t \leq T$, suppose that there exists positive real constant $N_1 > 0$ such that

$$\|f(t, \gamma) - f(t, \xi)\|^p + \|g(t, \gamma) - g(t, \xi)\|^p_{\mathcal{F}} \leq N_1 \|\gamma - \xi\|^p_{C_\alpha},$$

$$\|f(t, \xi)\|^p + \|g(t, \xi)\|^p_{\mathcal{F}} \leq N_1 (1 + \|\xi\|^p_{C_\alpha}).$$

**Assumption E.**

$$3^{p-1}M^p N_5 \|B\|^p T^{p/q} + 3^{p-1}M^p N_1 T^{p/q}$$

$$+ 3^{p-1}M^p \left(\frac{T^{(\beta - 1) + p/q}}{(q\beta - q + 1)^{p/q}} C^p (1 - 2\beta)^{p/2}\right) N_1 < 1.$$

Using Assumptions D, L and B, for an arbitrary process $Z_s$, define the control process

$$u(t, Z) = E\left\{ (\tilde{L})^{-1} \left[ \int_0^T \left( h - S(T)\phi(0) - \int_0^T S(T - s)f(s, Z_s) \, ds ight. ight. ight. \left. \left. - \int_0^T S(T - s)g(s, Z_s) \, dW(s) \right) \right\} \bigg| \mathcal{F}_t \right\}.$$
**Proof.** We will only prove the first inequality, since the proof of the second is similar the first one.

\[
E \left\| u(t, X) - u(t, Y) \right\|^p \leq 2^{p-1} E \left\| (\tilde{L})^{-1} \int_0^T S(T-s) \left[ f(s, X_s) - f(s, Y_s) \right] ds \right\|^p \\
+ 2^{p-1} E \left\| (\tilde{L})^{-1} \int_0^T S(T-s) \left[ g(s, X_s) - g(s, Y_s) \right] dW(s) \right\|^p
\]

\[
\leq 2^{p-1} \| (\tilde{L})^{-1} \|^p M^p N_1 \int_0^T E \| X_s - Y_s \|^p_{C_u} ds \\
+ 2^{p-1} \| (\tilde{L})^{-1} \|^p M^p N_1 E \left( \int_0^T \| g(s, X_s) - g(s, Y_s) \|^2_Q ds \right)^{p/2}
\]

\[
\leq 2^{p-1} \| (\tilde{L})^{-1} \|^p M^p N_1 \int_0^T E \| X_s - Y_s \|^p_{C_u} ds \\
+ 2^{p-1} \| (\tilde{L})^{-1} \|^p M^p N_1 E \left( \int_0^T \| X_s - Y_s \|^2_{C_u} ds \right)^{p/2}
\]

\[
\leq 2^{p-1} \| (\tilde{L})^{-1} \|^p M^p N_1 (1 + T^{p-2}) \int_0^T E \| X_s - Y_s \|^p_{C_u} ds
\]

Lemma is proved. ∎

We will show that, when using this control, the operator \( \Psi \), defined by

\[
\Psi Z = S(t)\phi(0) + \int_0^t S(t-s)Bu(s, Z) ds \\
+ \int_0^t S(t-s)f(s, Z_s) ds + \int_0^t S(t-s)g(s, Z_s) dW(s)
\]

has a fixed point \( Z \), which is a solution of (1).

**Theorem 14.** Assume that Assumptions D, L, B and E are satisfied. Then the system (1) is exactly controllable on \([0, T]\).
Proof. The proof is carried through by the Banach fixed point theorem. First, it have to be shown that \( \Psi \) maps \( \mathcal{B}_P \) into itself. It is similar to that of Lemma 10 and is omitted.

Now it is shown that \( \Psi \) is a contraction in \( \mathcal{B}_P \). Let \( X, Y \in \mathcal{B}_P \), then for any fixed \( t \in [0, T] \), we have

\[
E \| (\Psi X)_t - (\Psi Y)_t \|_C^P \leq E \sup_{-r \leq \theta \leq 0} \| (\Psi X)(t + \theta) - (\Psi Y)(t + \theta) \|_C^P
\]

\[
\leq 3^{p-1} E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} S(t + \theta - s) B \left( u(s, X) - u(s, Y) \right) ds \right)^p
\]

\[
+ 3^{p-1} E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} S(t + \theta - s) \left( f(s, X_s) - f(s, Y_s) \right) ds \right)^p
\]

\[
+ 3^{p-1} E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} S(t + \theta - s) \left( g(s, X_s) - g(s, Y_s) \right) dW(s) \right)^p
\]

\[= J_1 + J_2 + J_3\]

and

\[
J_1 \leq 3^{p-1} M^P E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} \| B \left( u(s, X) - u(s, Y) \right) \| ds \right)^P
\]

\[
\leq 3^{p-1} M^P E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} ds \right)^{p/q} \left( \int_0^{t+\theta} \| B \|_C^P \| u(s, X) - u(s, Y) \|_C^P ds \right)^{p/q}
\]

\[
\leq 3^{p-1} M^P N_5 \| B \|_C^P \| X_t - Y_t \|_C^P \int_0^t ds,
\]

\[
J_2 \leq 3^{p-1} M^P E \sup_{-r \leq \theta \leq 0} \left( \int_0^{t+\theta} \| f(s, X) - f(s, Y) \| ds \right)^P
\]

\[
\leq 3^{p-1} M^P N_1 t^{p/q} E \int_0^t \| X_s - Y_s \|_C^P ds,
\]

Next, let \( 1/p + \alpha \leq \beta < 1/2 \). Then by Lemma 7 we have

\[
J_3 \leq 3^{p-1} M^P t^{p(\beta-1)+1/p/q} C_{1-P} t^{-p^p + p/2} \left( q\beta - q + 1 \right)^{p/q} \left( 1 - 2\beta \right)^{p/2} N_1 E \int_0^t \| X_s - Y_s \|_C^P ds.
\]

Hence, we obtain a positive real number
\[ D = 3^{p-1} M^p \|N\|_p B \|T\|^{p/q} + 3^{p-1} M^p N_1 T^{p/q} \]
\[ + 3^{p-1} M^p \frac{T^{p(\beta-1)+p/q}}{(q\beta-q+1)p/q} C_p \frac{T^{-p^2+p/2}}{(1-2\beta)^{p/2}} N_1 < 1 \]
such that
\[ \sup_{0 \leq t \leq T} E \| (\Psi X)_t - (\Psi Y)_t \|_C^p \leq DE \int_0^T \| X_s - Y_s \|_C^p ds \]
\[ \leq DT \sup_{0 \leq t \leq T} E \| X_s - Y_s \|_C^p \]
for any \( X, Y \in \mathcal{F}_p \). Thus \( \Psi X \) is a contraction in \( \mathcal{F}_p \) and has a unique fixed point in \( \mathcal{F}_p \).
The theorem is proved. \( \square \)

5. Example

In this section we present an example of controllable stochastic partial functional differential equation. Let \( H = L^2[0, \pi] \) and \( U = L^2[0, T] \). Let \( A : L^2[0, \pi] \to L^2[0, \pi] \) be the linear operator defined by \( A \xi = (\partial^2/\partial x^2) \xi \), where \( D(A) = \{ \xi \in H : (d/\xi) \xi \) are absolutely continuous, and \((\partial^2/\partial x^2) \xi \) \in H, \( \xi(0) = \xi(\pi) = 0 \). Let \( B \in L(R, X) \) be defined as \((Bu)(x) = b(x)u, 0 \leq x \leq \pi, u \in R, b(x) \in L^2[0, \pi] \). Let \( p > 2, 0 < \theta < (p-2)/2 \) and suppose \( r > 0 \) is a real number. Set \( H_\theta = D(A^\theta) \) and \( C_\theta = C([-r, 0], H_\theta) \). It is well known that \( A \) is a closed, densely defined linear operator. \( \beta(t) \) denotes a one-dimensional standard Brownian motion.

Consider the stochastic delay reaction diffusion equation
\[
\begin{align*}
    dX(t, x) &= \left[ \frac{\partial^2}{\partial x^2} X(t, x) + b(x)u(t) + F(t, X(t-r_1(t), x)) \right] dt \\
    &\quad + G(t, X(t-r_2(t), x))d\beta(t), \\
    X(t, 0) &= X(t, \pi) = 0, \quad t \geq 0, \\
    X(s, x) &= \phi(s, x), \quad -r \leq s \leq 0, \quad 0 \leq x \leq \pi,
\end{align*}
\]
where \( r_1, r_2 \) are continuous functions with \( 0 < r_1(t) < r, 0 < r_2(t) < r \) for all \( t \geq 0 \) and \( \phi \in C_\theta \). Suppose \( F : [0, \infty) \times R \to R \) and \( G : [0, \infty) \times R \to R \) are continuous and global Lipschitz continuous in the second variable and uniformly bounded.

First of all, note that there exists a complete orthonormal set \( \{e_n\}, n \geq 1 \), of eigenvectors of \( A \) with \( e_n(x) = \sqrt{2/\pi} \sin nx \) and the analytic semigroup \( S(t), t \geq 0 \), that is generated by \( A \) such that
\[
A \xi = \sum_{n=1}^\infty n^2(\xi, e_n)e_n, \quad \xi \in D(A),
\]
\[
S(t)\xi = \sum_{n=1}^\infty \exp(-n^2 t)(\xi, e_n)e_n, \quad \xi \in H.
\]
We define $A^\theta$ for self-adjoint operator $A$ by the classical spectral theorem and it is easy to deduce that

$$|A^\theta S(t)\xi| = \sum_{n=1}^{\infty} (n^2)^\theta \exp(-n^2 t)(\xi, e_n)e_n,$$

which immediately implies

$$\|A^\theta S(t)\xi\|^2 = \sum_{n=1}^{\infty} n^{4\theta} \exp(-2n^2 t)\|\xi, e_n\|^2$$

$$= e^{-2at} -2\theta \sum_{n=1}^{\infty} (n^2)^\theta \exp(-2a n^2 t)\|\xi, e_n\|^2.$$ (20)

Now let $f(t, \psi)(x) = F(t, \psi(-r_1(t)))(x)$ and $g(t, \psi)(x) = G(t, \psi(-r_2(t)))(x)$ for all $\psi \in C_0 = C([-r, 0], H_0)$ and any $x \in [0, \pi]$. Then we have that for any fixed $s \in [-r, 0]$,

$$\|f(t, \psi) - f(t, \phi)\|^2 = \frac{\pi}{0} \int_0^\pi \left| F(t, \psi(-r_1(t)))(x) - F(t, \phi(-r_1(t)))(x) \right|^2 dx$$

$$\leq N^2 \int_0^\pi \left| \psi(-r_1(t)) - \phi(-r_1(t)) \right|^2 dx$$

$$= N^2 \sum_{n=1}^{\infty} (\psi(-r_1(t)) - \phi(-r_1(t)), e_n)^2$$

$$\leq N^2 \sum_{n=1}^{\infty} n^{4\theta} (\psi(-r_1(t)) - \phi(-r_1(t)), e_n)^2$$

$$= N^2 \|A^\theta (\psi(-r_1(t)) - \phi(-r_1(t)))\|^2 \leq N^2 \|\psi - \phi\|^2_{C_0},$$

where $N$ is appropriate constant. So the functions $f(t, \psi)$ and $g(t, \psi)$ are globally Lipschitz continuous in $\psi \in C_0$ and uniformly bounded. On the other hand, it is known that the deterministic linear system corresponding to (19) is approximately controllable on every $[0, t], t > 0$, provided that

$$\int_0^\pi b(x)e_n(x) dx \neq 0 \text{ for } n = 1, 2, 3, \ldots.$$

Hence, all conditions in Theorem 12 are satisfied, and consequently the system (19) is approximately controllable on $[0, T]$.

References