On exponent of indecomposability for primitive Boolean matrices

Bolian Liu a,b,1

a Department of Mathematics, South China Normal University, Guangzhou 510631, People’s Republic of China

b Department of Computer, Guangdong Polytechnical Normal University, Guangzhou 510633, People’s Republic of China

Received 4 August 1998; accepted 4 March 1999

Submitted by R.A. Brualdi

Abstract

Let \( r, n \) be integers, \(-n < r < n\), An \( n \times n \) Boolean matrix \( A \) is called \( r \)-indecomposable if it contains no \( k \times l \) zero submatrix with \( k + l = n - r + 1 \). \( A \) is primitive if one of its powers, \( A^k \), has all positive entries for some integer \( k \geq 1 \). If \( A \) is primitive, then there is a smallest positive integer \( h_r(A) = k \) such that \( A^k \) is \( r \)-indecomposable. There also is a smallest positive integer \( h^*_r(A) \), such that \( A^m \) is \( r \)-indecomposable for all \( m \geq h^*_r \). \( h_r \) and \( h^*_r \) are called exponent and strict exponent of \( r \)-indecomposability respectively. In this paper we obtain some new bounds of \( h^*_r(A) \) for primitive matrices and exact value of \( h^*_r(A) \) for symmetric primitive matrices. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

Let \( B_n \) denote the set of all matrices of order \( n \) over the Boolean algebra \{0,1\}. Let \( J_n \) denote the matrix in \( B_n \) each of whose entries equals 1. A matrix \( A \in B_n \) is primitive provided there is a positive integer \( k \) such that \( A^k = J_n \). The minimum such \( k \) is called the exponent of \( A \) and is denoted by \( \exp(A) \). We denote the set of primitive matrices in \( B_n \) by \( P_n \). A matrix \( A = (a_{ij}) \in B_n \) is

1 Supported by NSF of Guangdong and People’s Republic of China.

0024-3795/99/$ - see front matter © 1999 Elsevier Science Inc. All rights reserved.

PII: S 0 0 2 4 - 3 7 9 5 ( 9 9 ) 0 0 0 8 2 - 8
symmetric if \( a_{ij} = a_{ji} \) for all \( i, j \). \( A \) is called a microsymmetric matrix if there is a pair \( i, j \) with \( i \neq j \), such that \( a_{ij} = a_{ji} = 1 \). We denote the set of symmetric primitive matrices by \( SP_n \), set of microsymmetric primitive matrices by \( MSP_n \). Clearly \( SP_n \subseteq MSP_n \subseteq P_n \).

As we know, there is a one to one correspondence between set \( B_n \) and the set of digraphs \( D_n = (V, E) \) with vertex set \( V = \{1, 2, \ldots, n\} \) and arc set \( E = \{(i, j) \in V \times V : a_{ij} = 1\} \). We call a vertex \( i \) for a digraph a loop-vertex if \((i, i) \in E \) (or \( a_{ii} = 1 \)). For \( i, j \in V(D) \), the distance from \( i \) to \( j \) is denoted by \( d(i, j) \). If \( R = R(D) \) is the set of distinct lengths of the elementary cycles of \( D \), then \( d_R(i, j) \) is the length of the shortest walk from \( i \) to \( j \) which meets at least one circuit of each length of \( R(D) \). For \( X \subseteq V(D) \), vertex \( v \) is called neighbor of \( X \) if there exists a \( x \in X \) such that there is an arc \((x, v)\).

A digraph \( D \) corresponding to \( A \) is called an associated digraph of \( A \) and is denoted by \( D(A) \). If \( A \) is primitive, then \( D \) is called primitive graph. We shall make use of the following known results.

(1.1) \( A \) is primitive if and only if \( D(A) \) is strongly connected and the greatest common divisor of the lengths of all elementary cycles of \( D(A) \) is 1.

(1.2) If \( A \) is primitive and \( a \) and \( b \) are relatively prime lengths of circuits of \( D(A) \), then for any \( t \geq d_R(i, j) + \phi(a, b) \), there exists a walk of length \( t \) from \( i \) to \( j \). Here \( \phi(a, b) = (a - 1)(b - 1) \) is the Frobenius number of \( a \) and \( b \) (see [1]).

Let \( r \) be an integer, \( -n < r < n, A \in B_n \) is \( r \)-indecomposable (shortly, \( r \)-inde) if it contains no \( k \times l \) zero submatrix with \( 1 \leq k, l \leq n \) and \( k + l = n - r + 1 \). A \( 1 \)-inde matrix is called a fully indecomposable, while a \( 0 \)-inde matrix is called a Hall matrix. If \( A \) is \( r \)-inde, \( D(A) \) is said to be \( r \)-inde.

According to the definition of \( r \)-inde.

(1.3) \( A \) digraph \( D \) is \( r \)-inde if each subset \( X \subseteq V \) with \( 1 \leq |X| = k \leq \min\{n, n - r\} \) has at least \( k + r \) neighbors in \( D \).

In [2] the exponent of \( r \)-indecomposability \( h_r(D) \) (or \( h_r(A) \)) is defined as the smallest positive integer \( h \) for which \( D^h \) (or \( A^h \)) is \( r \)-inde. The strict exponent of \( r \)-indecomposability \( h'_r(D) \) (or \( h'_r(A) \)) is defined as the smallest positive integer \( h' \) for which \( D^{m} \) (or \( A^{m} \)) is \( r \)-inde for all \( m \geq h' \). In fact, those definitions are generalizations of fully indecomposability exponent and Hall exponent (see [3,4]).

Shen et al. [2] gave the following results about \( h'_r(D) \) and \( h_r(D) \).

(1.4) If \( D \) is a primitive digraph with \( n \geq 2 \) vertices and girth \( s \) (the length of the shortest cycle in \( D \)), then \( h'_r(D) \leq s(n - s + r - 1) + 1 \) whenever \(-n + s + 2 \leq r < n\).

For \(-n + s + 2 \leq r < n\), \( h'_r(D) = s(n - s + r - 1) + 1 \) if and only if \( r \leq -n + 2s + 1 \) and \( D \) is isomorphic to a unique digraph.

(1.5) If \( D \) is a primitive digraph with \( n \) vertices and a cycle of length \( s \) and that \( t \) of vertices of \( D \) are on \( s \)-cycles, then \( h_r(D) \leq \max\{s, s(n - t + r)\} \) whenever \(-n < r \leq n\).

In this paper we consider \( h'_r(A) \) for \( A \in MSP_n \), \( A \in SP_n \) and obtained some results which are improvements for conclusion (1.4), (1.5) for \( s = 2 \).
2. r-Indecomposability of primitive matrices

Let \( t \) be a nonnegative integer. For \( X \subseteq V, R_t(X) \) denotes the set of all those vertices which can be reached by a walk of length \( t \) in \( D(A) \) starting from a vertex in \( X \) (if \( t = 0, R_t(X) = X \)). A restatement of (1.3) is

\[
(2.1) \quad A' \text{ is } r\text{-inde if and only if } |R_t(X)| > |X| + r \text{ for all } X \text{ with }
\[
\emptyset \neq X \subset \{1, 2, \ldots, n\}, \text{ where } -n < r < n.
\]

According to the discussion of [2], \( h^*_r(G) = 1 \) whenever \( -n < r \leq -n + t \), for a digraph \( G \) of order \( n \) with a vertex disjoint union of cycles with total length \( t \). Noting that \( h^*_r(G) \leq h^*_r(G) \), we are allowed to assume that \( -n + 4 \leq r \leq n \) when determining \( h^*_r(A) \) for \( A \in SP_n \) and \( A \in MSP_n \).

For \( A \in P_n \), an upper bound of \( h^*_r(A) \) has been given in [2]. We now give another bound of \( h^*_r(A) \).

**Theorem 2.1.** Let \( A \in P_n \) and \( R = \{s_1, s_2, \ldots, s_t\} \) be a set of distinct lengths of all cycles \( C_{s_1}, C_{s_2}, \ldots, C_{s_t} \) in \( D(A) \), \( s_t > s_{t-1} > \cdots > s_1 > 1 \), \( \gcd(s_1, s_2, \ldots, s_t) = 1. \) Then

\[
h^*_r(A) \leq \sum_{i=1}^{t} (n - s_i) + \phi(R) + r.
\]

**Proof.** Let \( X \) be a set of vertices with \( \emptyset \neq X \subset V(D(A)). k = |X| < n. \)

Let \( x^* \) be a vertex in \( X \) and \( y_1 \) be a vertex in \( C_{s_1} \) such that

\[
d_1 = d(x^*, y_1) = \min \{d(x, y) \mid x \in X, y \in V(C_{s_1})\}.
\]

Then \( d_1 \leq n - (s_1 - 1) - k = n - s_1 - k + 1 \). Let \( y_2 \in C_{s_2} \) such that

\[
d_2 = d(y_1, y_2) = \min \{d(y_1, y) \mid y \in V(C_{s_2})\}.
\]

Then \( d_2 \leq n - s_2 \). Similarly we have

\[
d_i \leq n - s_i, \quad i = 3, 4, \ldots, t.
\]

Thus

\[
d_{q}(x^*, y_i) \leq \sum_{i=1}^{t} d_i \leq \sum_{i=1}^{t} (n - s_i) - k + 1.
\]

For each \( h \geq \sum_{i=1}^{t} (n - s_i) - k + 1 + \phi(R) \), there exists a walk of length \( h \) from \( x^* \) to \( y_i \). Thus for \( q \geq \sum_{i=1}^{t} (n - s_i) - k + 1 + \phi(R) + k + r - 1 \),

\[
|R_q(X)| \geq |R_q(x^*)| \geq \left| \bigcup_{a=0}^{k+r-1} R_a(y_i) \right| \geq k + r = |X| + r.
\]
Hence $|R_q(X)| \geq |X| + r$ for $q \geq \sum_{i=1}^{t}(n - s_i) + \phi(R) + r$. We have $h_r^*(A) \leq \sum_{i=1}^{t}(n - s_i) + \phi(R) + r$. \qed

According to Theorem 2.1 we have the following corollary.

**Corollary 2.2.** Let $R = \{s_1, s_2\}, \gcd(s_1, s_2) = 1, s_1 < s_2$. Then

$$h_r^*(A) \leq 2n + s_1s_2 + r - 2(s_1 + s_2) + 1.$$  

**Corollary 2.3.** If $A \in MSP_n$,

$$h_r^*(A) \leq 2n + r - 3.$$  

**Proof.** $A \in MSP_n, s_1 = 2, s_2$ is odd. By Corollary 2.2

$$h_r^*(A) \leq 2n + 2s_2 + r - 2(2 + s_2) + 1 = 2n + r - 3. \quad \square$$

By the bound of [2], for $A \in MSP_n$

$$h_r^*(A) \leq 2(n - 2 + r - 1) + 1 = 2n + 2r - 5.$$  

If $r \geq 2, 2n + r - 3 \leq 2n + 2r - 5$. The bound (2.3) is better than that of [2]. For $R = \{s_1, s_2\}, s_1 < s_2, A \in P_n$, by (1.4), the bound is

$$h_r^{(1)}(A) \leq s_1(n - s_1 + r - 1) + 1$$  

while by (2.2)

$$h_r^{(2)}(A) \leq 2n + s_1s_2 + r - 2(s_1 + s_2) + 1.$$  

It is not difficult to see that if $r \geq s_1 \geq 2$, then

$$[s_1(n - s_1 + r - 1) + 1] - [2n + s_1s_2 + r - 2(s_1 + s_2) + 1]$$

$$= (s_1 - 2)(n - s_2) + (s_1 - 1)(r - s_1) \geq 0.$$  

Thus bound (2.5) is better than bound (2.4).

We consider $h_r^*(A), A \in MSP_n$, again. In [5], we have established the following lemma.

**Lemma 2.4.** Let $D$ be a strongly connected digraph with vertex set $\{1, 2, \ldots, n\}$. Let $A \in D$ and $X \subseteq V(D)$. Then for positive integer $t$,

$$\bigcup_{i=0}^{t} R_i(X) \geq \min\{|X| + t, n\}.$$
Theorem 2.5. Let $A \in P_n$. If there exist distinct $i,j,k$ such that $a_{ij} = a_{ji} = a_{jk} = a_{ki} = 1$, then $h^*_a(A) \leq n + r$, $-n + 4 \leq r \leq n$.

Proof. $D(A)$ contains the 2-cycle $(i,j,i)$ and the 3-cycle $(i,j,h,i)$. Let $Z = \{i,j\}$, $X \subset \{1,2,\ldots,n\}, |X| = k$, $1 \leq k \leq \min\{n,n-r\}, k \neq n$. We will show that

$$|R_t(X)| \geq |X| + r \quad \text{for} \quad t \geq n + r.$$

Let $x^*$ be a vertex in $X$ and $z^*$ be a vertex in $Z$ such that $d = d(x^*,z^*) = \min\{d(x,z) \mid x \in X, z \in Z\}$ (if $z^* = x^*$, then $d = 0$). Then

$$d \leq n + 1 - |Z| - |X| = n + 1 - 2 - k = n - 1 - k.$$

Applying (1.2), there exists a walk of length

$$t \geq n - 1 - k + \phi(2,3) = n + 1 - k \quad \text{from} \quad x^* \quad \text{to} \quad z^*.$$

If $t \geq n + r$, then for any nonnegative integer $i \leq k + r - 1$, since $t - i \geq n + r - (k + r - 1) = n + 1 - k$, there is a walk of length $t - i$ from $x^*$ to $z^*$. Hence for $i \leq k + r - 1$

$$R_i(\{z^*\}) \subseteq R_i(\{x^*\}).$$

By Lemma 2.4 we have

$$|R_t(X)| \geq |R_t(\{x^*\})| \geq \bigcup_{i=0}^{k+r-1} R_i(\{z^*\}) \geq k + r = |X| + r$$

for $t \geq n + r$. □

3. The $r$-indecomposability of symmetric primitive matrices

We now investigate the strict exponent.

The theorem below is straightforward extension of results on $h^*_1(A)$ in [4].

Theorem 3.1. Let $s$ be a positive integer, and let $A$ be a matrix in $P_n$ having $s$ 1’s on its main diagonal. Then

$$h^*_r(A) \leq n - s + r, \quad -n + s < r < n.$$
In this section we will give an exact representation of strict exponent for $A \in SP_n$.

**Theorem 3.2.** If $A \in SP_n(n > 1)$, then

$$\max_{A \in SP_n} h^*_r(A) = \begin{cases} 
  n + r - 1, & n + r \text{ odd, } -n + 1 < r < n, \\
  n + r - 2, & n + r \text{ even, } -n + 2 < r < n.
\end{cases}$$

(1)

**Proof.** By (1.1), the associated digraph $D(A)$ of $A$ must contain an $s$-cycle $C_s$ ($s$ is odd).

Let $\emptyset \neq X \subset V(D)$ and $|X| = k < n$. Let $x^*$ be a vertex in $X$ and $y \in V(C_s)$ such that $x^*$ has the minimum distance $d$ to $y$ among all vertices in $X$ and vertices in $C_s$. Then

$$d \leq n - (s - 1) - k.$$ 

Applying (1.2) for

$$m \geq n - (s - 1) - k + \phi(2, s) = n - (s - 1) - k + (2 - 1)(s - 1) = n - k$$

there exists a walk meeting $C_s$ of length $m$ from $x^*$ to $y$ or some $u \in V(D)$.

(3.1) Taking

$$m \geq n - (s - 1) - k + \phi(2, s) = n - (s - 1) - k + (2 - 1)(s - 1) = n - k$$

we get some sets of vertices, which can be reached from $x^*$ by a walk of length $m$. Those sets are denoted by $S(n - k + i)$, $S(n - k + (k + r - 1))$ respectively.

By (3.1) $|S(n - k + i)| \geq i + 1$, we have

$$|S(n - k + (k + r - 1))| = |S(n + r - 1)| \geq k + r.$$ 

Thus for $t \geq n + r - 1$

$$|R_t(X)| \geq |R_t(x^*)| \geq |S(n + r - 1)| \geq k + r = |X| + r.$$ 

Hence $h^*_r(A) \leq n + r - 1$. We will show that $h^*_r(A) \leq n + r - 2$, if $n + r$ is even and $-n + 2 < r < n$. Since $A^2$ contains a loop at each vertex, $|R_t(X)| \geq |X| + t$ in $A^2$. Therefore $|R_{2t}(X)| \geq |X| + t$ in $A$. In particular

$$|R_{n+r-2}(X)| \geq |X| + \frac{n + r - 2}{2} \geq |X| + r$$

in $A$.

($n \geq r + 2$ since $n + r$ is even). As above result $|R_t(X)| \geq |X| + r$ for all $t \geq n + r - 1$. We have $|R_t(X)| \geq |X| + r$ for all $t \geq n + r - 2$, i.e.

$$h^*_r(A) \leq n + r - 2, \text{ if } n + r \text{ is even.}$$

If $n + r$ is odd, let $a = (n - r + 1)/2$. 

If \( n + r \) is even, let \( b = (n - r)/2 \),

\[
A_1 = \begin{bmatrix}
1 & & & & \vdots & & 1 \\
& 1 & & & & & \vdots \\
& & 1 & & & & 1 \\
& & & O_{a \times a} & & & \\
& & & & 1 & & 1 \\
& & & & & 1 & 1 \\
& & & & & & \vdots \\
& & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}_{n \times n}
\]

\[
A_2 = \begin{bmatrix}
1 & & & & \vdots & & 1 \\
& 1 & & & & & \vdots \\
& & 1 & & & & 1 \\
& & & O_{b \times b} & & & \\
& & & & 1 & & 1 \\
& & & & & 1 & 1 \\
& & & & & & \vdots \\
& & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}_{n \times n}
\]

It is easy to verify that

\( A_1^{n+r-2} = \begin{bmatrix} O_{a \times a} & J \\ J & J \end{bmatrix} \),

\( A_1^{n+r-1} = J \) (\( n + r \) odd), \( A_1^{n+r-2} \) contains an \( a \times a \) zero submatrix, \( a + a = n - r + 1 \), \( A_1^{n+r-2} \) is not \( r \)-inde. But \( A_1^{n+r-1} \) is \( r \)-inde.

\[
A_2^{n+r-3} = \begin{bmatrix}
1 & 0 & 1 & 1 \\
& \vdots & \vdots & \vdots \\
& 1 & 0 & 1 & 1 \\
& & 1 & 1 & 1 & 1 \\
& & 1 & 1 & 1 & 1 \\
& & 1 & 1 & 1 & 1 \\
\end{bmatrix}_{n \times n}
\]

(\( n + r \) even),

\( A_1^{n+r-1} = J \) (\( n + r \) odd), \( A_1^{n+r-2} \) contains an \( a \times a \) zero submatrix, \( a + a = n - r + 1 \), \( A_1^{n+r-2} \) is not \( r \)-inde. But \( A_1^{n+r-1} \) is \( r \)-inde.
\[ A_2^{n+r-2} = \begin{bmatrix} 
0 & 1 & 1 & 1 \\
J_{b\times b} & : & : & : \\
0 & 1 & 1 & 1 \\
0 & \cdots & 0 & 1 & 1 & 1 \\
1 & \cdots & 1 & : & : & : \\
1 & \cdots & 1 & : & : & : \\
1 & \cdots & 1 & 1 & 1 & 1 \\
\end{bmatrix}_{n\times n} \quad (n+r \text{ even}). \]

\( A_2^{n+r-3} \) contains an \( b \times (b+1) \) zero submatrix, \( b + (b+1) = n - r + 1 \). \( A_2^{n+r-3} \) is not \( r \)-inde. But \( A_2^{n+r-2} \) is \( r \)-inde. Hence we have

\[ h_r^+(A_1) = n + r - 1, \quad n + r \text{ odd}, \]
\[ h_r^+(A_2) = n + r - 2, \quad n + r \text{ even}. \]

The proof of the theorem is now completed. \( \square \)

Acknowledgements

The author is grateful to Dr. J. Shen for his careful reading and valuable suggestions.

References