Isometries of a generalized numerical radius

Maria Inez Cardoso Gonçalves a,*, Ahmed Ramzi Sourour b

a Departamento de Matemática, Universidade Federal de Santa Catarina, Trindade, Florianópolis, SC – 88.040-900, Brazil
b Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8P 5C2

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Abstract

For 0 < q < 1, the q-numerical range is defined on the algebra \( \mathcal{M}_n \) of all \( n \times n \) complex matrices by

\[
W_q(A) = \{x^*Ay : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1, \langle y, x \rangle = q \}.
\]

The q-numerical radius is defined by \( r_q(A) = \max\{|\mu| : \mu \in W_q(A)\} \). We characterize isometries of the metric space \((\mathcal{M}_n, r_q)\), i.e., the maps \( \varphi : \mathcal{M}_n \rightarrow \mathcal{M}_n \) that satisfy \( r_q(A - B) = r_q(\varphi(A) - \varphi(B)) \). We also characterize maps on \( \mathcal{M}_n \) that preserves the q-numerical range. The maps are not assumed to be linear.

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1. Introduction and statements of results

Let \( \mathcal{M}_n = \mathcal{M}_n(\mathbb{C}) \) be the algebra of all \( n \times n \) complex matrices. For 0 \( \leq q \leq 1 \), the q-numerical range is defined by

\[
W_q(A) = \{x^*Ay : x, y \in \mathbb{C}^n, x^*x = y^*y = 1, x^*y = q \}.
\]
This reduces to the classical numerical range when $q = 1$. The $q$-numerical radius is defined by $r_q(A) = \max\{|\mu| : \mu \in W_q(A)\}$. This is a norm [5] when $0 < q \leq 1$. For a matrix $C \in \mathcal{M}_n$, the $C$-numerical range is defined by

$$W_C(A) = \{\text{tr}(A^*U^*CU) : U \in \mathcal{U}_n\}.\quad (1)$$

where $\mathcal{U}_n$ is the group of all unitary matrices in $\mathcal{M}_n$ and $\text{tr}(\cdot)$ denotes the trace of a matrix. The $C$-numerical radius is defined by

$$r_C(A) = \max\{|z| : z \in W_C(A)\}.\quad (2)$$

When $C$ is a matrix of rank one, norm 1 and trace $q$, the $C$-numerical range reduces to the $q$-numerical range. In particular, $W_q(A) = W_{C_q}(A)$ where

$$C_q = qE_{11} + \sqrt{1 - q^2}E_{12}\quad (3)$$

and where the matrices $E_{ij}$ are the usual matrix units.

The $C$-numerical range is the image of the unitary group $\mathcal{U}_n$ under the continuous map $U \mapsto \text{tr}(CU^*AU)$, therefore it is a compact and connected subset of the complex plane $\mathbb{C}$. In particular the maximum in (2) exists. Another important property of the $q$-numerical range is that it is always convex [20].

The $C$-numerical radius is easily seen to be invariant under unitary conjugation, i.e., $r_C(U^*AU) = r_C(A)$ for every unitary $U$, but unlike the operator norm or the Hilbert–Schmidt (Frobenius) norm, it is not invariant under one-sided multiplication by a unitary. Indeed $C$-numerical radii may be viewed as building blocks of norms invariant under unitary conjugation, (see [15] or [2, p. 106]). They derive their importance, in part, from this and from applications to, e.g., quantum dynamics [19] or NMR spectroscopy [7].

As usual, by an isometry of a metric space, we mean a function that preserves distance. In particular, a $q$-numerical radius isometry is a mapping $f : \mathcal{M}_n \to \mathcal{M}_n$ that satisfies

$$r_q(f(A) - f(B)) = r_q(A - B),$$

for all $A, B \in \mathcal{M}_n$. In this paper, we characterize isometries of $q$-numerical radius and maps that preserve the $q$-numerical range for $0 < q < 1$. The case $q = 1$ (the classical numerical range and radius) are due to Lešnjak [10] and Li and Šemrl [16]. The complex-linear isometries of $r_q$ were characterized by Li et al. [12]. We assume no additivity or linearity of our maps. It will be shown below that the general case can be easily reduced to the study of real-linear isometries. Our proof is quite different from the proof in [12] and provides an alternative proof in the complex-linear case.

In Section 2, we prove our assertions for $n > 2$. In Section 3, we prove that the group of maps that preserve the $q$-numerical radius is generated by the groups of maps that preserve the $q$-numerical range together with multiplications by constants of modulus 1 and complex conjugations. Proofs for $n = 2$ are given in Section 4.

We pause to fix notation and terminology. All vectors in $\mathbb{C}_n$ are assumed to be column vectors so that, for $x, y \in \mathbb{C}_n$, the expression $x^*y$ is the usual inner product of $x$ and $y$ while $xy^*$ or $xy^t$ is a matrix of rank at most one. We also observe that every matrix of rank one is of this form.

The standard basis of $\mathcal{M}_n$ is denoted by $\{e_1, e_2, \ldots, e_n\}$, i.e., $e_j$ is the column vector whose $j$th entry is 1 and all other entries are 0. We denote the standard matrix units by $E_{ij}$, i.e., $E_{ij} = e_i e_j^*$. For a matrix $A = [a_{jk}]$, the matrix $\overline{A}$ denotes the matrix $\overline{A} = [\overline{a_{jk}}]$ obtained by taking the entry-wise complex-conjugate of $A$. 
The unitary group is denoted by $U_n$. The unitary orbit $U(A)$ of a matrix $A$ is the set

$$U(A) := \{ U^*AU : U \in U_n \}.$$ 

The set

$$SU(A) := \{ \mu U^*AU : \mu \in \mathbb{C}, |\mu| = 1, U \in U_n \}$$

will be called the saturated unitary orbit of $A$.

We denote the operator norm of a matrix $A$ by $\|A\|$ and its Hilbert–Schmidt (Frobenius) norm by $\|A\|_2$. We note that $\|xy^*\| = \|xy^*\|_2 = \|x\|\|y\|$ where the vector norms are the Euclidian ($\ell^2$) norms on $\mathbb{C}^n$.

A mapping $\varphi : M_n \to M_n$ is said to preserve the $q$-numerical range if $W_q(\varphi(A)) = W_q(A)$ for all $A \in M_n$ and is said to preserve the $q$-numerical radius if $r_q(\varphi(A)) = r_q(A)$ for all $A \in M_n$.

The space $M_n(\mathbb{C})$ is a Hilbert space under the (complex) inner product

$$\langle A, B \rangle_\mathbb{C} = \text{tr}(AB^*),$$

It is also a real Hilbert space under the (real) inner product

$$\langle A, B \rangle_\mathbb{R} = \text{Re}(\text{tr}(AB^*)),$$

where $\text{Re}(\cdot)$ denotes the real part of a complex number. By the dual $\tau'$ of a real-linear operator $\tau$ on $M_n$, we mean the “adjoint” relative to the real inner product $\langle \cdot, \cdot \rangle_\mathbb{R}$, i.e., the unique real-linear operator $\tau' : M_n \to M_n$ that satisfies:

$$\langle \tau(A), B \rangle_\mathbb{R} = \langle A, \tau'(B) \rangle_\mathbb{R} \text{ for all } A, B \in M_n.$$  

We now state our main results.

**Theorem 1.1.** Let $0 < q < 1$ and let $\varphi : M_n \to M_n$. Then $\varphi$ is an isometry of the $q$-numerical radius if and only if there exists a matrix $S_0 \in M_n$, a unitary $U \in U_n$ and a complex number $\mu$ with $|\mu| = 1$ such that for all $A \in M_n$

$$\varphi(A) = S_0 + \mu U^*A^\dagger U,$$

where $A^\dagger$ denotes either $A$ or $A^t$ or $A^* \text{ or } \overline{A}$. 

**Theorem 1.2.** Let $0 < q < 1$ and let $\varphi : M_n \to M_n$. Then $\varphi$ preserves the $q$-numerical range, i.e., $W_q(\varphi(A)) = W_q(A)$ for every $A \in M_n$, if and only if there exists a unitary $U \in U_n$ such that

$$\varphi(A) = U^*AU \text{ or } \varphi(A) = U^*A^tU$$

for every $A \in M_n$.

We end this section by describing the reduction to real-linear maps alluded to above. Let $(X, \| \cdot \|)$ be a finite dimensional (real or complex) Banach space and $\varphi : X \to X$ satisfy $\|\varphi(x) - \varphi(y)\| = \|x - y\|$. Define $\tau : X \to X$ by $\tau(x) = \varphi(x) - \varphi(0)$. Then $\|\tau(X)\| = \|X\|$. It is known ([3,17], [1, p. 66]) that such isometries are real-linear.

In view of the above, we shall assume henceforth that all isometries are real-linear.
2. Proofs of main results for \( n > 2 \)

We shall prove Theorem 1 directly for \( n > 2 \). Theorem 2 (for \( n > 2 \)) will then follow easily. In what follows we shall assume that \( \varphi : \mathcal{M}_n \to \mathcal{M}_n \) is real-linear and satisfies \( r_q(\varphi(A)) = r_q(A) \), for every \( A \in \mathcal{M}_n \). We denote the dual of \( \varphi \) by \( \psi \).

The following result describes the condition on the dual of \( \varphi \) that is equivalent to the isometric property of \( \varphi \). For complex-linear mapping \( \varphi \), this was proved in [14]. Although the proof given there can be easily modified to prove the real-linear case, we prefer to provide a different proof.

**Lemma 2.1.** Let \( 0 < q \leq 1 \), \( p = (1 - q^2)^{1/2} \) and \( C_q = qE_{11} + pE_{12} \). Assume that \( \varphi : \mathcal{M}_n \to \mathcal{M}_n, n \geq 2 \), is a real-linear map with dual \( \psi \). Then \( \varphi \) is an \( r_q \)-isometry if and only if \( \psi \) satisfies \( \psi(\mathcal{U}(C_q)) = \mathcal{U}(C_q) \) where \( \mathcal{U}(C_q) \) is the saturated unitary orbit of \( C_q \).

**Proof.** The map \( \varphi \) is an \( r_q \)-isometry if and only if its dual \( \psi \) maps the unit ball \( B' \) of the dual norm \( r_q' \) onto itself. Since \( B' \) is the convex hull of \( \text{extr}(B') \), the set of its extreme points, it follows that \( \psi \) maps the ball \( B' \) onto itself if and only if it maps the set \( \text{extr}(B') \) into itself. As observed in [12], it is easy to see that the set of extreme points of \( B' \) is precisely \( \mathcal{U}(C_q) \). \( \square \)

Let \( C \) be a matrix of rank one. If \( \psi \) maps \( \mathcal{U}(C) \) onto itself, then it maps the sum of two members of \( \mathcal{U}(C) \) to a sum of two such members. Now, if \( R \in \mathcal{M}_n \) is a sum of two matrices \( A, B \) in \( \mathcal{U}(C) \), then \( R \) can have rank two, one or zero. In the next two lemmas we give a tool to distinguish between the rank one case and rank two case by the dimension of the span of all matrices \( A, B \in \mathcal{U}(C) \) such that \( R = A + B \). This will allow us to prove that \( \varphi \) preserves the set of rank one matrices.

In the following Lemma by \( \text{span} \mathcal{S} \) we mean the real-linear span of the set \( \mathcal{S} \).

**Lemma 2.2.** Let \( n > 2 \), let \( R \) be a rank one matrix with \( \| R \| < \min\{2q, 2p\} \), where \( p = (1 - q^2)^{1/2} \), and let

\[
\mathcal{S}(R) := \{ A \in \mathcal{U}(C) : R = A + B \text{ for some } B \in \mathcal{U}(C) \},
\]

and let \( \mathcal{M}(R) = \text{span} \mathcal{S}(R) \). Then \( \mathcal{M}(R) \) is a vector space of dimension \( 4n - 2 \) over \( \mathbb{R} \).

**Proof.** Let \( R = uv^* \). We shall prove that \( \mathcal{M}(R) = \mathcal{U}_1 + \mathcal{U}_2 \), where

\[
\mathcal{U}_1 = \{ yv^* : y \in \mathbb{C}^n \}, \quad \mathcal{U}_2 = \{ xv^* : x \in \mathbb{C}^n \}.
\]

First assume that \( R = A + B \) for matrices \( A, B \) in \( \mathcal{U}(C) \). If \( A = ac^t \) and \( B = bd^t \) for vectors \( a, b, c, d \in \mathbb{C}^n \), then either \( a \) and \( b \) are linearly dependent (in which case, we may take \( a = b = u \)), or \( c \) and \( d \) are linearly dependent and again we may take \( c = d = v \). This means that either \( A \) and \( B \) are in \( \mathcal{U}_1 \) or both are in \( \mathcal{U}_2 \). Therefore \( \mathcal{M}(R) \subseteq \mathcal{U}_1 + \mathcal{U}_2 \).

For the converse, we may replace \( R \) by any matrix from its unitary orbit, hence we may assume that \( R = \xi E_{11} + \eta E_{12} \) and so \( u = e_1 \). By assumption, we have \( |\xi| < 2q \) and \( |\eta| < 2p \). For every \( q' \) and \( p' \) with \( |\xi|/2 < q' < q \) and \( |\eta|/2 < p' < p \), there exist complex numbers \( z_j \), \( 1 \leq j \leq 4 \), such that \( |z_1| = |z_2| = q' \), \( |z_3| = |z_4| = p' \). Let \( r = (1 - (p')^2 - (q')^2)^{1/2}, t \in \mathbb{R}, \)

\[
3 \leq k \leq n
\]

and

\[
A_t = z_1 E_{11} + z_2 E_{12} + re^{it} E_{1k}, \quad B_t = z_3 E_{11} + z_4 E_{12} - re^{it} E_{1k}.
\]

Then \( A_t, B_t \in \mathcal{U}(C) \) and \( A_t + B_t = R \). With all the various choices of \( p', q' \) and \( t \), it is easy to see that \( \mathcal{M}(R) \) contains \( E_{1j} + iE_{1j} \) for \( 1 \leq j \leq n \), hence it contains \( e_1 y^* \) for every
y ∈ C^n. This proves that L_1 ⊂ M(R). By symmetry we also get L_2 ⊂ M(R). This proves that M(R) = L_1 + L_2. Each of L_1 and L_2 is a space of dimension 2n over R and dim(L_1 ∩ L_2) = 2. Therefore M(R) has dimension 4n − 2 over R. □

**Lemma 2.3.** Let R be a rank two matrix in M_n, n ≥ 2, and let R denote the set of all rank one matrices in M_n. Then the set

\[ \mathfrak{M}(R) := \{ A ∈ R : R = A + B \text{ for some } B ∈ R \} \]

spans a real vector space of dimension 7 and so dim(\mathfrak{M}(R)) ≤ 7, where \mathfrak{M}(R) is as defined in Lemma 2.2.

**Proof.** We view matrices in M_n as complex-linear operators on the complex vector space C^n. If \( A = ab^* \) is a rank one matrix, then range(\( A \)) = span(a) and null(\( A \)) = \{ b \}^⊥. Therefore if \( A = ab^* \) and \( B = cd^* \) are rank one matrices, then

\[ \text{range}(A + B) ⊆ \text{span}(a, c); \quad \text{null}(A + B))^⊥ ⊆ \text{span}(b, d). \]

If A + B has rank 2, then we must have equality in each of the above inclusions. If we now start with a rank 2 matrix R, let \( R = \text{range}(R) \) and \( \mathfrak{M} = \text{null}(R) \). If R is a sum of two rank one matrices \( A = ab^* \) and \( B = cd^* \), then from the above, we must have that \( \mathfrak{M} = \text{span}(a, c) \) and \( \mathfrak{M}^⊥ = \text{span}(b, d) \). So the operators A and B vanish on \( \mathfrak{M} \) and map the two dimensional space \( \mathfrak{M}^⊥ \) into the two-dimensional space \( \mathfrak{M} \).

The above reduces the problem to the case n = 2. The assumption and the conclusion are unperturbed under the map \( T \mapsto T^{-1}T \), so we may assume that \( R = I \). In this case we show that the span of \( \mathfrak{M}(I) \) is the space of all 2 × 2 complex matrices with real trace, which clearly has dimension 7. First if \( I = A + B \), then the spectrum of \( A \) is \( \{0, 1\} \) and so \( A \) has real trace. Conversely assume that \( K \) is 2 × 2 complex matrix with real trace 2t. If \( K = tI \), then it is easy to write \( K \) as a sum of two rank one matrices. If \( K \) is not a scalar, then \( K \) is similar to a matrix with diagonal entries \( (0, 2t) \) and again it is easy to see that \( C \) is a sum of two rank one matrices.

Since \( \mathfrak{M}(R) ⊆ \text{span}(\mathfrak{M}(R)) \), we have dim(\( \mathfrak{M}(R) \)) ≤ 7. □

**Lemma 2.4.** Let n > 2. Assume that \( ψ \) is a bijective real-linear map on M_n and that \( ψ(\mathcal{H}(C_q)) = \mathcal{H}(C_q) \). Then \( ψ \) preserves rank one matrices.

**Proof.** Assume that \( R ∈ M_n \) has rank one and let \( R' = λR \), where 0 < λ < (min(2q, 2p))∥R∥−1. Then \( R' \) satisfies the norm condition of Lemma 2.2. Thus dim(\( \mathfrak{M}(R') \)) = 4n − 2. By (real) linearity, we have that \( ψ \) maps \( \mathfrak{M}(R') \) onto \( ψ(\mathfrak{M}(R')) \). Therefore dim(\( ψ(\mathfrak{M}(R')) \)) = 4n − 2. For \( n ≥ 3 \) one has 4n − 2 ≥ 10 > 7 and so, by Lemma 2.3, \( ψ(R') \) is not of rank 2. It is also nonzero since \( ψ \) is bijective. Therefore \( ψ(R') \) and hence also \( ψ(R) \) has rank one. □

**Corollary 2.5.** Let n > 2 and assume that \( φ : M_n → M_n \) preserves the q-numerical radius and let \( ψ \) be the dual of \( φ \). Then there exist invertible matrices \( P, Q \) such that

\[ ψ(A) = PA^tQ, \text{ for all } A ∈ M_n, \]  \hspace{1cm} (7)

where \( A^t \) denotes either \( A \) or \( A' \) or \( A^* \) or \( A \).

**Proof.** From the previous results we know that \( ψ \) preserves rank one matrices. Then, by a result of Omladić and Šemrl, [18], we get that \( ψ \) is of the form:

\[ ψ(A) = PA^tQ \text{ or } ψ(A) = P A^t Q, \]
where \( P \) and \( Q \) are invertible matrices, and \( \tilde{A} \) is obtained from \( A \) by applying a field automorphism \( c \mapsto \tilde{c} \) entrywise. But the only real-linear automorphisms of \( \mathbb{C} \) are the identity and complex conjugation. This proves the corollary. \( \square \)

Next, we aim to show that the matrices \( P \) and \( Q \) in Corollary 2.5 are unitary matrices and that their product is a scalar. We will write this part of the proof for any complex Hilbert space \( H \)(rather than \( \mathbb{C}^n \)). As usual \( L(H) \) denotes the space of all complex-linear operators from \( H \) into itself. In the following \( \langle \cdot, \cdot \rangle \) denotes the (complex) inner-product in the complex Hilbert space.

**Lemma 2.6.** Let \( H \) be a complex Hilbert space. If \( u, v \in H \) and \( \|u + e^{is}v\| = 1 \) for all \( s \in \mathbb{R} \), then \( u \) is orthogonal to \( v \).

**Proof.** From \( 1 = \langle u + e^{is}v, u + e^{is}v \rangle \), we get

\[
\|u\|^2 + \|v\|^2 - 1 + \langle v, u \rangle e^{is} + \langle u, v \rangle e^{-is} = 0 \text{ for every } s \in \mathbb{R}.
\]

Therefore \( \langle u, v \rangle = 0 \). \( \square \)

**Lemma 2.7.** Let \( \mathcal{L}(H) \) be a Hilbert space of dimension \( \geq 2 \) and let \( T \in \mathcal{L}(H) \). Then \( T \) is a scalar multiple of an isometry if and only if \( Tx \) is orthogonal to \( Ty \) whenever \( x \) is orthogonal to \( y \).

**Proof.** The “only if” part is clear. To prove the converse, assume that \( x, y \in H \) and that \( \|x\| = \|y\| = 1 \). If \( x \) is orthogonal to \( y \), then \( x + e^{it}y \) is orthogonal to \( x - e^{it}y \) for all \( t \in \mathbb{R} \). By hypothesis we have \( \langle Tx, Ty \rangle = 0 \) and \( \langle T(x + e^{it}y), T(x - e^{it}y) \rangle = 0 \), this implies that \( \|Tx\| = \|Ty\| \).

If \( x \) and \( y \) are not orthogonal, let \( u = y - \langle y, x \rangle x \). Then \( u \) is orthogonal to \( x \) and hence \( Tx \) is orthogonal to \( Tu \). Thus

\[
\langle Tx, Ty \rangle = \langle Tx, Tu \rangle + \langle y, x \rangle \langle Tx, Tx \rangle = \langle y, x \rangle \langle Tx, Tx \rangle.
\]

By symmetry, we get

\[
\langle Ty, Tx \rangle = \langle x, y \rangle \langle Ty, Ty \rangle.
\]

The two equations imply that once again we have \( \|Tx\| = \|Ty\| \). Thus \( T \) is a scalar multiple of an isometry. \( \square \)

**Proposition 2.8.** Assume that \( n \geq 2 \) and that \( \psi(A) = PA^\dagger Q \) for every \( A \in M_n \), where \( A^\dagger \) is either \( A \) or \( A^t \) or \( A^* \) or \( \overline{A} \), and \( P, Q \in M_n \). If \( \psi(\mathcal{U}(C_q)) = \mathcal{U}(C_q) \), then each of \( P \) and \( Q \) is a scalar multiple of a unitary matrix and \( PQ = \lambda I \) for a complex number \( \lambda \) of modulus 1. Consequently, there exists a unitary matrix \( U \) such that \( \psi(A) = \lambda U A^\dagger U^* \) for every \( A \in M_n \).

**Proof.** Since each of the transpose map, the adjoint map and complex conjugation maps \( \mathcal{U}(C_q) \) onto itself, it suffices to prove the proposition when \( \psi(A) = PAQ \).

Let \( x, y \in \mathbb{C}^n \) be such that \( x \) is orthogonal to \( y \) and \( \|x\| = \|y\| = 1 \). Let \( s \in \mathbb{R} \) and \( z = qx + e^{is}\sqrt{1 - q^2}y \). Then \( xz^* \) is a matrix of rank 1, norm 1 and trace \( q \), hence \( xz^* \in \mathcal{U}(C_q) \). Thus \( Pxz^*Q = \psi(xz^*) \in \mathcal{U}(C_q) \), which implies that:
(a) \( \| Pxz^* Q \| = 1 \)
(b) \( |\text{tr}(Pxz^* Q)| = q \)

From condition (a), we have \( 1 = \| Pxz^* Q \| = \| P \| \| Q^* z \| \), therefore \( q \| Q^* x + e^{is} \sqrt{1 - q^2} Q^* y \| = K \) for all \( s \in \mathbb{R} \), where \( K \) is a positive constant. By Lemma 2.6 we have that \( Q^* x \) is perpendicular to \( Q^* y \).

From the above, we conclude that \( Q^* \) preserves orthogonality. By Lemma 2.7, \( Q^* \) is a scalar multiple of an isometry. This easily implies that \( Q \) is a scalar multiple of a unitary matrix. Similarly, \( P \) is a scalar multiple of a unitary matrix.

The trace condition (b) implies that \( |q \langle QPx, x \rangle + q^2 e^{is} \langle QPx, y \rangle| = q \), for every \( s \in \mathbb{R} \). This implies that \( \langle QPx, x \rangle = 0 \) or \( |\langle QPx, x \rangle| = 1 \). As this is true for every unit vector \( x \), we get that \( W(QP) \), the classical numerical range of \( QP \) is included in the union of the unit circle and \( \{0\} \). The well-known convexity of the classical numerical range implies that \( W(QP) \) is a singleton of modulus 0 or 1. But \( QP \) is evidently nonzero, hence \( QP = \lambda I \), for a complex number \( \lambda \) of modulus 1. □

We are now in a position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1 (for \( n \geq 3 \)). Since \((C_q)^t, (C_q)^* \) and \( \overline{C_q} \) are in \( \mathcal{U}(C) \), it follows that:

\[ W_q(A^t) = W_q(A), \quad \text{and} \quad W_q(A^*) = W_q(\overline{A}) = \overline{W_q(A)}. \]

Thus \( r_q(A^t) = r_q(A^*) = r_q(\overline{A}) = r_q(A) \). The “if” part follows easily.

Conversely, if \( \varphi \) is an \( r_q \) isometry with \( \varphi(0) = 0 \), and if \( \psi \) is the dual of \( \varphi \) then by Corollary 2.5 and Proposition 2.8, there exists a unitary matrix \( U \) and a complex number \( \lambda \) of modulus 1 such that

\[ \psi(A) = \lambda U A^\dagger U^* \] for every \( A \in \mathcal{M}_n \).

It follows that

\[ \varphi(A) = \overline{\lambda} U^* A^\dagger U \] for every \( A \in \mathcal{M}_n \).

This finishes the proof in the case \( \varphi(0) = 0 \). The general case can be reduced to this by taking \( S_0 = \varphi(0) \). □

Proof of Theorem 1.2 (for \( n \geq 3 \)). This follows from Theorem 1.1 together with the facts that

\[ W_q(iI^*) = W_q(i\overline{I}) = \{-iq\} \neq \{iq\} = W_q(iI) \]

and that \( W_q(\mu I) = W_q(I) \) only if \( \mu = 1 \). □

3. The relationship between preservers of the \( q \)-numerical radius and the preservers of the \( q \)-numerical range

In this section, we prove that the group of isometries of the norm \( r_q \) is generated by the group of maps that preserve the \( q \)-numerical range \( W_q \) together with complex conjugation and multiplications by constants of modulus 1. More precisely, we prove (for all \( n \geq 2 \)) that \( \varphi \) preserves
the \( q \)-numerical radius if and only if there exists a map \( \varphi_1 \) that preserves the \( q \)-numerical range and a complex number \( \mu \) of modulus 1, such that \( \varphi = \mu \varphi_1 \) or \( \varphi = \mu \overline{\varphi_1} \), where \( \overline{\varphi_1} \) is defined by \( \overline{\varphi_1}(A) = \overline{\varphi(A)} \).

We observe that for \( n > 2 \), this relationship follows from the results in the previous section, but we shall give below an independent proof which is also valid for \( n = 2 \).

One step along the way is to show that if \( \varphi \) preserves the \( q \)-numerical radius, then \( \varphi(CI) = CI \). We start with a simple lemma.

**Lemma 3.1.** If \( \alpha, \beta \) and \( \gamma \in \mathbb{C} \) satisfy \( |\alpha \cos t + \beta \sin t + \gamma| = 1 \) for all \( t \in \mathbb{R} \), then either \( \alpha = \beta = 0 \) or \( \gamma = 0 \), \( \beta = \pm i \alpha \), \( |\alpha| = |\beta| = 1 \).

**Proof.** By assumption, we have \(|(\alpha - i\beta)e^{it} + (\alpha + i\beta)e^{-it} + 2\gamma| = 2\). By expanding this expression and using the orthogonality of the trigonometric system we get that two of \( \alpha - i\beta, \alpha + i\beta \) and \( \gamma \) are zero. This easily implies the conclusion of the lemma.

The following result of Fillmore [4], (cf. [8, pp. 18–19], is quite useful.

**Proposition 3.2.** Every matrix in \( \mathcal{M}_n \) is unitarily equivalent to a matrix with constant diagonal.

We now state and prove the “dual statement” to the statement that \( \varphi(CI) = CI \).

**Theorem 3.3.** Let \( \varphi : \mathcal{M}_n \to \mathcal{M}_n, n \geq 2 \), be a map that preserves \( q \)-numerical radius. Then \( \psi \), the dual of \( \varphi \), leaves the set of matrices of trace zero invariant.

**Proof.** By Proposition 3.2, every matrix of trace zero is unitarily equivalent to a matrix with zero diagonal and hence is a sum of rank one matrices of trace zero, so it suffices to prove that \( \text{tr}(\psi(R)) = 0 \) for every matrix \( R \) of rank one and trace zero. Every such matrix \( R \) is of the form \( \lambda uv^* \) where \( \lambda > 0 \) and \( u, v \) are orthonormal vectors in \( \mathbb{C}^n \). Hence, it suffices to prove that \( \text{tr}(\psi(uv^*)) = 0 \) for every pair of orthonormal vectors \( u \) and \( v \).

Towards this end, let \( u \) and \( v \) be orthonormal vectors and let \( A = \psi(uu^*) \), \( B = \psi(iuu^*) \) and \( D = \psi(uv^*) \). Clearly

\[
X_t := quu^* \cos t + iquu^* \sin t + puv^* \in \mathcal{S}\mathcal{U}(C_q)
\]

for every real number \( t \). By Lemma 2.1 \( \psi(X_q) \in \mathcal{S}\mathcal{U}(C_q) \), i.e.,

\[
qA \cos t + qB \sin t + pD \in \mathcal{S}\mathcal{U}(C_q) \text{ for all } t.
\]

Hence

\[
|\text{tr}(A) \cos t + \text{tr}(B) \sin t + \frac{p}{q}\text{tr}(D)| = 1
\]

for every \( t \in \mathbb{R} \).

By Lemma 3.1 we have that either \( \text{tr}(A) = \text{tr}(B) = 0 \) or \( \text{tr}(D) = 0 \), \( |\text{tr}(A)| = |\text{tr}(B)| = 1 \). We shall prove next that the former alternative is impossible.

The map \( u \mapsto \text{tr}(\psi(uu^*)) \) is evidently a continuous map from the unit sphere of \( \mathcal{M}_n \) into \( \mathbb{C} \), and from the above it takes values in the union of the unit circle and \{0\}. Since the unit sphere is connected, its image is also connected. Hence if \( \text{tr}(\psi(u_0u_0^*)) = 0 \) for any unit vector \( u_0 \), then by connectedness, we must have that \( \text{tr}(\psi(uu^*)) = 0 \) for every unit vector \( u \). From the above we would also have that \( \text{tr}(\psi(iuu^*)) = 0 \) for every unit vector \( u \). Thus \( \text{tr}(\psi(A)) = 0 \) for every \( A \) in
the real linear span of these two classes of matrices of rank one. By the spectral theorem, every hermitian matrix is a real linear combination of matrices of the form $uu^*$ and every skew-hermitian matrix is a real linear combination of matrices of the form $i uu^*$. Therefore we would have that $\text{tr}(\psi(A)) = 0$ for every $A \in \mathcal{M}_n$. This is impossible, since $\psi$ is surjective.

Now that $\text{tr}(A) \neq 0$, we must have that $\text{tr}(D) = 0$. □

**Theorem 3.4.** Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n, n \geq 2$, be a map that preserves $q$-numerical radius. Then there exists a complex number $\alpha$ with $|\alpha| = 1$ such that $\varphi(I) = \alpha I$ and $\varphi(iI) = \pm i\alpha I$.

**Proof.** By theorem 3.3, $\psi$ leaves the space $s\ell_n$ of zero-trace matrices invariant. It follow that $\varphi$ leaves its orthogonal complement $(s\ell_n)^\perp$ invariant. It is easy to see that $(s\ell_n)^\perp = \mathbb{C}I$, the space of scalar multiples of the identity. Thus $\varphi(CI) = CI$.

From the above, we have that $\varphi(I) = \alpha I$ and $\varphi(iI) = \beta I$, for some $\alpha, \beta \in \mathbb{C}$. Since $\varphi$ preserves the $q$-numerical radius and $r_q(I) = r_q(iI) = q$, we have that $|\alpha| = 1$ and $|\beta| = 1$. We also have that $r_q(I + iI) = \sqrt{2}$, and hence $|\alpha + \beta| = \sqrt{2}q$. This easily implies that $\beta = \pm i\alpha$. □

**Theorem 3.5.** Let $\varphi : \mathcal{M}_n \to \mathcal{M}_n, n \geq 2$, be a map that preserves $q$-numerical radius. Then there exists a map $\varphi_1 : \mathcal{M}_n \to \mathcal{M}_n$ that preserves the $q$-numerical range and a complex number $\alpha$ of modulus 1 such that $\varphi = \alpha \varphi_1$ or $\varphi = \alpha \overline{\varphi_1}$.

**Proof.** Let $\alpha$ be as in Theorem 3.5 and let $\varphi_1 = \overline{\alpha} \varphi$ or $\overline{\alpha} \varphi$ according as $\varphi(iI) = i\alpha I$ or $\varphi(iI) = -i\alpha I$. The map $\varphi_1$ is easily seen to preserve the $q$-numerical radius and, in addition, satisfies $\varphi_1(I) = I$ and $\varphi_1(iI) = iI$. It remains only to show that $\varphi_1$ preserves the $q$-numerical range. We prove this using the convexity of $W_q$ in the same way as done in [13,16].

If there is a $\mu \in W_q(\varphi_1(A)) \setminus W_q(A)$, then by a standard separation theorem for compact convex sets, there exists an $\eta \in \mathbb{C}$ such that $|\mu - \eta| > \max\{|z - \eta| : z \in W_q(A)\}$. Thus

$$r_q(\varphi_1(A - q^{-1}\eta I)) = r_q(\varphi_1(A) - q^{-1}\eta I) \geq |\mu - \eta| \geq \max_{z \in W_q(A)} |z - \eta|$$

$$= r_q(A - q^{-1}\eta I),$$

a contradiction which proves that $W_q(\varphi_1(A)) \subseteq W_q(A)$. A similar argument establishes the reverse inclusion. □

4. The case $n = 2$

In contrast to the case $n \geq 3$, we shall prove the characterization of preservers of the $q$-numerical range first and then use the results of Section 3 to establish the form of preservers of the $q$-numerical radius.

**Lemma 4.1.** Let $\varphi : \mathcal{M}_2 \to \mathcal{M}_2$ be a mapping that preserves the $q$-numerical range. Then $\varphi$ is real-linear and preserves spectrum and rank.

**Proof.** If $W_q(\varphi(A)) = W_q(A)$, then $r_q(\varphi(A)) = r_q(A)$ and as before, [3] implies that $\varphi$ is real-linear.
We shall use a description of the $C$-numerical range for $2 \times 2$ matrices due to Li [11]. By Proposition 3.2 combined with a permutation unitary, every $2 \times 2$ matrix $A$ is unitarily equivalent to a matrix $\hat{A}$ of the form:

$$
\hat{A} = \begin{bmatrix}
a & b \\
c & a \\
\end{bmatrix}
$$

with $|b| \geq |c|$.

Theorem 1 of [11] asserts that $W_C(A) = W(2\hat{A} \circ \hat{C})$, where $\circ$ denotes the Schur or Hadamard product, i.e., the entry-wise product, of two matrices. It is easy to see that the matrix

$$
\hat{C} = \begin{bmatrix}
q/2 & (1+p)/2 \\
(1-p)/2 & q/2 \\
\end{bmatrix}
$$

is in $U(C_q)$, where $p = \sqrt{1-q^2}$. Consequently, $W_q(A) = W(A_q)$, where

$$
A_q = \begin{bmatrix}
qa & (1+p)b \\
(1-p)c & qa \\
\end{bmatrix}.
$$

The classical numerical range of a $2 \times 2$ matrix is an ellipse whose foci are the eigenvalues (see [6]). If $\lambda_1, \lambda_2$ are the eigenvalues of $A$, then it is easy to see that the eigenvalues of $A_q$ are $q\lambda_1$ and $q\lambda_2$. Therefore $W_q(A)$ is an ellipse with eigenvalues $q\lambda_1$ and $q\lambda_2$. Since $W_q(\phi(A)) = W_q(A)$, it follows that $A$ and $\phi(A)$ have the same eigenvalues. This proves that $\phi$ preserves spectrum. Since a nonzero $2 \times 2$ matrix has rank one if and only if its spectrum includes zero, it follows that $\phi$ preserves rank. □

Remark. For every $n$, indeed for infinite-dimensional spaces as well, maps that preserve the spectrum also preserve the rank, cf. [9].

Proof of Theorems 1.1 and 1.2 (for $n = 2$). First, assume that $\phi$ preserves the $q$-numerical range. By Theorem 4.1, $\phi$ is real-linear and preserves rank. By [18], there exist invertible matrices $P, Q \in M_n$ such that $\phi(A) = PAQ$ or $PA^tQ$. Therefore $\psi(A) = P^*AQ^*$ or $P^*A^tQ^*$. Since $W_q(A) = \{q\}$ if and only if $A = I$, we conclude that $\phi(I) = I$ and hence $Q = P^{-1}$. By Proposition 2.8, $P$ is a scalar multiple of a unitary matrix $U^*$ and we get that $\phi$ is of the required form. The converse is obvious.

The characterization of maps that preserve the $q$-numerical radius follows from the above and Theorem 3.5. □

References